ECE 604, Lecture 23

November 29, 2018

In this lecture, we will cover the following topics:

- Rayleigh Scattering
- Mie Scattering
    - Separation of Variables in Spherical Coordinates

Additional Reading:

- Topic 7.1A, J.A. Kong, Electromagnetic Wave Theory.
- Section 1.2.5 Waves and Fields in Inhomogeneous Media.

You should be able to do the homework by reading the lecture notes alone. Additional reading is for references.

Printed on December 7, 2018 at 17:53: W.C. Chew and D. Jiao.

1 Rayleigh Scattering

Rayleigh scattering is a solution to the scattering of light by small particles. These particles are assumed to be much smaller than wavelength of light. Then a simple solution by the method of asymptotic matching. This single scattering solution can be used to explain a number of physical phenomena in nature. For instance, why the sky is blue, the sunset so magnificently beautiful, how birds and insects can navigate without the help of a compass. By the same token, it can also be used to explain why the Vikings can cross the Atlantic Ocean over to Iceland without the help of a magnetic compass.

Figure 1:

When a ray of light impinges on an object, we model the incident light as a plane electromagnetic wave (see Figure 2). Without loss of generality, we can assume that the electromagnetic wave is polarized in the z direction and propagating in the x direction. We assume the particle to be a small spherical particle with permittivity ε_s and radius a. Essentially, the particle sees a constant field as the plane wave impinges on it. In other words, the particle feels an almost electrostatic field in the incident field.

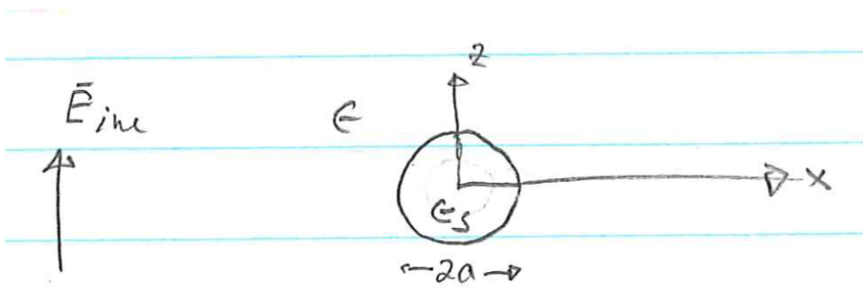

Figure 2:

The incident field polarizes the particle making it look like an electric dipole. Since the incident field is time harmonic, the small electric dipole will oscillate and radiate like a Hertzian dipole in the far field. Solving a boundary value problem by looking at the solutions in two different physical regimes, and then matching the solutions together is known as asymptotic matching.

A Hertzian dipole can be approximated by a small current source so that

$$\mathbf{J}(\mathbf{r}) = \hat{z} I l \delta(\mathbf{r}) \tag{1.1}$$

In the above, we can let the time-harmonic current $I = dq/dt = j\omega q$

$$Il = j\omega q l = j\omega p \tag{1.2}$$

where the dipole moment $p = ql$. The vector potential $\mathbf{A}$ due to a Hertzian dipole is

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu}{4\pi} \iiint_V d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{-j\beta|\mathbf{r}-\mathbf{r}'|} \\ &= \hat{z} \frac{\mu I l}{4\pi r} e^{-j\beta r}\end{aligned} \tag{1.3}$$

The corresponding potential $\Phi(\mathbf{r})$ is obtained from the Lorenz gauge that $\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\Phi$. Therefore,

$$\Phi(\mathbf{r}) = \frac{-1}{j\omega\mu\varepsilon} \nabla \cdot \mathbf{A} = -\frac{Il}{j\omega\varepsilon 4\pi} \frac{\partial}{\partial z} \frac{1}{r} e^{-j\beta r} \tag{1.4}$$

When we are close to the dipole, we can use a quasi-static approximation about the potential by assuming that $\beta r \ll 1$. Then

$$\frac{\partial}{\partial z} \frac{1}{r} e^{-j\beta r} \approx \frac{\partial}{\partial z} \frac{1}{r} = \frac{\partial r}{\partial z} \frac{\partial}{\partial r} \frac{1}{r} = -\frac{z}{r} \frac{1}{r^2} \tag{1.5}$$

or after using that $z/r = \cos\theta$,

$$\Phi(\mathbf{r}) \approx \frac{ql}{4\pi\varepsilon r^2} \cos\theta \tag{1.6}$$

This dipole induced in the small particle is formed in response to the incident field. The incident field can be approximated by a constant local static electric field,

$$\mathbf{E}_{inc} = \hat{z}E_i \tag{1.7}$$

The corresponding electrostatic potential for the incident field is

$$\Phi_{inc} = -\hat{z}E_i \tag{1.8}$$

so that $\mathbf{E}_{inc} = -j\omega\mathbf{A}_{inc} - \nabla\Phi_{inc} \approx -\nabla\Phi_{inc} = \hat{z}E_i$, $\omega \to 0$. The scattered dipole potential from the spherical particle in the vicinity of it is given by

$$\Phi_{sca} = E_s \frac{a^3}{r^2} \cos\theta \tag{1.9}$$

The electrostatic boundary value problem has been previously solved and[1]

$$E_s = \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} E_i \tag{1.10}$$

Using (1.10) in (1.9), and comparing with(1.6), one can see that the dipole moment induced by the incident field is that

$$p = ql = 4\pi\varepsilon \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} a^3 E_i \tag{1.11}$$

In the far field of the Hertzian dipole, we can start with

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi = -j\omega\mathbf{A} - \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{A} \tag{1.12}$$

But when we are in the far field, $\mathbf{A}$ behaves like a spherical wave which in turn behaves like a plane wave. Therefore, $\nabla \to -j\boldsymbol{\beta} = -j\beta\hat{r}$. Using this approximation in (1.12), we arrive at

$$\mathbf{E} = -j\omega\left(\mathbf{A} - \frac{\boldsymbol{\beta}\boldsymbol{\beta}}{\beta^2}\cdot\mathbf{A}\right) = -j\omega(\mathbf{A} - \hat{r}\hat{r}\cdot\mathbf{A}) = -j\omega(\hat{\theta}A_\theta + \hat{\phi}A_\phi) \tag{1.13}$$

where we have used $\hat{r} = \boldsymbol{\beta}/\beta$. From (1.3), we see that $A_\phi = 0$ while

$$A_\theta = -\frac{j\omega\mu ql}{4\pi r} e^{-j\beta r}\sin\theta \tag{1.14}$$

Consequently, using (1.11) for ql, we have in the far field that[2]

$$E_\theta \cong -j\omega A_\theta = -\frac{\omega^2\mu ql}{4\pi r} e^{-j\beta r}\sin\theta = -\omega^2\mu\varepsilon\left(\frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon}\right)\frac{a^3}{r}E_i e^{-j\beta r}\sin\theta \tag{1.15}$$

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[1] It was one of the exam problems.

[2] The ω^2 dependence of the following function implies that the radiated electric field in the far zone is proportional to the acceleration of the charges on the dipole.

$$H_\phi \cong \sqrt{\frac{\varepsilon}{\mu}} E_\theta = \frac{1}{\eta} E_\theta \tag{1.16}$$

where $\eta = \sqrt{\mu/\varepsilon}$. The total scattered power is

$$P_s = \frac{1}{2} \int_0^\pi r^2 \sin\theta d\theta \int_0^{2\pi} d\phi E_\theta H_\theta^* \tag{1.17}$$

$$= \frac{1}{2\eta} \beta^4 \left(\frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} \right)^2 \frac{a^6}{r^2} |E_i|^2 r^2 \left(\int_0^\pi \sin^3\theta d\theta \right) 2\pi \tag{1.18}$$

But

$$\int_0^\pi \sin^3\theta d\theta = -\int_0^\pi \sin^2\theta d\cos\theta = -\int_0^\pi (1-\cos^2\theta) d\cos\theta$$

$$= -\int_1^{-1} (1-x^2) dx = \frac{4}{3} \tag{1.19}$$

Therefore

$$P_s = \frac{4\pi}{3\eta} \left(\frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon_s} \right)^2 \beta^4 a^6 |E_i|^2 \tag{1.20}$$

The scattering cross section is defined as

$$\Sigma_s = \frac{P_s}{\frac{1}{2\eta}|E_i|^2} = \frac{8\pi a^2}{3} \left(\frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon} \right)^2 (\beta a)^4 \tag{1.21}$$

In other words, the first equality is the definition for scattering cross section: Namely

$$\langle S_{\mathrm{inc}} \rangle \times \Sigma_s = P_s$$

It is seen that the scattering cross section grows as the fourth power of frequency since $\beta = \omega/c$. The radiated field grows as the second power because it is proportional to the acceleration of the charges on the particle. The higher the frequency, the more the scattered power. this mechanism can be used to explain why the sky is blue. It also can be used to explain why sunset has a brilliant hue of red and orange. The above also explain the brilliant glitter of gold plasmonic nano-particles. For gold, the medium resembles a plasma, and hence, we can have $\varepsilon_s < 0$, and the denominator can be very small. Also, since

$$\langle S \rangle = \frac{1}{2\eta} E_\theta H_\phi^* \sim \sin^2\theta \tag{1.22}$$

the scattering pattern of a small particle is not isotropic. In other words, these dipoles radiate predominantly in the broadside direction but not in their end-fire directions. Therefore, insects and sailors can use this to figure out where the sun is even in a cloudy day. In fact, it is like a rain bow: If the sun is rising or setting in the horizon, there will be a bow across the sky where the scattered field is predominantly linearly polarized.

For a perfect electric conductor immersed in a time varying electromagnetic field, the magnetic field in the long wavelength limit induces eddy current in PEC sphere. Hence, a PEC sphere behaves like a magnetic dipole and scatters like one.

When the size of the dipole becomes larger, quasi-static approximation is insufficient to approximate the solution. Then one has to solve boundary value problem in its full glory usually called the full-wave theory or Mie theory. With this theory, the scattering cross section does not grow indefinitely with frequency. For a sphere of radius a, the scattering cross section becomes πa^2. This physics in shown in Figure 3, and also explains why the sky is not purple.

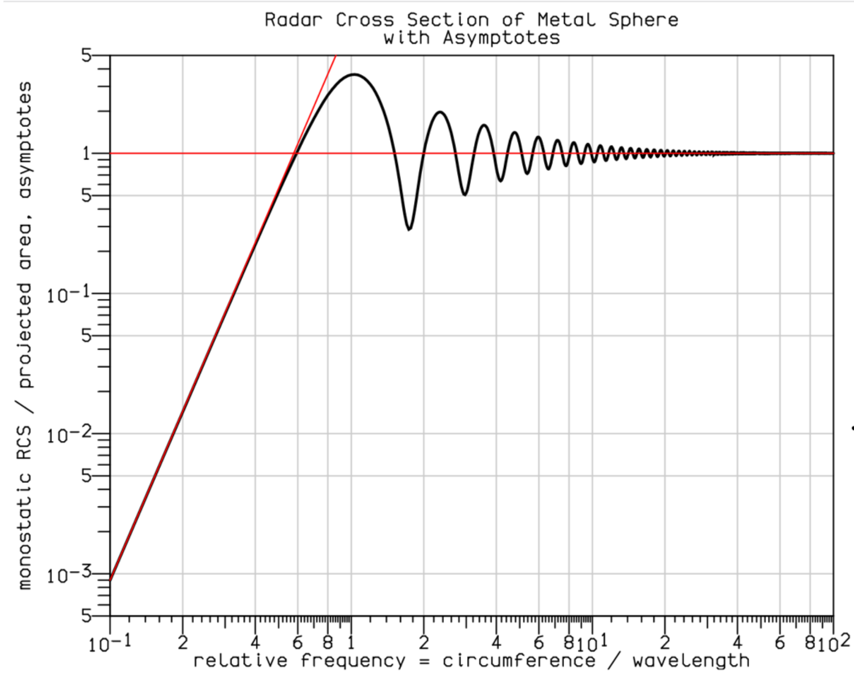

Figure 3:

2 Mie Scattering

The Mie scattering solution by a sphere is beyond the scope of this course.[3] This problem have to solved by the separation of variables in spherical coordinates. The separation of variables in spherical coordinates is not the only useful for Mie scattering, it is also useful for analyzing spherical cavity. So we will present

[3]But it is treated in J.A. Kong's book and Chapter 3 of my book, Waves and Fields in Inhomogeneous Media.

the precursor knowledge so that you can read further into Mie scattering theory if you need to in the future.

2.1 Separation of Variables in Spherical Coordinates

To this end, we look at the scalar wave equation $(\nabla^2+\beta^2)\Psi(\mathbf{r})=0$ in spherical coordinates. A lookup table can be used to evaluate $\nabla\cdot\nabla$, or divergence of a gradient in spherical coordinates. Hence, the Helmholtz wave equation becomes[4]

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\beta^2\right)\Psi(\mathbf{r})=0 \tag{2.1}$$

Noting the $\partial^2/\partial\phi^2$ derivative, by using separation of variables, we assume $\Psi(\mathbf{r})$ to be

$$\Psi(\mathbf{r})=F(r,\theta)e^{jm\phi} \tag{2.2}$$

where $\frac{\partial}{\partial\phi^2}e^{jm\phi}=-m^2e^{jm\phi}$. Then (2.1) becomes

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}-\frac{m^2}{r^2\sin^2\theta}+\beta^2\right)F(r,\theta)=0 \tag{2.3}$$

Again, by using separation of variables, and letting further that

$$F(r,\theta)=b_n(\beta r)P_n^m(\cos\theta) \tag{2.4}$$

where we require that

$$\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\right\}P_n^m(\cos\theta)=0 \tag{2.5}$$

when $P_n^m(\cos\theta)$ is the associate Legendre polynomial. Note that (2.5) is an eigenvalue problem, and $|m|\le|n|$. Then $b_n(kr)$ satisfies

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}-\frac{n(n+1)}{r^2}+\beta^2\right]b_n(\beta r)=0 \tag{2.6}$$

The above is the spherical Bessel equation where $b_n(\beta r)$ is either the spherical Bessel function $j_n(\beta r)$, spherical Newmann function $n_n(\beta r)$, or the sphereical Hankel functions, $h_n^{(1)}(\beta r)$ and $h_n^{(2)}(\beta r)$. The spherical functions are related to the cylindrical functions via[5]

$$b_n(\beta r)=\sqrt{\frac{\pi}{2\beta r}}B_{n+\frac{1}{2}}(\beta r) \tag{2.7}$$

[4]By quirk of mathematics, it turns out that the first term on the right-hand side below can be simplified by observing that $\frac{1}{r^2}\frac{\partial}{\partial r}r^2=\frac{1}{r}\frac{\partial}{\partial r}r$.

[5]By a quirk of nature, the spherical Bessel functions are in fact simpler than cylindrical Bessel functions. One can say that 3D is real, but 2D is surreal.

It is customary to define the spherical harmonic

$$Y_{nm}(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)e^{jm\phi} \tag{2.8}$$

The above is normalized such that[6]

$$Y_{n,-m}(\theta, \phi) = (-1)^m Y_{nm}^*(\theta, \phi) \tag{2.9}$$

and that

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y_{n'm'}^*(\theta, \phi) Y_{nm}(\theta, \phi) = \delta_{n'n}\delta_{m'm} \tag{2.10}$$

These functions are also complete like Fourier series, so that

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_{nm}^*(\theta', \phi') Y_{nm}(\theta, \phi) = \delta(\phi - \phi^{'})\delta(\cos\theta - \cos\theta^{'}) \tag{2.11}$$

[6]See J.D. Jackson, for instance.