

ECE 604, Lecture 18

November 1, 2018

In this lecture, we will cover the following topics:

- Homomorphism of Uniform Plane Waves and Transmission Line Equations:
 - TE or TE_z Waves
 - TM or TM_z Waves
- Wave Polarization:
 - Arbitrary Polarization Case and Axial Ratio
 - More about Polarization and Power Flow
- A Few Words about Faraday Rotation

Additional Reading:

- Sections 6.6, 6.8 of Ramo, Whinnery, and Van Duzer.
- Lecture Notes 11, Prof. Dan Jiao.
- Section 2.5, J.A. Kong, Electromagnetic Wave Theory.
- Lecture 18, ECE 350X.

You should be able to do the homework by reading the lecture notes alone. Additional reading is for references.

1 Homomorphism of Uniform Plane Waves and Transmission Lines Equations

It turns out that the plane waves through layered medium can be mapped into the multi-section transmission problem due to mathematical homomorphism between the two problems. Hence, we can kill two birds with one stone: apply all the transmission line techniques and equations that we have learnt so solve the waves through layered medium problems.

For uniform plane wave, we know that with $\nabla \rightarrow -j\beta$, we have

$$\beta \times \mathbf{E} = \omega\mu\mathbf{H} \quad (1.1)$$

$$\beta \times \mathbf{H} = -\omega\varepsilon\mathbf{E} \quad (1.2)$$

for a general isotropic homogeneous medium. We will specialize these equations for different polarizations.

1.1 TE or TE_z Waves

For this, one assumes a TE wave traveling in z direction with electric field polarized in the y direction, or $\mathbf{E} = \hat{y}E_y$, $\mathbf{H} = \hat{x}H_x + \hat{z}H_z$, then we have from (1.1)

$$\beta_z E_y = -\omega\mu H_x \quad (1.3)$$

$$\beta_x E_y = \omega\mu H_z \quad (1.4)$$

From (1.2), we have

$$\beta_z H_x - \beta_x H_z = -\omega\varepsilon E_y \quad (1.5)$$

Then, expressing H_z in terms of E_y from (1.4), we can show from (1.5) that

$$\beta_z H_x = -\omega\varepsilon \cos^2 \theta E_y \quad (1.6)$$

where $\beta \cos \theta = \beta_z$. Eqns. (1.3) and (1.6) can be written to look like the telegrapher's equation by letting $-j\beta_z \rightarrow \partial/\partial z$ to get

$$\frac{\partial}{\partial z} E_y = j\omega\mu H_x \quad (1.7)$$

$$\frac{\partial}{\partial z} H_x = j\omega\varepsilon \cos^2 \theta E_y \quad (1.8)$$

If we let $E_y \rightarrow V$, $H_x \rightarrow -I$, $\mu \rightarrow L$, $\varepsilon \cos^2 \theta \rightarrow C$, the above is exactly analogous to the telegrapher's equation. The equivalent characteristic impedance of these equations above is then

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon} \frac{1}{\cos^2 \theta}} = \sqrt{\frac{\mu}{\varepsilon} \frac{\beta}{\beta_z}} = \frac{\omega\mu}{\beta_z} \quad (1.9)$$

The above is the wave impedance for a propagating plane wave with propagation direction or the β inclined with an angle θ respect to the z axis. When $\theta = 0$, the wave impedance becomes the intrinsic impedance of space.

A two region, single interface reflection problem can then be mathematically mapped to a single-junction two transmission line problem. The equivalent characteristic impedances of these two regions are then

$$Z_{01} = \frac{\omega\mu}{\beta_{1z}}, \quad Z_{02} = \frac{\omega\mu}{\beta_{2z}} \quad (1.10)$$

We can use the above to find Γ_{12} as given by

$$\Gamma_{12} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} = \frac{(\mu_2/\beta_{2z}) - (\mu_1/\beta_{1z})}{(\mu_2/\beta_{2z}) + (\mu_1/\beta_{1z})} \quad (1.11)$$

The above is the same as the Fresnel reflection coefficient found earlier for TE waves after some simple re-arrangement.

1.2 TM or TM_z Waves

For the TM polarization, from duality principle, the corresponding equations are, from (1.7) and (1.8),

$$\frac{\partial}{\partial z} H_y = -j\omega\epsilon E_x \quad (1.12)$$

$$\frac{\partial}{\partial z} E_x = -j\omega\mu \cos^2 \theta H_y \quad (1.13)$$

Just for consistency of units, we may chose the following map to convert the above into the telegraphers equations, viz;

$$E_y \rightarrow V, \quad H_y \rightarrow I, \quad \mu \cos^2 \theta \rightarrow L, \quad \epsilon \rightarrow C \quad (1.14)$$

Then, the equivalent characteristic impedance is now

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\epsilon}} \cos \theta = \sqrt{\frac{\mu}{\epsilon}} \frac{\beta_z}{\beta} = \frac{\beta_z}{\omega\epsilon} \quad (1.15)$$

The above is also termed the wave impedance of a TM propagating wave making an inclined angle θ with respect to the z axis. Notice again that this wave impedance becomes the intrinsic impedance of space when $\theta = 0$.

Now,

$$\Gamma_{12} = \frac{(\beta_{2z}/\epsilon_2) - (\beta_{1z}/\epsilon_1)}{(\beta_{2z}/\epsilon_2) + (\beta_{1z}/\epsilon_1)} \quad (1.16)$$

Notice that (1.16) has a sign difference from the definition of R^{TM} derived earlier in the last lecture. The reason is that R^{TM} is for the reflection coefficient of magnetic field while Γ_{12} above is for the reflection coefficient of the voltage or

the electric field. This difference is also seen in the definition for transmission coefficients.

Because of the above homomorphism, one can easily use the multi-section transmission line formulas to study electromagnetic waves in layered media shown in Figures 1 and 2. Figure 2 shows the case of a normally incident wave into a layered media. For this case, the wave impedance becomes the intrinsic impedance.

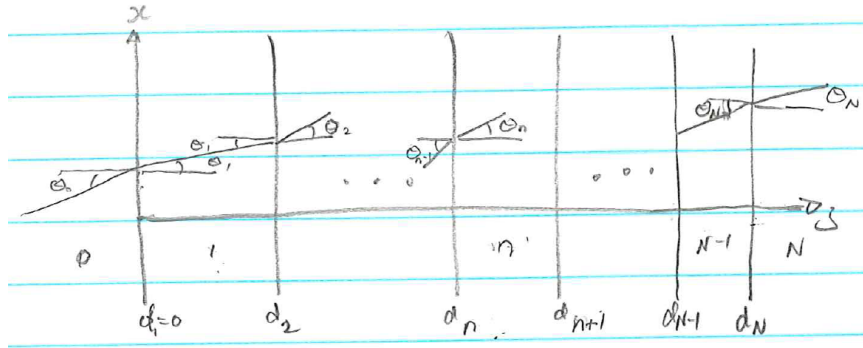


Figure 1:

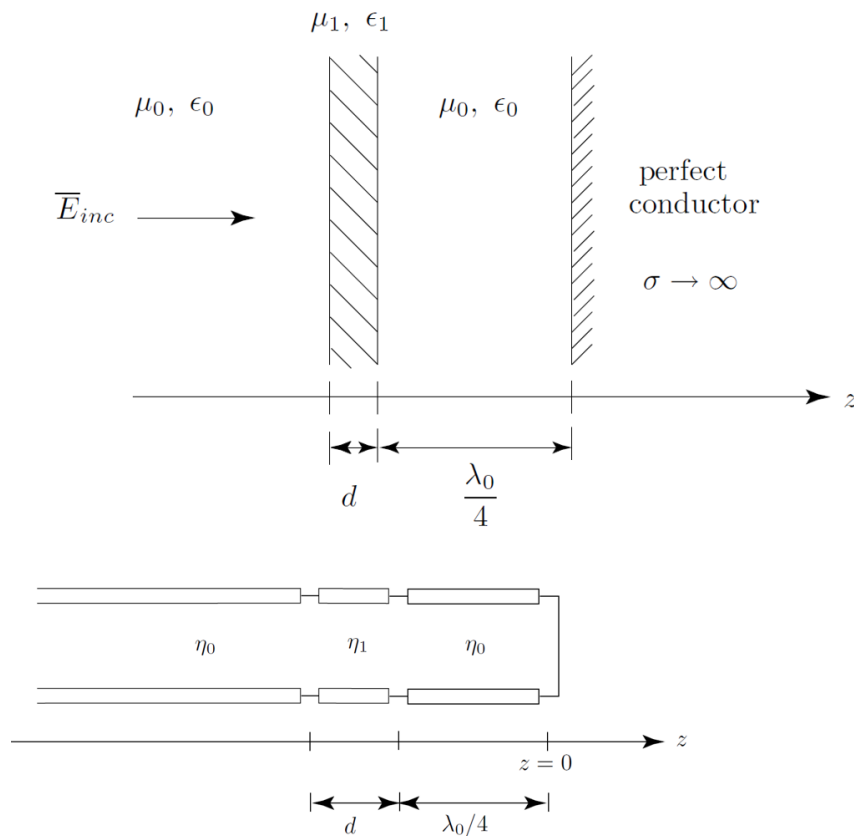


Figure 2: For normal incidence, the wave impedances becomes intrinsic impedances (Courtesy of J.A. Kong, Electromagnetic Wave Theory).

2 Wave Polarization

Studying wave polarization is very important for communication purposes. A wave whose electric field is pointing in the x direction while propagating in the z direction is a linearly polarized (LP) wave. The same can be said of one with electric field polarized in the y direction. It turns out that a linearly polarized wave suffers from Faraday rotation when it propagates through the ionosphere. For instance, an x polarized wave can become a y polarized due to Faraday rotation. So its polarization becomes ambiguous: to overcome this, Earth to satellite communication is done with circularly polarized (CP) waves. So even if the electric field vector is rotated by Faraday's rotation, it remains to be a CP wave. We will study these polarized waves next.

We can write a general uniform plane wave propagating in the z direction as

$$\mathbf{E} = \hat{x}E_x(z, t) + \hat{y}E_y(z, t) \quad (2.1)$$

Clearly, $\nabla \cdot \mathbf{E} = 0$, and $E_x(z, t)$ and $E_y(z, t)$ are solutions to the one-dimensional wave equation. For a time harmonic field, the two components may not be in phase, and we have in general

$$E_x(z, t) = E_1 \cos(\omega t - \beta z) \quad (2.2)$$

$$E_y(z, t) = E_2 \cos(\omega t - \beta z + \alpha) \quad (2.3)$$

where α denotes the phase difference between these two waves components. We shall study how the linear superposition of these two components behaves for different α . First, we set $z = 0$ to observe this field. Then

$$\mathbf{E} = \hat{x}E_1 \cos(\omega t) + \hat{y}E_2 \cos(\omega t + \alpha) \quad (2.4)$$

For $\alpha = \frac{\pi}{2}$

$$E_x = E_1 \cos(\omega t), \quad E_y = E_2 \cos(\omega t + \pi/2) \quad (2.5)$$

We evaluate the above for different ωt 's

$$\omega t = 0, \quad E_x = E_1, \quad E_y = 0 \quad (2.6)$$

$$\omega t = \pi/4, \quad E_x = E_1/\sqrt{2}, \quad E_y = -E_2/\sqrt{2} \quad (2.7)$$

$$\omega t = \pi/2, \quad E_x = 0, \quad E_y = -E_2 \quad (2.8)$$

$$\omega t = 3\pi/4, \quad E_x = -E_1/\sqrt{2}, \quad E_y = -E_2/\sqrt{2} \quad (2.9)$$

$$\omega t = \pi, \quad E_x = -E_1, \quad E_y = 0 \quad (2.10)$$

The tip of the vector field \mathbf{E} travels out an ellipse as show in Figure 3. With the thumb pointing in the z direction, and the wave rotating in the direction of the fingers, such a wave is called left-hand elliptically polarized (LHEP) wave.

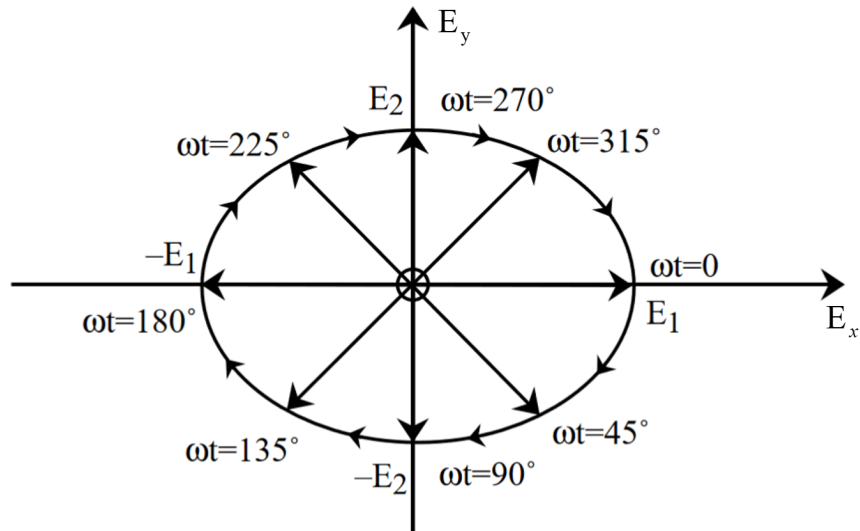


Figure 3:

When $E_1 = E_2$, the ellipse becomes a circle, and we have a left-hand circularly polarized (LHCP) wave. When $\alpha = -\pi/2$, the wave rotates in the counter-clockwise direction, and the wave is either right-hand elliptically polarized (RHEP), or right-hand circularly polarized (RHCP) wave depending on the ratio of E_1/E_2 . Figure 4 shows the different polarizations of the wave for different phase differences and amplitude ratio.

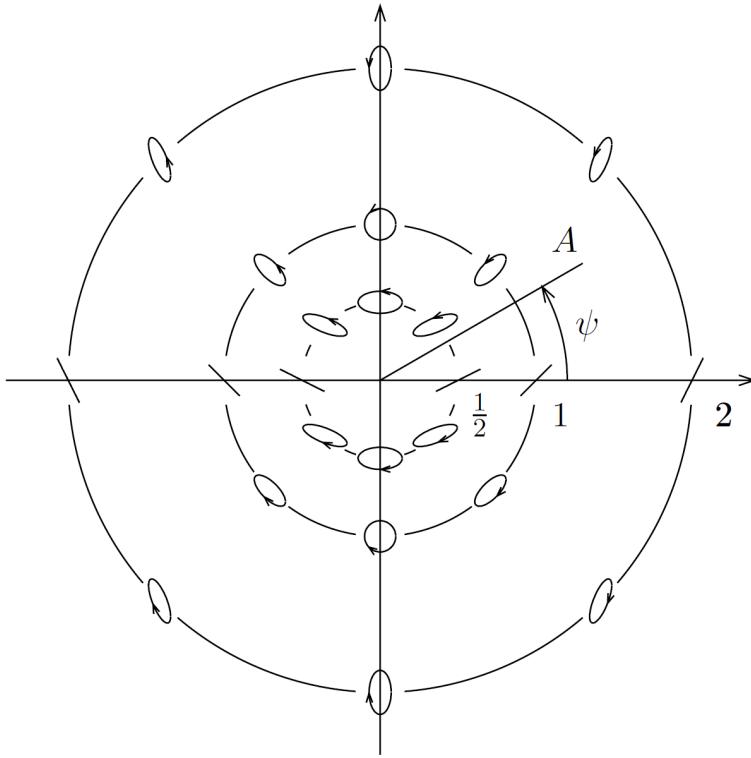


Figure 4: In this figure, $\psi = -\alpha$ in our notes, and $A = E_2/E_1$ (Courtesy of J.A. Kong, Electromagnetic Wave Theory).

Figure 5 shows a graphic picture of a CP wave propagating through space.

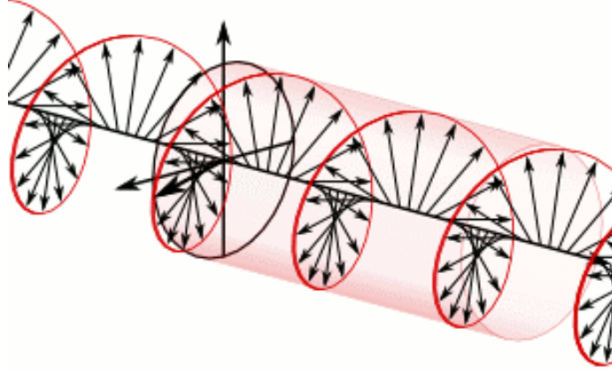


Figure 5: Courtesy of Wikipedia.

2.1 Arbitrary Polarization Case and Axial Ratio

The axial ratio (AR) is an important figure of merit for designing CP antennas (antennas that will radiate CP waves). The closer is this ratio to 1, the better the antenna design. We will discuss the general polarization and the axial ratio of a wave.

For the general case for arbitrary α , we let

$$E_x = E_1 \cos \omega t, \quad E_y = E_2 \cos(\omega t + \alpha) = E_2(\cos \omega t \cos \alpha - \sin \omega t \sin \alpha) \quad (2.11)$$

Then

$$E_y = \frac{E_2}{E_1} E_x \cos \alpha - E_2 \left[1 - \left(\frac{E_x}{E_1} \right)^2 \right]^{1/2} \sin \alpha \quad (2.12)$$

Rearranging and squaring, we get

$$aE_x^2 - bE_x^2E_y^2 + cE_y^2 = 1 \quad (2.13)$$

where

$$a = \frac{1}{E_1^2 \sin^2 \alpha}, \quad b = \frac{2 \cos \alpha}{E_1 E_2 \sin \alpha}, \quad c = \frac{1}{E_2^2 \sin^2 \alpha} \quad (2.14)$$

After letting $E_x \rightarrow x$, and $E_y \rightarrow y$, equation (2.13) is of the form,

$$ax^2 - bxy + cy^2 = 1 \quad (2.15)$$

The equation of an ellipse in its self coordinates is

$$\left(\frac{x'}{A} \right)^2 + \left(\frac{y'}{B} \right)^2 = 1 \quad (2.16)$$

where A and B are axes of the ellipse as shown in Figure 6. We can transform the above back to the (x, y) coordinates by letting

$$x' = x \cos \theta - y \sin \theta \quad (2.17)$$

$$y' = x \sin \theta + y \cos \theta \quad (2.18)$$

to get

$$x^2 \left(\frac{\cos^2 \theta}{A^2} + \frac{\sin^2 \theta}{B^2} \right) - xy \sin 2\theta \left(\frac{1}{A^2} - \frac{1}{B^2} \right) + y^2 \left(\frac{\sin^2 \theta}{A^2} + \frac{\cos^2 \theta}{B^2} \right) = 1 \quad (2.19)$$

Comparing (2.13) and (2.19), one gets

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2 \cos \alpha E_1 E_2}{E_2^2 - E_1^2} \right) \quad (2.20)$$

$$\text{AR} = \left(\frac{1 + \Delta}{1 - \Delta} \right)^{1/2} > 1 \quad (2.21)$$

where AR is the axial ratio where

$$\Delta = \left(1 - \frac{4E_1^2 E_2^2 \sin^2 \alpha}{(E_1^2 + E_2^2)^2} \right)^{1/2} \quad (2.22)$$

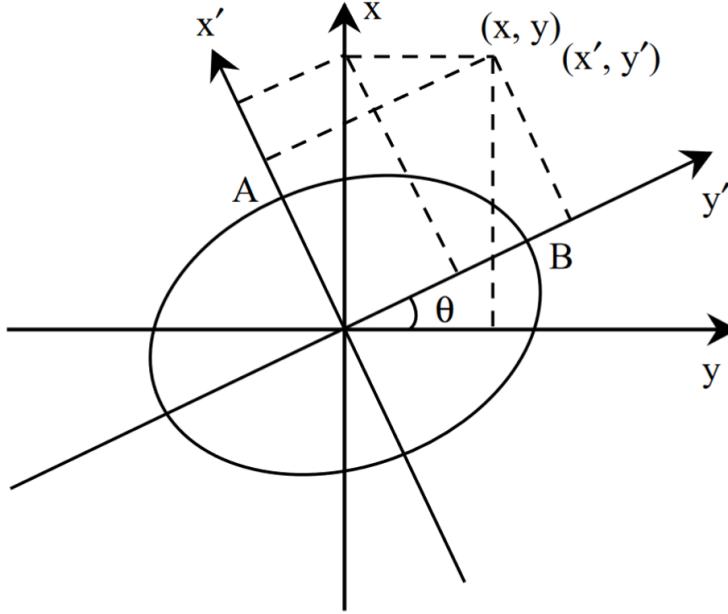


Figure 6:

2.2 More about Polarization and Power Flow

It is to be noted that in the phasor world, (2.1) becomes

$$\mathbf{E}(z, \omega) = \hat{x}E_1e^{-j\beta z} + \hat{y}E_2e^{-j\beta z + j\alpha} \quad (2.23)$$

For LHEP,

$$\mathbf{E}(z, \omega) = e^{-j\beta z}(\hat{x}E_1 + j\hat{y}E_2) \quad (2.24)$$

whereas for LHCP

$$\mathbf{E}(z, \omega) = e^{-j\beta z}E_1(\hat{x} + j\hat{y}) \quad (2.25)$$

For RHEP, the above becomes

$$\mathbf{E}(z, \omega) = e^{-j\beta z}(\hat{x}E_1 - j\hat{y}E_2) \quad (2.26)$$

whereas for RHCP, it is

$$\mathbf{E}(z, \omega) = e^{-j\beta z}E_1(\hat{x} - j\hat{y}) \quad (2.27)$$

For a linearly polarized wave,

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z} \frac{E_0^2}{\eta} \cos^2(\omega t - \beta z) \quad (2.28)$$

indicating that for a linearly polarized wave, the instantaneous power is function of both time and space. In the above E_0 is the amplitude of the linearly polarized wave.

For a circularly polarized wave,

$$\mathbf{E} = (\hat{x} \pm j\hat{y})E_0e^{-j\beta z} \quad (2.29)$$

$$\mathbf{H} = (\mp\hat{x} - j\hat{y})j\frac{E_0}{\eta}e^{-j\beta z} \quad (2.30)$$

Then

$$\mathbf{E}(t) = \hat{x}E_0 \cos(\omega t - \beta z) \pm \hat{y}E_0 \sin(\omega t - \beta z) \quad (2.31)$$

$$\mathbf{H}(t) = \mp\hat{x}\frac{E_0}{\eta} \sin(\omega t - \beta z) + \hat{y}\frac{E_0}{\eta} \cos(\omega t - \beta z) \quad (2.32)$$

Then the instantaneous power becomes

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z} \frac{E_0^2}{\eta} \cos^2(\omega t - \beta z) + \hat{z} \frac{E_0^2}{\eta} \sin^2(\omega t - \beta z) = \hat{z} \frac{E_0^2}{\eta} \quad (2.33)$$

In other words, a CP wave delivers constant power independent of space and time.

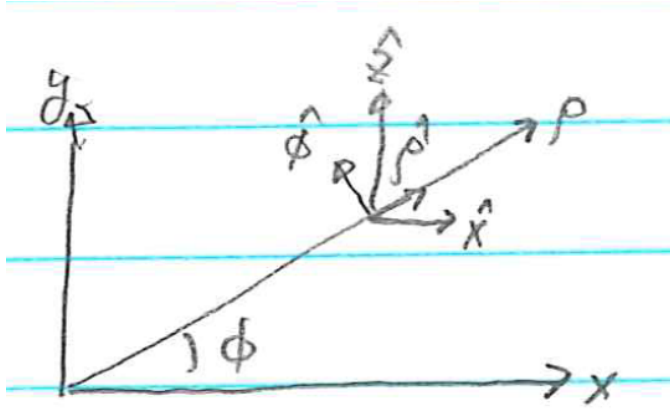


Figure 7:

It is to be noted that in cylindrical coordinates, as shown in Figure 7 $\hat{x} = \hat{\rho} \cos \phi - \hat{\phi} \sin \phi$, $\hat{y} = \hat{\rho} \sin \phi + \hat{\phi} \cos \phi$, then

$$(\hat{x} \pm j\hat{y}) = \hat{\rho}e^{\pm j\phi} \pm j\hat{\phi}e^{\pm j\phi} = e^{\pm j\phi}(\hat{\rho} \pm \hat{\phi}) \quad (2.34)$$

Therefore, the $\hat{\rho}$ and $\hat{\phi}$ of a CP is also a traveling wave in the $\hat{\phi}$ direction in addition to being a traveling wave $e^{-j\beta z}$ in the \hat{z} direction. Thus, the wave possesses angular momentum called the spin angular momentum (SAM), just as a traveling wave $e^{-j\beta z}$ possesses linear angular momentum in the \hat{z} direction.

In optics research, the generation of cylindrical vector beam is in vogue. Figure 8 shows a method to generate such a beam. A CP light passes through a radial analyzer that will only allow the radial component (2.34). Then a spiral phase element (SPE) compensates for the $\exp(\pm j\phi)$ phase shift in the azimuthal direction. Finally, the light is a cylindrical vector beam which is radially polarized.

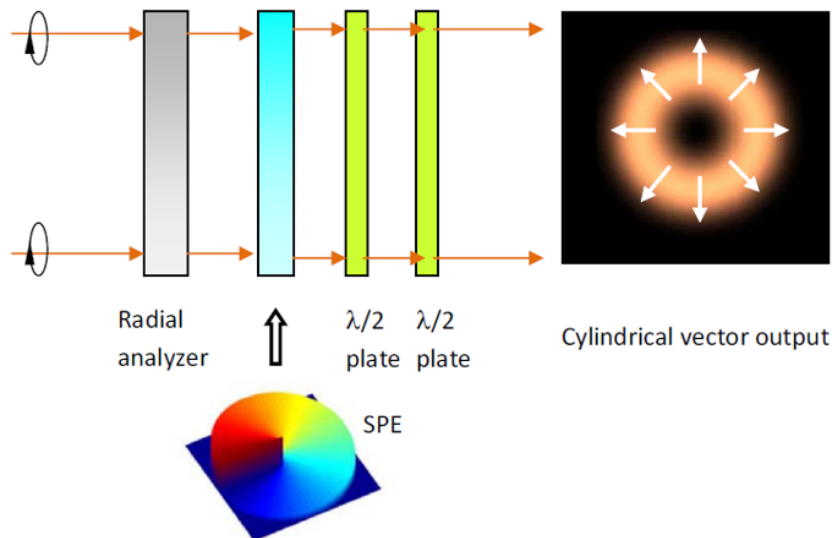


Figure 8: Courtesy of Zhan, Q. (2009). Cylindrical vector beams: from mathematical concepts to applications. *Advances in Optics and Photonics*, 1(1), 1-57.

3 A Few Words about Faraday Rotation

The prime reason why engineers have to design CP antennas is for Earth-to-satellite communication. Hence, it is important to understand why Faraday rotation exists.

Faraday rotation occurs for wave propagation through the ionosphere because of the Earth's static magnetic field. According to Lorentz force law

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}_0 \quad (3.1)$$

where \mathbf{B}_0 is a static magnetic field. Thus, when an electric field is applied, the force acting on the electron is not in the direction of the \mathbf{E} field, but has an addition force component orthogonal to the velocity of the electron. This causes the derivation of the effective permittivity to be rather complicated. At the end, the medium becomes an anisotropic medium where the electric flux \mathbf{D} does not point in the same direction as the applied electric field. The simplest form of an anisotropic permittivity is

$$\bar{\epsilon} = \begin{pmatrix} \epsilon_1 & jg & 0 \\ -jg & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix} \quad (3.2)$$

Such a medium is also known as a gyrotropic medium, and that the ionosphere is gyrotropic because of the static biasing magnetic field of the Earth.

A linearly polarized wave can be written as a linear superposition of LHCP and RHCP using the identity that

$$\hat{x} = \frac{1}{2}(\hat{x} + j\hat{y}) + \frac{1}{2}(\hat{x} - j\hat{y}) \quad (3.3)$$

Assuming the $\mathbf{E}_{\pm} = (\hat{x} \pm j\hat{y})E_0$ then it can be shown that for a gyrotropic medium, $\mathbf{D}_{\pm} = \varepsilon_{\pm}\mathbf{E}_{\pm}$. In other words, the RHCP and LHCP wave will see a different effective permittivity. Hence, they will have different phase velocity given by

$$v_{\pm} = \frac{1}{\sqrt{\mu_0\varepsilon_{\pm}}} \quad (3.4)$$

The LHCP wave and the RHCP wave will propagate through a gyrotropic medium with different phase velocities. Each of them will acquire a different phase shift and hence, there is a phase difference between LHCP and RHCP after they have emerged from a gyrotropic medium. Furthermore, it can be shown that this phase difference causes the wave vector not to point in the \hat{x} direction anymore, but in another direction. This is the effect of Faraday rotation.