

ECE 604, Lecture 22

November 27, 2018

In this lecture, we will cover the following topics:

- Reciprocity Theorem
 - Conditions for Reciprocity
 - Application to a Two-Port Network
 - Epilogue
- Paraxial Wave Equation

Additional Reading:

- Prof. Dan Jiao's Lecture 16.
- Sections 11.3 of Ramo, Whinnery, and Van Duzer.
- Topic 6.2, J.A. Kong, Electromagnetic Wave Theory.
- Section 1.6, Lecture on Theory of Optical and Microwave Waveguide.
- Section 1.3.2 Waves and Fields in Inhomogeneous Media.
- Section 3.1, Haus, Electromagnetic Noise and Quantum Optical Measurements.

You should be able to do the homework by reading the lecture notes alone. Additional reading is for references.

1 Reciprocity Theorem

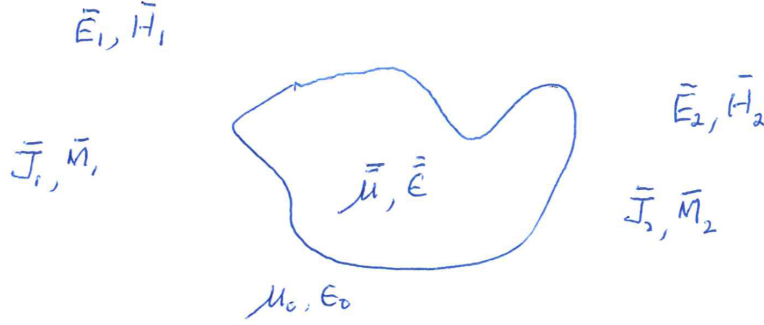


Figure 1:

Reciprocity theorem is like “tit-for-tat” relationship in humans: good-will is reciprocated with good will while ill-will is reciprocated with ill-will. Not exactly as in electromagnetics, this relationship can be expressed exactly and succinctly using mathematics. We shall see how this is done.

Consider a general anisotropic homogeneous medium where both $\bar{\mu}$ and $\bar{\epsilon}$ are described by permeability tensor and permittivity tensor over a finite part of space as shown in Figure 1. When only \mathbf{J}_1 and \mathbf{M}_1 are turned on, they generate fields \mathbf{E}_1 and \mathbf{H}_1 in this medium.¹ On the other hand, when only \mathbf{J}_2 and \mathbf{M}_2 are turned on, they generate \mathbf{E}_2 and \mathbf{H}_2 in this medium. Therefore, the pertinent equations for these two cases are²

$$\nabla \times \mathbf{E}_1 = -j\omega\bar{\mu} \cdot \mathbf{H}_1 - \mathbf{M}_1 \quad (1.1)$$

$$\nabla \times \mathbf{H}_1 = j\omega\bar{\epsilon} \cdot \mathbf{E}_1 + \mathbf{J}_1 \quad (1.2)$$

$$\nabla \times \mathbf{E}_2 = -j\omega\bar{\mu} \cdot \mathbf{H}_2 - \mathbf{M}_2 \quad (1.3)$$

$$\nabla \times \mathbf{H}_2 = j\omega\bar{\epsilon} \cdot \mathbf{E}_2 + \mathbf{J}_2 \quad (1.4)$$

From the above, we can show that

$$\mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 = -j\omega\mathbf{H}_2 \cdot \bar{\mu} \cdot \mathbf{H}_1 - \mathbf{H}_2 \cdot \mathbf{M}_1 \quad (1.5)$$

$$\mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 = j\omega\mathbf{E}_1 \cdot \bar{\epsilon} \cdot \mathbf{E}_2 + \mathbf{E}_1 \cdot \mathbf{J}_2 \quad (1.6)$$

¹This medium can be a highly complex one involving PEC and anisotropic materials of any shape. Hence, it can be a highly complex electronic circuit or antenna structure.

²The current sources are impressed currents so that they are immutable, and not changed by the environment they are immersed in.

Then,

$$\begin{aligned}\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2) &= \mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 \\ &= -j\omega \mathbf{H}_2 \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H}_1 - j\omega \mathbf{E}_1 \cdot \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_2 - \mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2\end{aligned}\quad (1.7)$$

By the same token,

$$\nabla \cdot (\mathbf{E}_2 \times \mathbf{H}_1) = -j\omega \mathbf{H}_1 \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H}_2 - j\omega \mathbf{E}_2 \cdot \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_1 - \mathbf{H}_1 \cdot \mathbf{M}_2 - \mathbf{E}_2 \cdot \mathbf{J}_1 \quad (1.8)$$

Subtracting (1.7) and (1.8), and using the fact that $\mathbf{H}_1 \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H}_2 = \mathbf{H}_2 \cdot \bar{\boldsymbol{\mu}}^t \cdot \mathbf{H}_1$, then³

$$\begin{aligned}\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) &= -j\omega \mathbf{H}_1 \cdot (\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}^t) \cdot \mathbf{H}_2 - j\omega \mathbf{E}_1 \cdot (\bar{\boldsymbol{\epsilon}} - \bar{\boldsymbol{\epsilon}}^t) \cdot \mathbf{E}_2 \\ &\quad - \mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2 + \mathbf{H}_1 \cdot \mathbf{M}_2 + \mathbf{E}_2 \cdot \mathbf{J}_1\end{aligned}\quad (1.9)$$

If

$$\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}^t, \quad \bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}^t \quad (1.10)$$

or when the tensors are symmetric, then the right-hand side of (1.8) simplifies as the terms involving the permeability tensors and permeability tensors disappear.

Now, integrating (1.9) over a volume V bounded by a surface S , and invoking Gauss' divergence theorem, we have the reciprocity theorem that

$$\begin{aligned}\oint_S d\mathbf{S} \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \\ = - \iiint_V dV [\mathbf{H}_2 \cdot \mathbf{M}_1 + \mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2 - \mathbf{E}_2 \cdot \mathbf{J}_1]\end{aligned}\quad (1.11)$$

When the volume V contains no sources, the reciprocity theorem reduces to

$$\oint_S d\mathbf{S} \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = 0 \quad (1.12)$$

The above is also called Lorentz reciprocity theorem by some authors.⁴

On the other hand, when the surface $S \rightarrow \infty$, \mathbf{E}_1 and \mathbf{H}_2 becomes spherical waves sharing the same $\boldsymbol{\beta}$ vector. Moreover $\omega\mu_0\mathbf{H}_2 = \boldsymbol{\beta} \times \mathbf{E}_2$, $\omega\mu_0\mathbf{H}_1 = \boldsymbol{\beta} \times \mathbf{E}_1$, then

$$\mathbf{E}_1 \times \mathbf{H}_2 \sim \mathbf{E}_1 \times (\boldsymbol{\beta} \times \mathbf{E}_2) = \mathbf{E}_1(\boldsymbol{\beta} \cdot \mathbf{E}_2) - \boldsymbol{\beta}(\mathbf{E}_1 \cdot \mathbf{E}_2) \quad (1.13)$$

$$\mathbf{E}_2 \times \mathbf{H}_1 \sim \mathbf{E}_2 \times (\boldsymbol{\beta} \times \mathbf{E}_1) = \mathbf{E}_2(\boldsymbol{\beta} \cdot \mathbf{E}_1) - \boldsymbol{\beta}(\mathbf{E}_2 \cdot \mathbf{E}_1) \quad (1.14)$$

But $\boldsymbol{\beta} \cdot \mathbf{E}_2 = \boldsymbol{\beta} \cdot \mathbf{E}_1 = 0$ in the far field because the spherical waves emanated by the sources resemble a plane wave, and the $\boldsymbol{\beta}$ vectors are parallel to each other.

³It is to be noted that in matrix algebra, the dot product between two vectors are often written as $\mathbf{a}^t \cdot \mathbf{b}$, but in the physics literature, the transpose on \mathbf{a} is implied. Therefore, the dot product between two vectors is just written as $\mathbf{a} \cdot \mathbf{b}$.

⁴Harrington, Time-Harmonic Electric Field.

Therefore, the two terms on the left-hand side of (1.11) cancel each other, and it vanishes when $S \rightarrow \infty$, and (1.11) can be rewritten as

$$\int_V dV[\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{M}_1] = \int_V dV[\mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2] \quad (1.15)$$

The inner product symbol is often used to rewrite the above as

$$\langle \mathbf{E}_2, \mathbf{J}_1 \rangle - \langle \mathbf{H}_2, \mathbf{M}_1 \rangle = \langle \mathbf{E}_1, \mathbf{J}_2 \rangle - \langle \mathbf{H}_1, \mathbf{M}_2 \rangle \quad (1.16)$$

The above inner product is also called reaction, a concept introduced by Rumsey. The above is rewritten as

$$\langle 2, 1 \rangle = \langle 1, 2 \rangle \quad (1.17)$$

where

$$\langle 2, 1 \rangle = \langle \mathbf{E}_2, \mathbf{J}_1 \rangle - \langle \mathbf{H}_2, \mathbf{M}_1 \rangle \quad (1.18)$$

1.1 Conditions for Reciprocity

It is seen that the above proof hinges on (1.10). In other words, the anisotropic medium has to be described by a symmetric tensor. Moreover, our starting equations (1.1) to (1.4) assume that the medium and the equations are linear time invariant so that Maxwell's equations can be written down in the frequency domain easily.

1.2 Application to a Two-Port Network

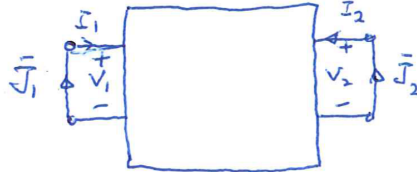


Figure 2:

Focusing on a two-port network as shown in Figure 2, we have

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Then

$$\langle \mathbf{E}_2, \mathbf{J}_1 \rangle = I_1 \int_{\text{Port 1}} \mathbf{E}_2 \cdot d\mathbf{l} = -I_1 V_1^{oc} \quad (1.19)$$

$$\langle \mathbf{E}_1, \mathbf{J}_2 \rangle = I_2 \int_{\text{Port 2}} \mathbf{E}_1 \cdot d\mathbf{l} = -I_2 V_2^{oc} \quad (1.20)$$

But $V_1^{oc} = Z_{12}I_2$, $V_2^{oc} = Z_{21}I_1$. Since $I_1V_1^{oc} = I_2V_2^{oc}$ by the reaction concept or by reciprocity, then $Z_{12} = Z_{21}$. In the above, we assume that \mathbf{J}_1 is constant in the input port 1 when it is turned on, so is \mathbf{J}_2 when it is in the input port 2. When the currents are constant of space when they are on, then the currents \mathbf{J}_1 and \mathbf{J}_2 can be factored out of the spatial integrals in (1.19) and (1.20), and the evaluation of the spatial integrals can be easily performed to yield the open-circuit voltages. The above analysis can be easily generalized to N -port network.

The simplicity of the above belies its importance. In the above derivation, \mathbf{J}_1 and \mathbf{J}_2 are impressed current sources. They need to be constant when turned on. This is easy to achieve if Port 1 and Port 2 are very small compared to wavelength so that circuit theory prevails. So the above result can also be applied to an experiment where two antennas are communicating over a vast terrain as shown in Figure 3. The terrain can also be replaced by complex circuits as in a circuit board, as long as the materials are reciprocal, linear and time invariant.



Figure 3:

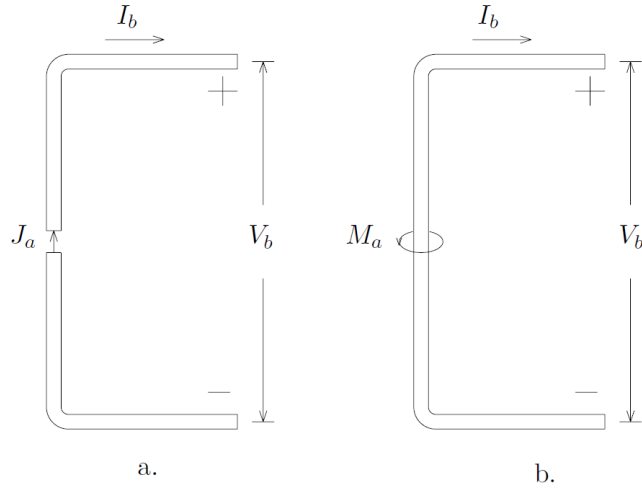


Figure 4: Courtesy of Kong, ELelectromagnetic Wave Theory.

The use of the impressed currents so that circuit concepts can be applied is shown in Figure 4. A magnetic current can be used as a voltage source in circuit theory as shown by Figure 4b.

1.3 Epilogue

The proof of uniqueness for Maxwell's equations is very deeply to the symmetry of the operator involved. We can see this from linear algebra. Given a matrix equation driven by two different sources, they can be written as

$$\overline{\mathbf{A}} \cdot \mathbf{x}_1 = \mathbf{b}_1 \quad (1.21)$$

$$\overline{\mathbf{A}} \cdot \mathbf{x}_2 = \mathbf{b}_2 \quad (1.22)$$

We can left dot multiply the first equation with \mathbf{x}_2 and do the same with the second equation with \mathbf{x}_1 to arrive at

$$\mathbf{x}_2^t \cdot \overline{\mathbf{A}} \cdot \mathbf{x}_1 = \mathbf{x}_2^t \cdot \mathbf{b}_1 \quad (1.23)$$

$$\mathbf{x}_1^t \cdot \overline{\mathbf{A}} \cdot \mathbf{x}_2 = \mathbf{x}_1^t \cdot \mathbf{b}_2 \quad (1.24)$$

If $\overline{\mathbf{A}}$ is symmetric, the left-hand side of both equations are equal to each other. Subtracting the two equations, we arrive at

$$\mathbf{x}_2^t \cdot \mathbf{b}_1 = \mathbf{x}_1^t \cdot \mathbf{b}_2 \quad (1.25)$$

The above is analogous to the statement of the reciprocity theorem. So if the operators in Maxwell's equations are symmetrical, then reciprocity theorem applies.

2 Paraxial Wave Equation

We have seen previously that in a source free space

$$\nabla^2 \mathbf{A} + \omega^2 \mu \varepsilon \mathbf{A} = 0 \quad (2.1)$$

$$\nabla^2 \Phi + \omega^2 \mu \varepsilon \Phi = 0 \quad (2.2)$$

The above are four scalar equations with the Lorenz gauge

$$\nabla \cdot \mathbf{A} = -j\omega \mu \varepsilon \Phi \quad (2.3)$$

connecting \mathbf{A} and Φ . We can examine the solution of \mathbf{A} such that

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_0(\mathbf{r})e^{-j\beta z} \quad (2.4)$$

where $\mathbf{A}_0(\mathbf{r})$ is a slowly varying function while $e^{-j\beta z}$ is rapidly varying in the z direction. This is primarily a quasi-plane wave propagating in the z -direction. We know to be the case in the far field of a source, but let us assume that this form persists less than the far field. Taking the x component of (2.4), we have

$$A_x(\mathbf{r}) = \Psi(\mathbf{r})e^{-j\beta z} \quad (2.5)$$

where $\Psi(\mathbf{r}) = \Psi(x, y, z)$ is a slowly varying function of x , y , and z . Substituting (2.5) into (2.1), and taking the double z derivative first, we arrive at

$$\frac{\partial^2}{\partial z^2} [\Psi(x, y, z)e^{-j\beta z}] = \left[\frac{\partial^2}{\partial z^2} \Psi(x, y, z) - 2j\beta \frac{\partial}{\partial z} \Psi(x, y, z) - \beta^2 \Psi(x, y, z) \right] \quad (2.6)$$

Consequently, after substituting the above into the x component of (2.1), we obtain an equation for $\Psi(\mathbf{r})$, the slowly varying envelope as

$$\frac{\partial^2}{\partial x^2} \Psi + \frac{\partial^2}{\partial y^2} \Psi - 2j\beta \frac{\partial}{\partial z} \Psi + \frac{\partial^2}{\partial z^2} \Psi = 0 \quad (2.7)$$

When $\beta \rightarrow \infty$, or in the high frequency limit,

$$\left| 2j\beta \frac{\partial}{\partial z} \Psi \right| \gg \left| \frac{\partial^2}{\partial z^2} \Psi \right| \quad (2.8)$$

In the above, we assume the envelope to be slowly varying and β large, so that $|\beta\Psi| \gg |\partial/\partial z\Psi|$. And then (2.7) can be approximated by

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - 2j\beta \frac{\partial \Psi}{\partial z} = 0 \quad (2.9)$$

The above is called the paraxial wave equation. It is also called the parabolic wave equation. It implies that the β vector of the wave is approximately parallel to the z axis, and hence, the name.

A closed form solution to the paraxial wave equation can be obtained by a simple trick. It is known that

$$A_x(\mathbf{r}) = \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (2.10)$$

is the solution to

$$\nabla^2 A_x + \beta^2 A_x = 0 \quad (2.11)$$

if $\mathbf{r} \neq \mathbf{r}'$. If we make $\mathbf{r}' = -\hat{z}jb$, a complex number, then (2.10) is always a solution to (2.10) for all \mathbf{r} , because $|\mathbf{r} - \mathbf{r}'| \neq 0$ always. Then

$$\begin{aligned} |\mathbf{r} - \mathbf{r}'| &= \sqrt{x^2 + y^2 + (z + jb)^2} \\ &\approx (z + jb) \left[1 + \frac{x^2 + y^2}{(z + jb)^2} + \dots \right]^{1/2} \\ &\approx (z + jb) + \frac{x^2 + y^2}{2(z + jb)} + \dots \end{aligned} \quad (2.12)$$

And then

$$A_x(\mathbf{r}) \approx \frac{e^{-j\beta(z+jb)}}{4\pi(z + jb)} e^{-j\beta \frac{x^2+y^2}{2(z+jb)}} \quad (2.13)$$

By comparing the above with (2.5), we can identify

$$\Psi(x, y, z) = A_0 \frac{jb}{z + jb} e^{-j\beta \frac{x^2+y^2}{2(z+jb)}} \quad (2.14)$$

By separating the exponential part into the real part and the imaginary part, we have

$$\Psi(x, y, z) = \frac{A_0}{\sqrt{1 + z^2/b^2}} e^{j \tan^{-1}(z/b)} e^{-j\beta \frac{x^2+y^2}{2(z^2+b^2)} z} e^{-b\beta \frac{x^2+y^2}{2(z^2+b^2)}} \quad (2.15)$$

The above can be rewritten as

$$\Psi(x, y, z) = \frac{A_0}{\sqrt{1 + z^2/b^2}} e^{-j\beta \frac{x^2+y^2}{2R}} e^{-\frac{x^2+y^2}{w^2}} e^{j\psi} \quad (2.16)$$

where

$$w^2 = \frac{2b}{\beta} \left(1 + \frac{z^2}{b^2} \right), \quad R = \frac{z^2 + b^2}{z}, \quad \psi = \tan^{-1} \left(\frac{z}{b} \right) \quad (2.17)$$

For a fixed z , the parameters w , R , and ψ are constants. Here, w is the beam waist which varies with z , and it is smallest when $z = 0$, or $w = w_0 = \sqrt{\frac{2b}{\beta}}$. And R is the radius of curvature of the constant phase front. This can be

appreciated by studying a spherical wave front $e^{-j\beta R}$, and make a paraxial wave approximation, namely, $x^2 + y^2 \ll z^2$ to get

$$\begin{aligned} e^{-j\beta R} &= e^{-j\beta(x^2+y^2+z^2)^{1/2}} = e^{-j\beta z \left(1 + \frac{x^2+y^2}{z^2}\right)^{1/2}} \\ &\approx e^{-j\beta z - j\beta \frac{x^2+y^2}{2z}} \approx e^{-j\beta z - j\beta \frac{x^2+y^2}{2R}} \end{aligned} \quad (2.18)$$

In the last approximation, we assume that $z \approx R$ in the paraxial approximation. The phase ψ changes rapidly with z .

A cross section of the electric field due to a Gaussian beam is shown in Figure 5.

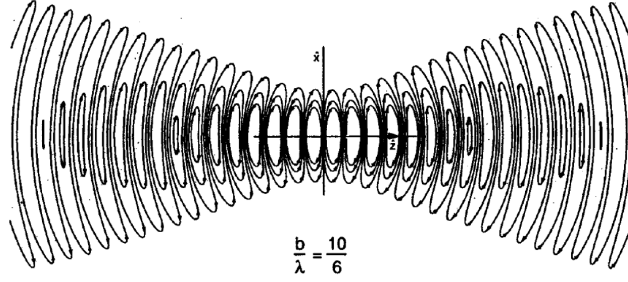


Figure 5: Electric field of a Gaussian beam in the $x - z$ plane frozen in time. The wave moves to the right as time increases; $b/\lambda = 10/6$ (Courtesy of Haus, Electromagnetic Noise and Quantum Optical Measurements).

In general, the paraxial wave equation has solution of the form

$$\Psi_{nm}(x, y, z) = \left(\frac{2}{\pi n! m!}\right)^{1/2} 2^{-N/2} \left(\frac{1}{w}\right) e^{-(x^2+y^2)/w^2} e^{-j\frac{\beta}{2R}(x^2+y^2)} e^{j(m+n+1)\Psi} \quad (2.19)$$

$$\cdot H_n\left(x\sqrt{2}/w\right) H_m\left(y\sqrt{2}/w\right) \quad (2.20)$$

where $H_n(\xi)$ is a Hermite polynomial of order n . The solution can also be express in terms of Laguerre polynomials, namely,

$$\begin{aligned} \Psi_{nm}(x, y, z) &= \left(\frac{2}{\pi n! m!}\right)^{1/2} \min(n, m)! \frac{1}{w} e^{-j\frac{\beta}{2R}\rho^2} e^{-\rho^2/w^2} e^{j(n+m+1)\Psi} e^{j\phi} \\ &\quad (-1)^{\min(n, m)} \left(\frac{\sqrt{2}\rho}{w}\right) L_{\min(n, m)}^{n-m} \left(\frac{2\rho^2}{w^2}\right) \end{aligned} \quad (2.21)$$

where $L_n^k(\xi)$ is the associated Laguerre polynomial.

These gaussian beams have rekindled recent excitement in the community because, in addition to carrying spin angular momentum as in a plane wave, they can carry orbital angular momentum due to the complex transverse field distribution of the beams.⁵ They harbor potential for optical communications as well as optical tweezers to manipulate trapped nano-particles. Figure 6 shows some examples of the cross section (xy plane) field plots for some of these beams.

Laguerre–Gaussian Beams and Orbital Angular Momentum

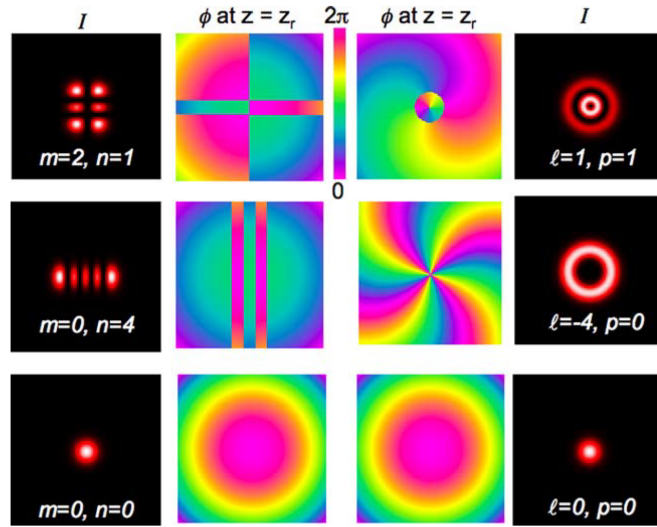


Figure 1.1 Examples of the intensity and phase structures of Hermite–Gaussian modes (*left*) and Laguerre–Gaussian modes (*right*), plotted at a distance from the beam waist equal to the Rayleigh range. See color insert.

Figure 6: Courtesy of L. Allen and M. Padgett’s chapter in J.L. Andrew’s book on structured light.

⁵See D.L. Andrew, Structured Light and Its Applications and articles therein.