ECE 604, Lecture 21

November 15, 2018

In this lecture, we will cover the following topics:

- Uniqueness Theorem
  - Isotropic Case
  - General Anisotropic Case
- Image Theorem
  - Electric Charges and Electric Dipoles
  - Magnetic Charges and Magnetic Dipoles
  - Some Special Geometries

Additional Reading:

- Prof. Dan Jiao's Lecture 15.
- Sections 1.18, 12.8 of Ramo, Whinnery, and Van Duzer.
- Topic 6.1A, J.A. Kong, Electromagnetic Wave Theory.
- Section 1.11, Lecture on Theory of Optical and Microwave Waveguide.

You should be able to do the homework by reading the lecture notes alone. Additional reading is for references.

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1 Uniqueness Theorem

Previously, we have derived the uniqueness theorem for Poisson's equation. If proper boundary conditions are stipulated for the boundary value problem, then the solution is unique. It turns out that there is a similar theorem for electrodynamics or time-varying Maxwell's equations driven by sources. But the solution is only guaranteed to be unique under certain conditions. We will derive those conditions under which uniqueness is guaranteed.

Assume that there exist two solutions in the presence of one common source, namely, these two solutions are $\mathbf{E}^a$, $\mathbf{H}^a$, $\mathbf{E}^b$, $\mathbf{H}^b$. Both of them satisfy Maxwell's equations and boundary conditions. Then, considering general anisotropic inhomogeneous media, where the tensors $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\varepsilon}}$ can be complex so that lossy media can be included, it follows that

$$\nabla \times \mathbf{E}^a = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^a - \mathbf{M}_i \tag{1.1}$$
$$\nabla \times \mathbf{H}^a = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}^a + \mathbf{J}_i \tag{1.2}$$
$$\nabla \times \mathbf{E}^b = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^b - \mathbf{M}_i \tag{1.3}$$
$$\nabla \times \mathbf{H}^b = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}^b + \mathbf{J}_i \tag{1.4}$$

By taking the difference of the two solutions, we have

$$\nabla \times (\mathbf{E}^a - \mathbf{E}^b) = -j\omega\overline{\boldsymbol{\mu}} \cdot (\mathbf{H}^a - \mathbf{H}^b) \tag{1.5}$$
$$\nabla \times (\mathbf{H}^a - \mathbf{H}^b) = j\omega\overline{\boldsymbol{\varepsilon}} \cdot (\mathbf{E}^a - \mathbf{E}^b) \tag{1.6}$$

Or alternatively, defining $\delta\mathbf{E} = \mathbf{E}^a - \mathbf{E}^b$ and $\delta\mathbf{H} = \mathbf{H}^a - \mathbf{H}^b$, we have

$$\nabla \times \delta\mathbf{E} = -j\omega\overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} \tag{1.7}$$
$$\nabla \times \delta\mathbf{H} = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \delta\mathbf{E} \tag{1.8}$$

By taking the left dot product of $\delta\mathbf{H}^*$ with (1.7), and then the left dot product of $\delta\mathbf{E}^*$ with the complex conjugation of (1.8), we obtain

$$\begin{aligned}\delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H}\\ \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= -j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*\end{aligned} \tag{1.9}$$

Now, taking the difference of the above, we get

$$\begin{aligned}\delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} - \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= \nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*)\\ &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*\end{aligned} \tag{1.10}$$

Next, integrating the above equation over a volume V bounded by a surface S, and making use of Gauss' divergence theorem, we arrive at

$$\begin{aligned}\iiint_V \nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*) dV &= \oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot d\mathbf{S}\\ &= \iiint_V [-j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*] dV\end{aligned} \tag{1.11}$$

And next, we would like to know what kind of boundary conditions would make the left-hand side equal to zero. It is seen that the surface integral on the left-hand side will be zero if:

1. If $\hat{n} \times \mathbf{E}$ is specified over S so that $\mathbf{E}_a = \mathbf{E}_b$, then $\hat{n} \times \delta\mathbf{E} = 0$, and then[1]
$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \hat{n} dS = \oiint_S (\hat{n} \times \delta\mathbf{E}) \cdot \delta\mathbf{H}^* dS = 0.$

2. If $\hat{n} \times \mathbf{H}$ is specified over S so that $\mathbf{H}_a = \mathbf{H}_b$, then $\hat{n} \times \delta\mathbf{H} = 0$, and then
$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \hat{n} dS = -\oiint_S (\hat{n} \times \delta\mathbf{H}^*) \cdot \delta\mathbf{E} dS = 0.$

3. If $\hat{n} \times \mathbf{E}$ is specified over S_1, and $\hat{n} \times \mathbf{H}$ is specified over S_2 ($S_1 \cup S_2 = S$), then $\hat{n} \times \delta\mathbf{E} = 0$ on S_1, and $\hat{n} \times \delta\mathbf{H} = 0$ on S_2, then the left-hand side becomes
$$\oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot \hat{n} dS = \iint_{S_1} + \iint_{S_2} = \iint_{S_1} (\hat{n} \times \delta\mathbf{E}) \cdot \delta\mathbf{H}^* dS - \iint_{S_2} (\hat{n} \times \delta\mathbf{H}^*) \cdot \delta\mathbf{E} dS = 0.$$

Thus, under the above three scenarios, the left-hand side of (1.11) is zero, and then the right-hand side of (1.11) becomes

$$\iiint_V [-j\omega \delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega \delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*] dV = 0 \tag{1.12}$$

For lossless media, $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\varepsilon}}$ are hermitian tensors (or matrices[2]), then it can be seen, using the properties of hermitian matrices or tensors, that $\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H}$ and $\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*$ are purely real. Taking the imaginary part of the above equation yields

$$\iiint_V [-\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + \delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*] dV = 0 \tag{1.13}$$

The above two terms correspond to stored magnetic field energy and stored electric field energy in the difference solutions $\delta\mathbf{H}$ and $\delta\mathbf{E}$, respectively. The above being zero does not imply that $\delta\mathbf{H}$ and $\delta\mathbf{E}$ are zero. For resonance solutions, the stored electric energy can balance the stored magnetic energy. When $\mathbf{H}_a$, $\mathbf{H}_b$, $\mathbf{E}_a$ and $\mathbf{E}_b$ are resonance solutions, their differences $\delta\mathbf{H} = \mathbf{H}_a - \mathbf{H}_b$ and $\delta\mathbf{E} = \mathbf{E}_a - \mathbf{E}_b$ are also resonance solutions. Therefore, $\delta\mathbf{H}$ and $\delta\mathbf{E}$ need not be zero, even though (1.13) is zero.

Uniqueness can only be guaranteed if the medium is lossy as shall be shown later. First we begin with the isotropic case.

1.1 Isotropic Case

It is easier to see this for lossy isotropic media. Then (1.12) simplifies to

$$\iiint_V [-j\omega\mu |\delta\mathbf{H}|^2 + j\omega\varepsilon^* |\delta\mathbf{E}|^2] dV = 0 \tag{1.14}$$

[1] Using the vector identity that $\mathbf{a} \cdot (\mathbf{b} \times c) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$.

[2] Tensors are a special kind of matrices.

For isotropic lossy media, $\mu = \mu' - j\mu''$ and $\varepsilon = \varepsilon' - j\varepsilon''$. Taking the real part of the above, we have from (1.14) that

$$\iiint_V [-\omega\mu''|\delta\mathbf{H}|^2 - \omega\varepsilon''|\delta\mathbf{E}|^2] dV = 0 \tag{1.15}$$

Since the integrand in the above is always negative definite, the integral can be zero only if

$$\delta\mathbf{E} = 0, \quad \delta\mathbf{H} = 0 \tag{1.16}$$

everywhere in V, implying that $\mathbf{E}_a = \mathbf{E}_b$, and that $\mathbf{H}_a = \mathbf{H}_b$. Hence, it is seen that uniqueness is guaranteed only if the medium is lossy.

Notice that the same conclusion can be drawn if we make μ'' and ε'' negative. This corresponds to active media, and uniqueness can be guaranteed for a time-harmonic solution.

1.2 General Anisotropic Case

The proof for general anisotropic media is more complicated, and is only for the interested readers. For the lossless anisotropic media, we see that (1.12) is purely imaginary. However, when the medium is lossy, this same equation will have a real part. Hence, we need to find the real part of (1.12) for the general lossy case. To this end, we need to find the complex conjugate[3] of (1.12), which is scalar, and add it to itself to get its real part. The complex conjugate of the scalar $c = \delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H}$ is $c^* = \delta\mathbf{H} \cdot \overline{\boldsymbol{\mu}}^* \cdot \delta\mathbf{H}^* = \delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}}^\dagger \cdot \delta\mathbf{H}$. Similarly, the complex conjugate of the scalar $d = \delta\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \delta\mathbf{E}^*$ is $d^* = \delta\mathbf{E}^* \cdot \overline{\varepsilon}^\dagger \cdot \delta\mathbf{E}$.[4]

Finally, after taking the complex conjugate of the scalar quantity (1.12) and add it to itself, we have

$$\iiint_V [-j\omega\delta\mathbf{H}^* \cdot (\overline{\boldsymbol{\mu}} - \overline{\boldsymbol{\mu}}^\dagger) \cdot \delta\mathbf{H} - j\omega\delta\mathbf{E}^* \cdot (\overline{\varepsilon} - \overline{\varepsilon}^\dagger) \cdot \delta\mathbf{E}] dV = 0 \tag{1.17}$$

For lossy media, $-j(\overline{\boldsymbol{\mu}} - \overline{\boldsymbol{\mu}}^\dagger)$ and $-j(\overline{\varepsilon} - \overline{\varepsilon}^\dagger)$ are hermitian negative matrices. Hence the integrand is always negative definite, and the above equation cannot be satisfied unless $\delta\mathbf{H} = \delta\mathbf{E} = 0$ everywhere in V. Thus, uniqueness is guaranteed in a lossy anisotropic medium.

Similar statement can be made as the isotropic case if the medium is active. Then the integrand is positive definite, and the above equation cannot be satisfied unless $\delta\mathbf{H} = \delta\mathbf{E} = 0$ everywhere in V and hence, uniqueness is satisfied.

[3] Also called hermitian conjugate.

[4] To arrive at these expressions, one makes use of the matrix algebra rule that if $\overline{\mathbf{D}} = \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \cdot \overline{\mathbf{C}}$, then $\overline{\mathbf{D}}^t = \overline{\mathbf{C}}^t \cdot \overline{\mathbf{B}}^t \cdot \overline{\mathbf{A}}^t$. This is true even for non-square matrices. But for our case here, $\overline{\mathbf{A}}$ is a 1×3 row vector, and $\overline{\mathbf{C}}$ is a 3×1 column vector, and $\overline{\mathbf{B}}$ is a 3×3 matrix. In vector algebra, the transpose of a vector is implied. Also, in our case here, $\overline{\mathbf{D}}$ is a scalar, and hence, its transpose is itself.

1.3 Epilogue

The proof of uniqueness for Maxwell's equations is very similar to the proof of uniqueness for a matrix equation

$$\overline{\mathbf{A}} \cdot \mathbf{x} = \mathbf{b} \tag{1.18}$$

If a solution to a matrix equation exists without excitation, namely, when $\mathbf{b} = 0$, then the solution is the null space solution, namely, $\mathbf{x} = \mathbf{x}_N$. In other words,

$$\overline{\mathbf{A}} \cdot \mathbf{x}_N = 0 \tag{1.19}$$

These null space solutions exist without a "driving term" $\mathbf{b}$ on the right-hand side. For Maxwell's Equations, it corresponds to the source terms. They are like the homogeneous solution of an ordinary differential equation or a partial differential equation. In an enclosed region of volume V bounded by a surface S, homogeneous solutions are the resonant solutions of this Maxwellian system. When these solutions exist, they give rise to non-uniqueness.

Also, notice that (1.7) and (1.8) are Maxwell's equations without the source terms. In a closed region V bounded by a surface S, only resonance solutions can exist when there are no source terms.

One way to ensure that these resonant solutions do not exist is to put in loss or gain. When loss or gain is present, then the resonant solutions are decaying sinusoids or growing sinusoids. Since we are looking for solutions in the frequency domain, or time harmonic solutions, these non-sinusoidal solutions are outside the solution space: They are not part of the time-harmonic solutions we are looking for. Therefore, there are no resonant null-space solutions.

2 Image Theory

Image theory can be used to derived closed form solution to boundary value problems when the geometry is simple and has a lot of symmetry. The closed form solutions in turn offer physical insight to the problem.

2.1 Electric Charges and Electric Dipoles

Image theory for a flat conductor surface is quite easy to derive. To see that, we can start with electro-static theory of putting a positive charge above a flat plane. For electrostatics, the plane does not have to be a perfect conductor, but only a conductor (a metal). The electric charges will move around until they come to rest. This is only possible if there is no electric field inside the conductor. As a result, the tangential electric field on the surface of the conductor has to be zero due to the continuity of tangential electric field across an interface. Hence, the static electric field cannot have a tangential component of the electric field on the surface.

The tangential electric field can be canceled by putting an image charge of opposite sign at the mirror location of the original charge. This is shown in

Figure 1. Now we can mentally add the total field due to these two charges. When the total electric field due to the original charge and image charge is sketched, it will look like that in Figure 2. It is seen that the electric field satisfies the boundary condition at the conductor interface due to symmetry.

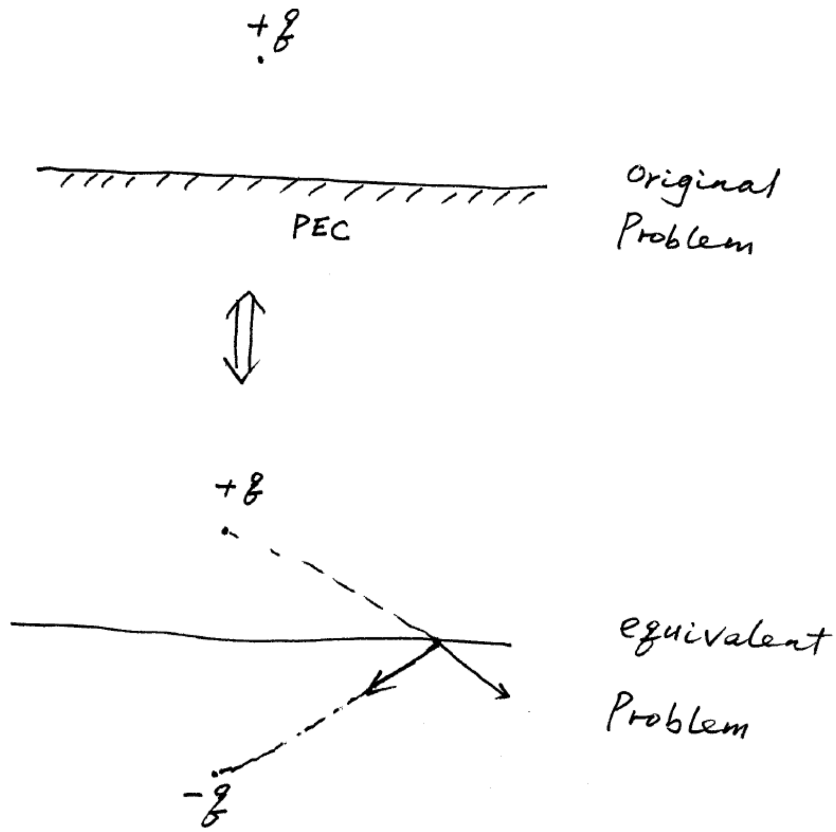

Figure 1:

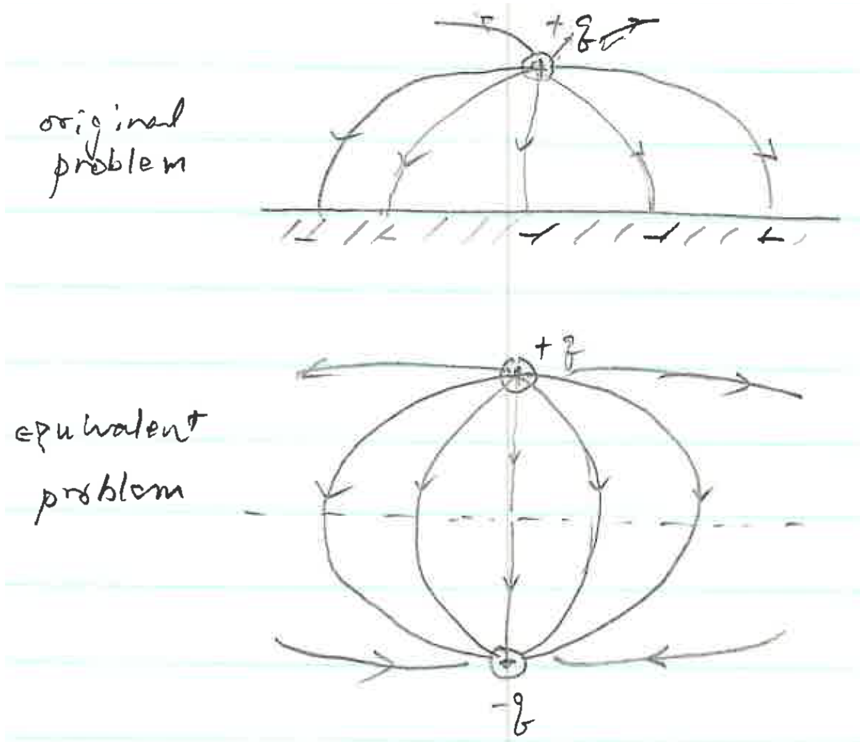

Figure 2:

An electric dipole is made from a positive charge placed in close proximity to a negative charge. Using that a charge reflects to a charge of opposite polarity, from the above, one can easily see that a horizontal electric dipole reflects to a horizontal electric dipole of opposite polarity while a vertical electric dipole reflects to vertical electric dipole of the same polarity.

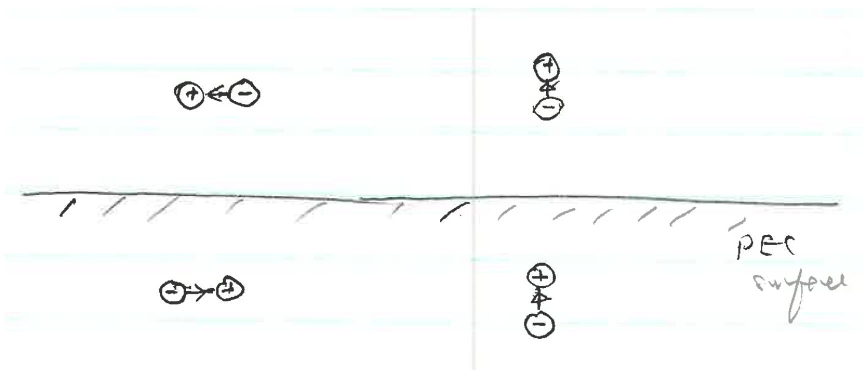

Figure 3:

For electrostatic problems, a conductive medium suffices to produce surface charges that shield out the electric field from the bottom half space so that $\hat{n} \times \mathbf{E} = 0$ on the conductor surface, and no current flows in the conductor.

For a perfect electric conductor (PEC), becasue $\mathbf{J} = \sigma \mathbf{E}$ where $\sigma \to \infty$, an infinitesimal electric field will yield an infinite current, and hence an infinite magnetic field. A time-varying magnetic field yields an electric field that will drive an electric current, and these fields and current will be infinitely large. Hence, the only possibility is for the time-varying electromagnetic fields to be zero inside a PEC.

Thus, the charges can re-orient themselves instantaneously on the PEC surface when the dilpoles are time varying. This is not the case when interface is between a finite conductor medium and free space. Therefore, the electric field $\mathbf{E}$ is always zero inside PEC, and hence $\hat{n} \times \mathbf{E} = 0$ on the surface. Consequently, the reflected images in Figure 3 are valid even for a time-varying dipole over a PEC surface.

2.2 Magnetic Charges and Magnetic Dipoles

A static magnetic field can penetrate a conductive medium. However, a time-varying magnetic field inside a conductive medium will produce an electric field, which in turn produces the conduction current via $\mathbf{J} = \sigma \mathbf{E}$. This is termed eddy current, which by Lenz's law, repels the magnetic field from the conductive medium.

Now, consider a static magnetic field penetrating into a perfect electric conductor, an infinitesimal amount of time variation will produce an electric field, which in turn produces an infinitely large eddy current. So the stable state for a static magnetic field is to be expelled from the perfect electric conductor. This in fact is what we observe when a magnetic field is brought near a superconductor. Therefore, for the static magnetic field, $\hat{n} \cdot \mathbf{B} = 0$ on the PEC surface.

Now, assuming a magnetic monopole exists, it will reflect to itself on a PEC surface so that $\hat{n} \cdot \mathbf{B} = 0$ as shown in Figure 4. Therefore, a magnetic charge reflects to a charge of similar polarity on the PEC surface.

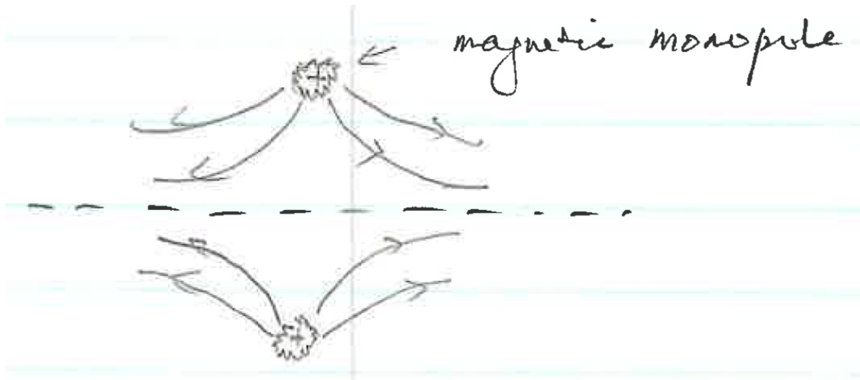

Figure 4:

By extrapolating this to magnetic dipoles, they will reflect themselves to the magnetic dipoles as shown in Figure 5. A horizontal magnetic dipole reflects to a horizontal magnetic dipole of the same polarity, and a vertical magnetic dipole reflects to a vertical magnetic dipole of opposite polarity.

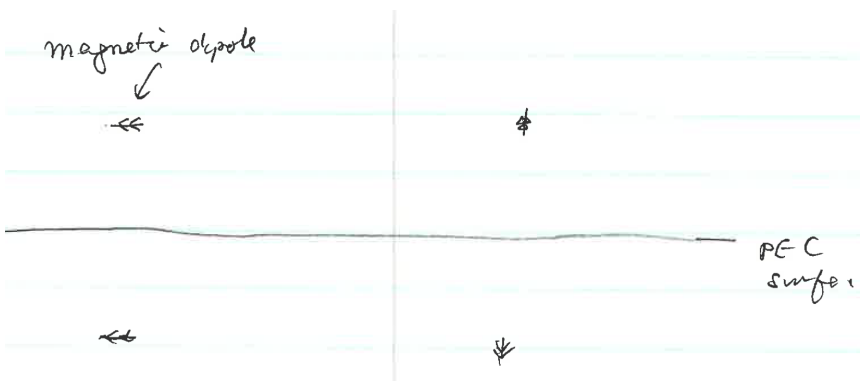

Figure 5:

A surface that is dual to the PEC surface is the perfect magnetic conductor (PMC) surface. The magnetic dipole is also dual to the electric dipole. Thus, on a PMC surface, these electric and magnetic dipoles will reflect differently as shown in Figure 6. One can go through Gedanken experiments and verify that the reflection rules are as shown in the figure.

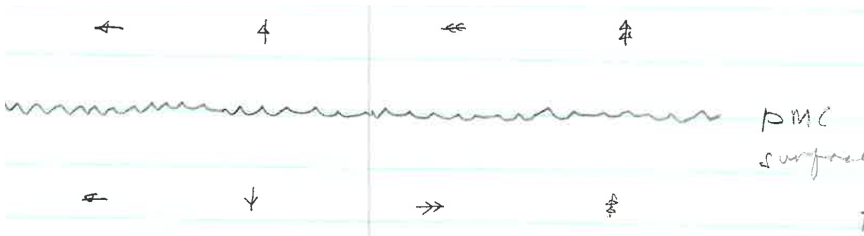

Figure 6:

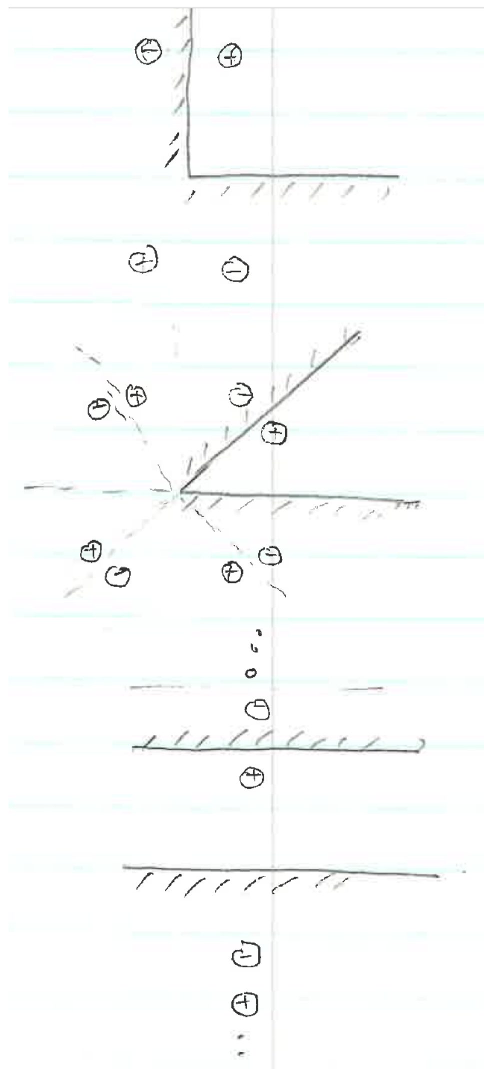

Figure 7:

2.3 Some Special Geometries

Even though for conductive medium, a time-varying electric field is not necessary, the image theorem can be generalized to a PEC medium easily. For the geometry shown in Figure 7, one can start with electrostatic theory, and convince oneself that $\hat{n} \times \mathbf{E} = 0$ on the metal surface with the placement of charges as shown. For PEC surfaces, one can extend these cases to time-varying dipoles because the charges in the PEC medium can re-orient instantaneously to shield out or expel the **E** and **H** fields. Again, one can repeat the above exercise for magnetic charges, magnetic dipoles, and PMC surface.

Figure 8:

One curious case is for a static charge placed near a conductive sphere as shown in Figure 8. As worked out in the textbook (p. 48 and p. 49, Ramo et al.), a charge of $+Q$ reflects to a charge of $-Q_I$ inside the sphere. However, this cannot be generalized to electrodynamics, because a time-harmonic, charge conserving dipole will reflect to a non-charge conserving dipole. Moreover, due to the retardation effect, an time-varying dipole or charge will be felt at different points asymmetrically on the surface of the sphere from the original and image charges.

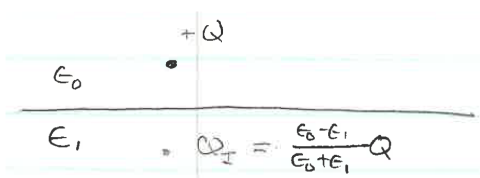

Figure 9:

When a static charge is placed over a dielectric interface, image theory can be used to find the closed form solution. This solution can be derived using Fourier transform technique which is outside the scope of this course. It can also be extended to multiple interfaces. But image theory cannot be used for the electrodynamic case due to the different speed of light in different media, giving rise to different retardation effect.