ECE 604, Lecture 20

November 8, 2018

In this lecture, we will cover the following topics:

- Radiation of Electromagnetic Fields
- Radiation or Far-Field Approximation
- Linear Array of Dipole Antennnas
- When is the Far-Field Approximation Valid?

Additional Reading:

- Sections 12.1 - 12.7 of Ramo, Whinnery, and Van Duzer.
- Sections 5.2, 5.3, 5.4, 6.1D J.A. Kong, Electromagnetic Wave Theory.
- Lectures 25, 26, and 27, ECE 350X.

You should be able to do the homework by reading the lecture notes alone. Additional reading is for references.

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1 Radiation of Electromagnetic Fields

Electromagnetic fields are used for communications, sensing, wireless power transfer applications, and many more. Hence, it is imperative to understand how electromagnetic fields radiate from sources. The fundamental reason for the radiation of electromagnetic sources is the acceleration of electric charges associated with the electric currents.

To this end, wee will start with frequency domain Maxwell's equations with sources $\mathbf{J}$ and ϱ included, and see how these sources $\mathbf{J}$ and ϱ can radiate electromagnetic fields. Maxwell's equations in the frequency domain are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{1.1}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \tag{1.2}$$

$$\nabla \cdot \mu\mathbf{H} = 0 \tag{1.3}$$

$$\nabla \cdot \varepsilon\mathbf{E} = \varrho \tag{1.4}$$

In order to satisfy the third Maxwell's equation, we let

$$\mu\mathbf{H} = \nabla \times \mathbf{A} \tag{1.5}$$

Since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, the third equation will be automatically satisfied. Now, using (1.5) in (1.1), we have

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0 \tag{1.6}$$

Since $\nabla \times (\nabla\Phi) = 0$, the above implies that

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi \tag{1.7}$$

Hence, the above shows that $\mathbf{A}$ and Φ uniquely determine the fields $\mathbf{E}$ and $\mathbf{H}$.[1] To this end, we derive expressions for $\mathbf{A}$ and Φ in terms of the sources $\mathbf{J}$ and ϱ which are given. Substituting (1.5) and (1.7) into (1.2) gives

$$\nabla \times \nabla \times \mathbf{A} = -j\omega\mu\varepsilon(-j\omega\mathbf{A} - \nabla\Phi) + \mu\mathbf{J} \tag{1.8}$$

or upon rearrangement, after using that $\nabla \times \nabla \times \mathbf{A} = \nabla\nabla \cdot \mathbf{A} - \nabla \cdot \nabla\mathbf{A}$, we have

$$\nabla^2\mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = -\mu\mathbf{J} + j\omega\mu\varepsilon\nabla\Phi + \nabla\nabla \cdot \mathbf{A} \tag{1.9}$$

Using (1.7) in (1.4), we have

$$\nabla \cdot (j\omega\mathbf{A} + \nabla\Phi) = -\frac{\varrho}{\varepsilon} \tag{1.10}$$

In the above, (1.9) and (1.10) represent two equations for the two unknowns $\mathbf{A}$ and Φ, expressed in terms of the known quantities, the sources $\mathbf{J}$ and ϱ

[1] Notice that when $\omega = 0$, the above reduces to the previous definition of scalar potential for electric field.

which are given. But these equations are coupled to each other, and are rather difficult to solve at this point. Furthermore, in (1.5), we can add a gradient term to $\mathbf{A}$, and then $\mathbf{H}$ remains invariant. In other words, $\mathbf{A}' = \mathbf{A} + \nabla\Psi$, and $\mu\mathbf{H} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}'$.

To make $\mathbf{E}$ invariant, we can let

$$\Phi' = \Phi - j\omega\Psi \tag{1.11}$$

Therefore, the pair of $\mathbf{A}$ and Φ that determines $\mathbf{H}$ and $\mathbf{E}$ are not unique.

To make them unique, in addition to specifying what $\nabla \times \mathbf{A}$ should be in (1.5), we need to specify its divergence. One way is to specify

$$\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\Phi \tag{1.12}$$

The above is judiciously chosen so that the pertinent equations will be simplified. Then using the above in (1.9) and (1.10), they become

$$\nabla^2\mathbf{A} + \omega^2\mu\varepsilon\mathbf{A} = -\mu\mathbf{J} \tag{1.13}$$

$$\nabla^2\Phi + \omega^2\mu\varepsilon\Phi = -\frac{\varrho}{\varepsilon} \tag{1.14}$$

Equation (1.12) is known as the Lorenz gauge. Not only are the equations simplified, they can be solved independently of each other since they are decoupled from each other.

Equations (1.13) and (1.14) can be solved using the Green's function method. They together constitute four scalar equations similar to each other. Hence, we need only to solve their point-source response, or the Green's function of these equations by solving

$$\nabla^2 g(\mathbf{r},\mathbf{r}') + \beta^2 g(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{1.15}$$

where $\beta^2 = \omega^2\mu\varepsilon$. In Lecture 5, we have shown that when $\beta = 0$,

$$g(\mathbf{r},\mathbf{r}') = g(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

When $\beta \neq 0$, the correct solution is

$$g(\mathbf{r},\mathbf{r}') = g(\mathbf{r}-\mathbf{r}') = \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{1.16}$$

which can be verified by back substitution. By using the principle of linear superposition, or convolution, the solutions to (1.13) and (1.14) are then

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\iiint d\mathbf{r}'\frac{\mathbf{J}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-j\beta|\mathbf{r}-\mathbf{r}'|} \tag{1.17}$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon}\iiint d\mathbf{r}'\frac{\varrho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}e^{-j\beta|\mathbf{r}-\mathbf{r}'|} \tag{1.18}$$

2 Radiation Field or Far-Field Approximation

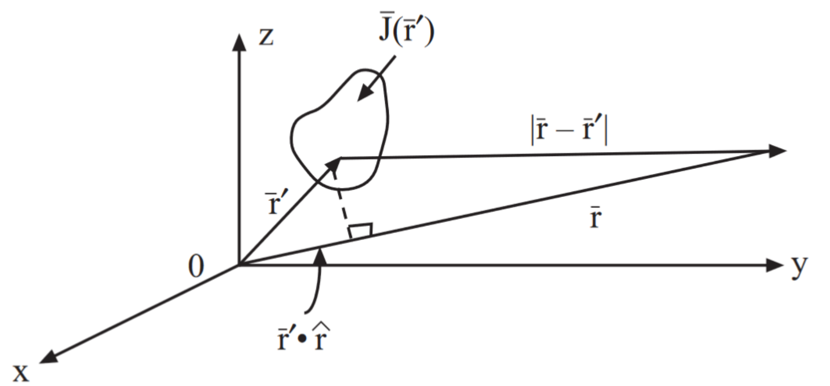

Figure 1:

The integrals in (1.17) and (1.18) are normally untenable, but when the observation point is far from the source, approximation to the integral can be made giving it a nice physical interpretation. When $|\mathbf{r}| \gg |\mathbf{r}'|$, then $|\mathbf{r}-\mathbf{r}'| \approx r - \mathbf{r}' \cdot \hat{r}$, where $r = |\mathbf{r}|$ and $r' = |\mathbf{r}'|$. Thus (1.17) previously derived becomes

$$\mathbf{A}(\mathbf{r}) \approx \iiint_V d\mathbf{r}' \frac{\mu \mathbf{J}(\mathbf{r}')}{r - \mathbf{r}' \cdot \hat{r}} e^{-j\beta r + j\beta \mathbf{r}' \cdot \hat{r}} \approx \frac{\mu e^{-j\beta r}}{4\pi r} \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') e^{j\beta \mathbf{r}' \cdot \hat{r}} \tag{2.1}$$

In the above we have made used of that $1/(1-\Delta) \approx 1$ when Δ is small, but $e^{j\beta\Delta} \neq 1$, unless $j\beta\Delta \ll 1$. Hence, we keep the exponential term in (2.1) but simplify the denominator to arrive at the last expression above.

If we let $\boldsymbol{\beta} = \beta \hat{r}$, and $\mathbf{r}' = \hat{x}x' + \hat{y}y' + \hat{z}z'$, then

$$e^{j\beta \mathbf{r}' \cdot \hat{r}} = e^{j\boldsymbol{\beta} \cdot \mathbf{r}'} = e^{j\beta_x x' + j\beta_y y' + j\beta_z z'} \tag{2.2}$$

Therefore (2.1) resembles a 3D Fourier transform integral, namely

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu e^{-j\beta r}}{4\pi r} \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') e^{j\boldsymbol{\beta} \cdot \mathbf{r}'} \tag{2.3}$$

and (2.3) can be rewritten as

$$\mathbf{A}(\mathbf{r}) \cong \frac{\mu e^{-j\beta r}}{4\pi r} \mathbf{F}(\boldsymbol{\beta}) \tag{2.4}$$

where $\mathbf{F}(\boldsymbol{\beta})$ is the 3D Fourier transform of $\mathbf{J}(\mathbf{r}')$ with $\boldsymbol{\beta} = \hat{r}\beta$.

This is not a normal 3D Fourier transform because $|\boldsymbol{\beta}|^2 = {\beta_x}^2 + {\beta_y}^2 + {\beta_z}^2 = \beta^2$. In other words, the length of the vector $\boldsymbol{\beta}$ is fixed to be β. It is the 3D

Fourier transform of the current source $\mathbf{J}(\mathbf{r}')$ with Fourier variables, β_x, β_y, β_z lying on a sphere of radius β and $\boldsymbol{\beta} = \beta\hat{r}$. This spherical surface in the Fourier space is also called the Ewald's sphere.

We can write $\hat{r}$ or $\boldsymbol{\beta}$ in terms of direction cosines in spherical coordinates or that

$$\hat{r} = \hat{x}\cos\phi\sin\theta + \hat{y}\sin\phi\sin\theta + \hat{z}\cos\theta \tag{2.5}$$

Hence

$$\mathbf{F}(\boldsymbol{\beta}) = \mathbf{F}(\beta\hat{r}) = \mathbf{F}(\beta, \theta, \phi) \tag{2.6}$$

Also in (2.4), when $r \gg \mathbf{r}' \cdot \hat{r}$, $e^{-j\beta r}$ is now a rapidly varying function of r while, $\mathbf{F}(\boldsymbol{\beta})$ is only a slowly varying function of θ and ϕ, the observation angles. Hence, we can write $e^{-j\beta r} = e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$ where $\boldsymbol{\beta} = \hat{r}\beta$ and $\mathbf{r} = \hat{r}r$ so that a spherical wave resembles a plane wave locally. Then, it is clear that $\nabla \to -j\boldsymbol{\beta} = -j\beta\hat{r}$, using the plane-wave approximation, and

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \approx -j\frac{\beta}{\mu}\hat{r} \times (\hat{\theta}A_\theta + \hat{\phi}A_\phi) = j\frac{\beta}{\mu}(\hat{\theta}A_\phi - \hat{\phi}A_\theta) \tag{2.7}$$

Similarly

$$\mathbf{E} = \frac{1}{j\omega\varepsilon}\nabla \times \mathbf{H} \cong -j\omega(\hat{\theta}A_\theta + \hat{\phi}A_\phi) \tag{2.8}$$

The above shows that in the far field, the wave radiated by a finite source resembles a spherical wave. Moreover, a spherical wave resembles a plane wave when one is sufficiently far from the source. Notice that $\boldsymbol{\beta} = \beta\hat{r}$ is orthogonal to $\mathbf{E}$ and $\mathbf{H}$ in the far field, a property of a plane wave.

Hence, it can be shown that in the far field, using the plane-wave approximation,

$$|\mathbf{E}|/|\mathbf{H}| \approx \eta \tag{2.9}$$

where η is the intrinsic impedance of free space, which is a property of a plane wave. Moreover, one can show that the time average Poynting's vector in the far field is

$$\langle \mathbf{S} \rangle \approx \frac{1}{2\eta}|\mathbf{E}|^2\hat{r} \tag{2.10}$$

which resembles also the property of a plane wave. Since the radiated field is a spherical wave, the Poynting's vector is radial. Therefore,

$$\langle \mathbf{S} \rangle = \hat{r}S_r(\theta, \phi) \tag{2.11}$$

The plot of $|\mathbf{E}(\theta, \phi)|$ is termed the far-field pattern or the radiation pattern of an antenna or the source, while the plot of $|\mathbf{E}(\theta, \phi)|^2$ is its far-field power pattern.

Once the far-field power pattern S_r is known, the total power radiated by the antenna can be found by

$$P_T = \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta d\theta d\phi S_r(\theta, \phi) \tag{2.12}$$

The above evaluates to a constant independent of r due to energy conservation. Now assume that this same antenna is radiating isotropically in all directions, then the average power density of this fictitious isotropic radiator as $r \to \infty$ is

$$S_{av} = \frac{P_T}{4\pi r^2} \tag{2.13}$$

A dimensionless directive gain pattern can be defined such that

$$G(\theta, \phi) = \frac{S_r(\theta, \phi)}{S_{av}} = \frac{4\pi r^2 S_r(\theta, \phi)}{P_T} \tag{2.14}$$

The above function is independent of r in the far field since $S_r \sim 1/r^2$ in the far field. The directivity of an antenna $D = max(G(\theta, \phi))$, is the maximum value of the directive gain.

An antenna also has an effective area or aperture, such that if a plane wave carrying power density denoted by S_{inc} impinges on the antenna, then the power received by the antenna, $P_{received}$ is given by

$$P_{received} = S_{inc} A_e \tag{2.15}$$

A wonderful relationship exists between the directive gain pattern $G(\theta, \phi)$ and the effective aperture, namely that[2]

$$A_e = \frac{\lambda^2}{4\pi} G(\theta, \phi) \tag{2.16}$$

Therefore, the effective aperture of an antenna is also direction dependent.

3 Linear Array of Dipole Antennas

Antenna array can be designed so that the constructive and destructive interference in the far field can be used to steer the direction of radiation of the antenna, or the far-field radiation pattern of an antenna array. A simple linear dipole array is shown in Figure 2.

[2] The proof of this formula is beyond the scope of this course.

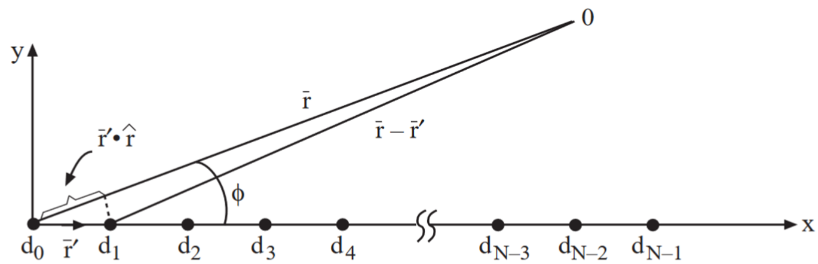

Figure 2:

First, we assume that this is a linear array of Hertzian dipoles (see ECE 350X for the definition of a Hertzian dipole),

$$\begin{aligned}\mathbf{J}(\mathbf{r}') = \hat{z}Il[&A_0\delta(x') + A_1\delta(x'-d_1) + A_2\delta(x'-d_2) + \cdots \\ &+ A_{N-1}\delta(x'-d_{N-1})]\delta(y')\delta(x')\end{aligned} \tag{3.1}$$

The vector potential on the xy-plane in the far field is derived to be

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &\cong \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\iiint d\mathbf{r}'[A_0\delta(x') + A_1\delta(x'-d_1) + \cdots]\delta(y')\delta(x')e^{j\beta\mathbf{r}'\cdot\hat{r}} \\ &= \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}[A_0 + A_1e^{j\beta d_1\cos\phi} + A_2e^{j\beta d_2\cos\phi} + \cdots + A_{N-1}e^{j\beta d_{N-1}\cos\phi}\end{aligned} \tag{3.2}$$

In the above, we have assumed that the observation point is on the xy plane, or that $\mathbf{r} = \boldsymbol{\rho} = \hat{x}x + \hat{y}y$. Also, the sources are aligned on the x axis, or that $\mathbf{r}' = \hat{x}x'$, and $\mathbf{r}'\cdot\hat{r} = x'\cos\phi$.

If $d_n = nd$, and $A_n = e^{jn\psi}$, then the antenna array, which assumes a progressively increasing phase shift between different elements, is called a linear phase array. Then (3.2) in the above becomes

$$\begin{aligned}\mathbf{A}(\mathbf{r}) \cong \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}[1 + e^{-j(\beta d\cos\psi+\phi)} + e^{-j2(\beta d\cos\phi+\psi)} + \cdots \\ +e^{-j(N-1)(\beta d\cos\phi+\psi)}]\end{aligned} \tag{3.3}$$

The above can be summed in closed form using

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x} \tag{3.4}$$

Then in the far field,

$$\mathbf{A}(\mathbf{r}) \cong \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\frac{1-e^{jN(\beta d\cos\phi+\psi)}}{1-e^{j(\beta d\cos\phi+\psi)}} \tag{3.5}$$

Since on the xy plane, $E_\theta = -j\omega A_\theta = j\omega A_z$. Then,

$$|E_\theta| = |E_0| \left| \frac{1 - e^{jN(\beta d \cos\phi + \psi)}}{1 - e^{j(\beta d \cos\phi + \psi)}} \right| = |E_0| \left| \frac{\sin \frac{N}{2}(\beta d \cos\phi + \psi)}{\sin \frac{1}{2}(\beta d \cos\phi + \psi)} \right| \tag{3.6}$$

The above can be used to plot the far-field pattern of an antenna array (see ECE 350X notes).

Figures 3 and 4 show some radiation patterns from different array designs. The radiation patterns can be changed by adjusting the spacings of the elements as well as the phase shift between them. The direction along the axis of an array is the "endfire" direction, while the direction orthogonal to the axis is known as the "broadside" direction. The radiation patterns have "lobes" as shown. The idea of antenna array design is to make the main lobe of the pattern to be much higher than the side lobes so that the radiated power of the antenna can be directed along the main lobe or lobes rather than the side lobes. So side-lobe level suppression is an important goal of designing a highly directive antenna design.

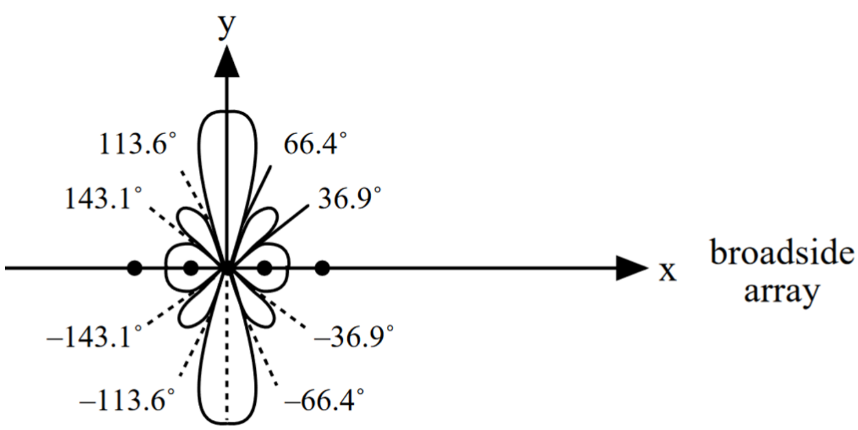

Figure 3:

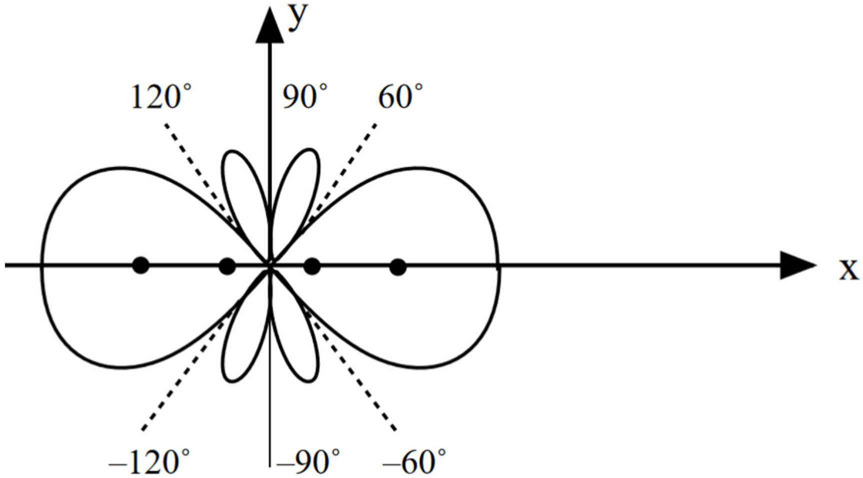

Figure 4:

4 When is Far-Field Approximation Valid?

In making the far-field approximation in (2.1), it will be interesting to ponder when the far-field approximation is valid? That is, when we can approximate

$$e^{-j\beta|\mathbf{r}-\mathbf{r}'|} \approx e^{-j\beta r + j\beta \mathbf{r}'\cdot\hat{r}} \tag{4.1}$$

This is especially important because when we integrate over $\mathbf{r}'$, it can range over large values especially for a large array. To answer this question, we need to study (4.1) more carefully. First, we have

$$|\mathbf{r}-\mathbf{r}'|^2 = (\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}'^2) = r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2 \tag{4.2}$$

We can take the square root of the above to get

$$\begin{aligned}
|\mathbf{r}-\mathbf{r}'| &= r\left(1 - \frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right)^{1/2} \\
&\approx r\left[1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{1}{2}\frac{r'^2}{r^2} - \frac{1}{2}\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)^2 + \cdots\right] \\
&= r - \frac{\hat{r}\cdot\mathbf{r}'}{r} + \frac{1}{2}\frac{r'^2}{r} - \frac{1}{2}\frac{(\mathbf{r}\cdot\mathbf{r}')^2}{r^3} + \\
&= r - \mathbf{r}\cdot\mathbf{r}' + \frac{1}{2}\frac{r'^2}{r} - \frac{1}{2r}(\hat{r}\cdot\mathbf{r}')^2 + \cdots
\end{aligned} \tag{4.3}$$

In the above, binomial expansion or Taylor series expansion have been used in making the approximation in the second line. The last two terms in the last line are of the same order. Moreover, their sum is bounded by ${r'}^2/(2r)$ since $\hat{r} \cdot \mathbf{r}'$ is always less than r'. Hence, the far field approximation is valid if

$$\beta \frac{{r'}^2}{2r} \ll 1 \tag{4.4}$$

In the above, β is involved because the approximation has to be valid in the exponent, namely $\exp(-j\beta|\mathbf{r} - \mathbf{r}'|)$. Consequently, we need that

$$r \gg \frac{\pi}{\lambda} {r'}^2 \tag{4.5}$$

If the aperture of the antenna is of radius W, then $r' < {r_{\max}}' \cong W$ and the far field approximation is valid if

$$r \gg \frac{\pi}{\lambda} W^2 = r_R \tag{4.6}$$

If r is larger than this distance, then a focus antenna beam behaves like a spherical wave and starts to diverge. This distance r_R is also known as the Rayleigh distance.

Hence, when a source radiates, the radiation field is divided into the near zone, the Fresnel zone, and the far zone (also known as the Fraunhofer zone in optics). The Rayleigh distance is the demarcation boundary between the Fresnel zone and the far zone. The larger the aperture of an antenna array is, the further one has to be to reach the far zone of an antenna. This distance becomes larger too when the wavelength is short. In the far zone, the far field behaves like a spherical wave, and its radiation pattern is proportional to the Fourier transform of the current.