ECE 604, Lecture 17

October 30, 2018

In this lecture, we will cover the following topics:

- Duality Principle
- Reflection and Transmission–Single Interface Case
- Interesting Physical Phenomena:
  - Total Internal Reflection
  - Brewster Angle
  - Surface Plasmonic Polariton

Additional Reading:

- Section 9.5 of Ramo, Whinnery, and Van Duzer.
- Lecture Notes 16, Prof. Dan Jiao.
- Section 4.1, Topic 6.1B, J.A. Kong, Electromagnetic Wave Theory.
- Lecture 20, ECE 350X.

You should be able to do the homework by reading the lecture notes alone. Additional reading is for references.

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1 Duality Principle

Duality principle exploits the inherent symmetry of Maxwell's equations. Once a set of $\mathbf{E}$, $\mathbf{H}$ has been found to solve Maxwell's equations for a certain geometry, another set for a similar geometry can be found by invoking this principle. Maxwell's equations in the frequency domain, including the fictitious magnetic sources, are

$$\nabla \times \mathbf{E}(\mathbf{r},\omega) = -j\omega\mathbf{B}(\mathbf{r},\omega) - \mathbf{M}(\mathbf{r},\omega) \tag{1.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r},\omega) = j\omega\mathbf{D}(\mathbf{r},\omega) + \mathbf{J}(\mathbf{r},\omega) \tag{1.2}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},\omega) = \varrho_m(\mathbf{r},\omega) \tag{1.3}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},\omega) = \varrho(\mathbf{r},\omega) \tag{1.4}$$

One way to make Maxwell's equations invariant is to do the following substitution.

$$\mathbf{E} \to \mathbf{H}, \quad \mathbf{H} \to -\mathbf{E}, \quad \mathbf{D} \to \mathbf{B}, \quad \mathbf{B} \to -\mathbf{D} \tag{1.5}$$

$$\mathbf{M} \to -\mathbf{J}, \quad \mathbf{J} \to \mathbf{M}, \quad \varrho_m \to \varrho, \quad \varrho \to \varrho_m \tag{1.6}$$

The above swaps retain the right-hand rule for plane waves. When material media is included, such that $\mathbf{D} = \overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}$, $\mathbf{B} = \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$, for anisotropic media, Maxwell's equations become

$$\nabla \times \mathbf{E} = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{M} \tag{1.7}$$

$$\nabla \times \mathbf{H} = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E} + \mathbf{J} \tag{1.8}$$

$$\nabla \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} = \varrho_m \tag{1.9}$$

$$\nabla \cdot \overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E} = \varrho \tag{1.10}$$

In addition to the above swaps, one need further to swap

$$\overline{\boldsymbol{\mu}} \to \overline{\boldsymbol{\varepsilon}}, \quad \overline{\boldsymbol{\varepsilon}} \to \overline{\boldsymbol{\mu}} \tag{1.11}$$

1.1 Unusual Swaps

If one adopts swaps where seemingly the right-hand rule is not preserved, e.g.,

$$\mathbf{E} \to \mathbf{H}, \mathbf{H} \to \mathbf{E}, \mathbf{M} \to -\mathbf{J}, \mathbf{J} \to -\mathbf{M}, \tag{1.12}$$

$$\varrho_m \to -\varrho, \varrho \to -\varrho_m, \overline{\boldsymbol{\mu}} \to -\overline{\boldsymbol{\varepsilon}}, \overline{\boldsymbol{\varepsilon}} \to -\overline{\boldsymbol{\mu}} \tag{1.13}$$

The above swaps will leave Maxwell's equations invariant, but when applied to a plane wave, the right-hand rule seems violated.

The deeper reason is that solutions to Maxwell's equations are not unique, since there is a time-forward as well as a time-reverse solution. In the frequency domain, this shows up in the choice of the sign of the $\mathbf{k}$ vector where in a plane wave $k = \pm\omega\sqrt{\mu\varepsilon}$. When one does a swap of $\mu \to -\varepsilon$ and $\varepsilon \to -\mu$, k is still indeterminate, and one can always choose a root where the right-hand rule is retained.

2 Reflection and Transmission—Single Interface case

We will derive the reflection coefficients for the single interface case. These reflection coefficients are also called the Fresnel reflection coefficients because they were first derive by Austin-Jean Fresnel (1788-1827). Note that he lived before the completion of Maxwell's equations in 1865. But when Fresnel derived the reflection coefficients in 1823, it was based on the elastic theory of light, and hence, the formulas are not exactly the same as what we are going to derive (see Born and Wolf, Principles of Optics, p. 40).

2.1 TE Polarization (Perpendicular or E Polarization)

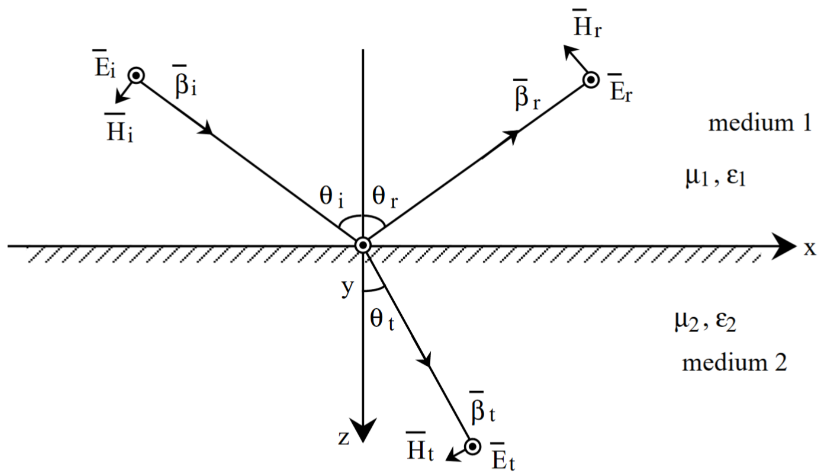

Figure 1:

To set up the above problem, the wave in Region 1 can be written as $\mathbf{E}_i + \mathbf{E}_r$. We assume plane wave polarized in the y direction where the wave vectors are $\boldsymbol{\beta}_i = \hat{x}\beta_{ix} + \hat{z}\beta_{iz}$, $\boldsymbol{\beta}_r = \hat{x}\beta_{rx} - \hat{z}\beta_{rz}$, $\boldsymbol{\beta}_t = \hat{x}\beta_{tx} + \hat{z}\beta_{tz}$, respectively for the incident, reflected, and transmitted waves. Then

$$\mathbf{E}_i = \hat{y}E_0 e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}} = \hat{y}E_0 e^{-j\beta_{ix}x - j\beta_{iz}z} \tag{2.1}$$

and

$$\mathbf{E}_r = \hat{y}R^{TE}E_0 e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}} = \hat{y}R^{TE}E_0 e^{-j\beta_{rx}x + j\beta_{rz}z} \tag{2.2}$$

In Region 2, we only have transmitted wave; hence

$$\mathbf{E}_t = \hat{y}T^{TE}E_0 e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}} = \hat{y}T^{TE}E_0 e^{-j\beta_{tx}x - j\beta_{tz}z} \tag{2.3}$$

In the above, the incident wave is known and hence, E_0 is known. From (2.2) and (2.3), R^{TE} and T^{TE} are unknowns yet to be sought. To find them, we need two boundary conditions to yield two equations. These are tangential $\mathbf{E}$ continuous and tangential $\mathbf{H}$ continuous, which are $\hat{n} \times \mathbf{E}$ continuous and $\hat{n} \times \mathbf{H}$ continuous conditions at the interface.

Imposing $\hat{n} \times \mathbf{E}$ continuous at $z = 0$, we get

$$E_0 e^{-j\beta_{ix}x} + R^{TE} E_0 e^{-j\beta_{rx}x} = T^{TE} E_0 e^{-j\beta_{tx}x}, \quad \forall x \tag{2.4}$$

In order for the above to be valid for all x, it is necessary that $\beta_{ix} = \beta_{rx} = \beta_{tx}$, which is also known as the phase matching condition.[1] Thus, canceling common terms on both sides of the equation, the above simplifies to

$$1 + R^{TE} = T^{TE} \tag{2.5}$$

To impose $\hat{n} \times \mathbf{H}$ continuous, one needs to find the $\mathbf{H}$ field using $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$, or that $\mathbf{H} = -j\boldsymbol{\beta} \times \mathbf{E}/(-j\omega\mu) = \boldsymbol{\beta} \times \mathbf{E}/(\omega\mu)$. By so doing

$$\mathbf{H}_i = \frac{\boldsymbol{\beta}_i \times \mathbf{E}_i}{\omega\mu_1} = \frac{\boldsymbol{\beta}_i \times \hat{y}}{\omega\mu_1} E_0 e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}} = \frac{\hat{z}\beta_{ix} - \hat{x}\beta_{iz}}{\omega\mu_1} E_0 e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}} \tag{2.6}$$

$$\mathbf{H}_r = \frac{\boldsymbol{\beta}_r \times \mathbf{E}_r}{\omega\mu_1} = \frac{\boldsymbol{\beta}_r \times \hat{y}}{\omega\mu_1} R^{TE} E_0 e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}} = \frac{\hat{z}\beta_{rx} + \hat{x}\beta_{rz}}{\omega\mu_2} R^{TE} E_0 e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}} \tag{2.7}$$

$$\mathbf{H}_t = \frac{\boldsymbol{\beta}_t \times \mathbf{E}_t}{\omega\mu_2} = \frac{\boldsymbol{\beta}_t \times \hat{y}}{\omega\mu_2} T^{TE} E_0 e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}} = \frac{\hat{z}\beta_{tx} - \hat{x}\beta_{tz}}{\omega\mu_2} T^{TE} E_0 e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}} \tag{2.8}$$

Imposing $\hat{n} \times \mathbf{H}$ continuous or H_x continuous at $z = 0$, we have

$$\frac{\beta_{iz}}{\omega\mu_1} E_0 e^{-j\beta_{ix}x} - \frac{\beta_{rz}}{\omega\mu_1} R^{TE} E_0 e^{-j\beta_{rx}x} = \frac{\beta_{tz}}{\omega\mu_2} T^{TE} E_0 e^{-j\beta_{tx}x} \tag{2.9}$$

As mentioned before, the phase-matching condition requires that $\beta_{ix} = \beta_{rx} = \beta_{tx}$. The dispersion relation for plane waves requires that

$$\beta_{ix}^2 + \beta_{iz}^2 = \beta_{rx}^2 + \beta_{rz}^2 = \omega^2\mu_1\varepsilon_1 = \beta_1^2 \tag{2.10}$$

$$\beta_{tx}^2 + \beta_{tz}^2 = \omega^2\mu_2\varepsilon_2 = \beta_2^2 \tag{2.11}$$

Since $\beta_{ix} = \beta_{rx} = \beta_{tx} = \beta_x$, it implies that $\beta_{iz} = \beta_{rz}$. Moreover, $\beta_{tz} = \beta_{2z} \neq \beta_{1z}$ usually. Then (2.9) simplifies to

$$\frac{\beta_{1z}}{\mu_1}(1 - R^{TE}) = \frac{\beta_{2z}}{\mu_2} T^{TE} \tag{2.12}$$

where $\beta_{1z} = \sqrt{\beta_1^2 - \beta_x^2}$, and $\beta_{2z} = \sqrt{\beta_2^2 - \beta_x^2}$.

Solving (2.5) and (2.12) yields

$$R^{TE} = \left(\frac{\beta_{1z}}{\mu_1} - \frac{\beta_{2z}}{\mu_2}\right) \Bigg/ \left(\frac{\beta_{1z}}{\mu_1} + \frac{\beta_{2z}}{\mu_2}\right) \tag{2.13}$$

$$T^{TE} = 2\left(\frac{\beta_{1z}}{\mu_1}\right) \Bigg/ \left(\frac{\beta_{1z}}{\mu_1} + \frac{\beta_{2z}}{\mu_2}\right) \tag{2.14}$$

[1] The phase-matching condition can also be proved by taking the Fourier transform of the equation with respect to x.

2.2 TM Polarization (Parallel or H Polarization)

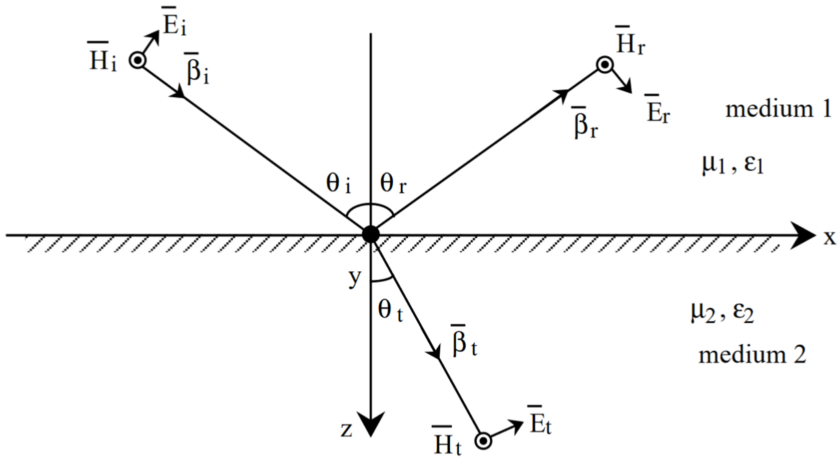

Figure 2:

The solution to the TM polarization case can be obtained by invoking duality principle where we do the substitution $\mathbf{E} \to \mathbf{H}$, $\mathbf{H} \to -\mathbf{E}$, and $\mu \rightleftharpoons \varepsilon$ as shown in Figure 2. The reflection coefficient for the TM magnetic field is then

$$R^{TM} = \left(\frac{\beta_{1z}}{\varepsilon_1} - \frac{\beta_{2z}}{\varepsilon_2}\right) \Bigg/ \left(\frac{\beta_{1z}}{\varepsilon_1} + \frac{\beta_{2z}}{\varepsilon_2}\right) \tag{2.15}$$

$$T^{TM} = 2\left(\frac{\beta_{1z}}{\varepsilon_2}\right) \Bigg/ \left(\frac{\beta_{1z}}{\varepsilon_1} + \frac{\beta_{2z}}{\varepsilon_2}\right) \tag{2.16}$$

3 Interesting Physical Phenomena

Three interesting physical phenomena emerge from the solutions of the single-interface problem. They are total internal reflection, Brewster angle effect, and surface plasmonic resonance. We will look at them next.

3.1 Total Internal Reflection

Total internal reflection comes about because of phase matching also called momentum matching. This phase-matching condition can be illustrated using β-surfaces (same as k-surfaces in some literature), as shown in Figure **??**.

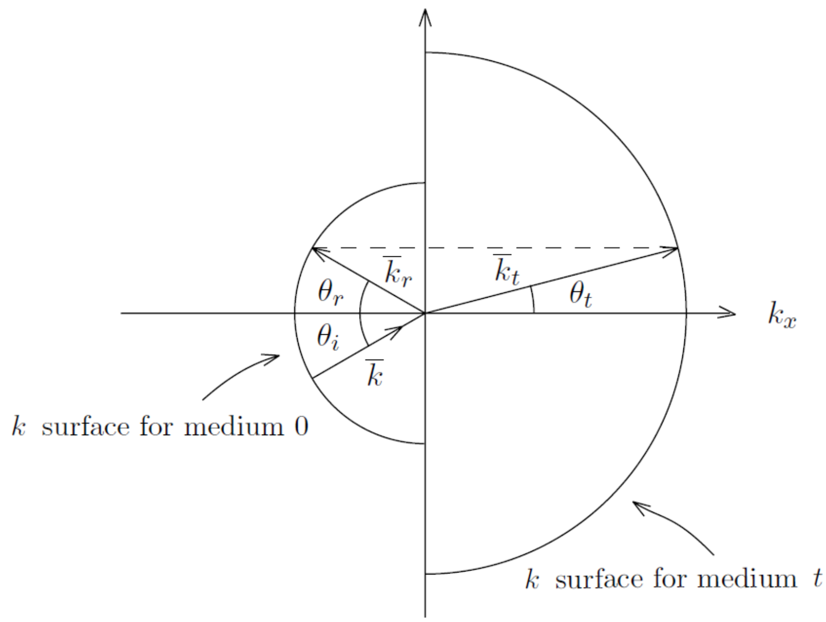

Figure 3: Courtesy of J.A. Kong, Electromagnetic Wave Theory. Here, k is synonymous with β.

It turns out that because of phase matching, for certain interfaces, β_{2z} becomes pure imaginary.

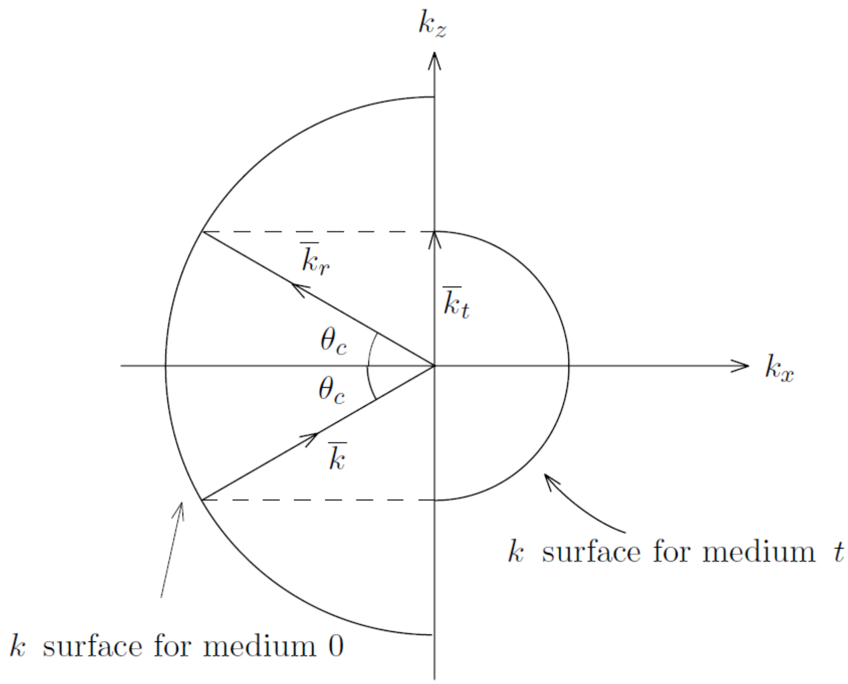

Figure 4: Courtesy of J.A. Kong, Electromagnetic Wave Theory. Here, k is synonymous with β.

As shown in Figures 3 and 4, because of the dispersion relation that $\beta_{rx}^2 + \beta_{rz}^2 = \beta_{ix}^2 + \beta_{iz}^2 = \beta_1^2$, $\beta_{tx}^2 + \beta_{tz}^2 = \beta_2^2$, they are equations of two circles in 2D whose radii are β_1 and β_2, respectively. The tips of the $\boldsymbol{\beta}$ vectors for Regions 1 and 2 have to be on a spherical surface in the β_x, β_y, and β_z space in the general 3D case, but in this figure, we only show a cross section of the sphere assuming that $\beta_y = 0$.

Phase matching implies that the x-component of the $\boldsymbol{\beta}$ vectors are equal to each other as shown. One sees that $\theta_i = \theta_r$ in Figure 4, and also as θ_i increases, θ_t increases. For an optically less dense medium where $\beta_2 < \beta_1$, according to the Snell's law of refraction, the transmitted $\boldsymbol{\beta}$ will refract away from the normal, as seen in the figure. Therefore, eventually the vector $\boldsymbol{\beta}_t$ becomes parallel to the x axis when $\beta_{ix} = \beta_{rx} = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$ and $\theta_t = \pi/2$. The incident angle at which this happens is termed the critical angle θ_c.

Since $\beta_{ix} = \beta_1 \sin\theta_i = \beta_{rx} = \beta_1 \sin\theta_r = \beta_2$, or

$$\sin\theta_r = \sin\theta_i = \sin\theta_c = \frac{\beta_2}{\beta_1} = \frac{\sqrt{\mu_2\varepsilon_2}}{\sqrt{\mu_1\varepsilon_1}} = \frac{n_2}{n_1} \tag{3.1}$$

where n_1 is the reflective index defined as $c_0/v_i = \sqrt{\mu_i\varepsilon_i}/\sqrt{\mu_0\varepsilon_0}$ where v_i is the phase velocity of the wave in Region i. Hence,

$$\theta_c = \sin^{-1}(n_2/n_1) \tag{3.2}$$

When $\theta_i > \theta_c$. $\beta_x > \beta_2$ and $\beta_{2z} = \sqrt{{\beta_2}^2 - {\beta_x}^2}$ becomes pure imaginary. When β_{2z} becomes pure imaginary, the wave cannot propagate in Region 2, or $\beta_{2z} = -j\alpha_{2z}$, and the wave becomes evanescent. The reflection coefficient (2.13) becomes of the form

$$R^{TE} = (A - jB)/(A + jB) \tag{3.3}$$

It is clear that $|R^{TE}| = 1$ and that $R^{TE} = e^{j\theta_{TE}}$. Therefore, a total internally reflected wave suffers a phase shift. A phase shift in the frequency domain corresponds to a time delay in the time domain. Such a time delay is achieved by the wave traveling laterally in Region 2 before being refracted back to Region 1. Such a lateral shift is called the Goos-Hanschen shift as shown in Figure 5.

Please be reminded that total internal reflection comes about entirely due to the phase-matching condition when Region 2 is a faster medium than Region 1. Hence, it will occur with all manner of waves, such as elastic waves, sound waves, seismic waves, quantum waves etc.

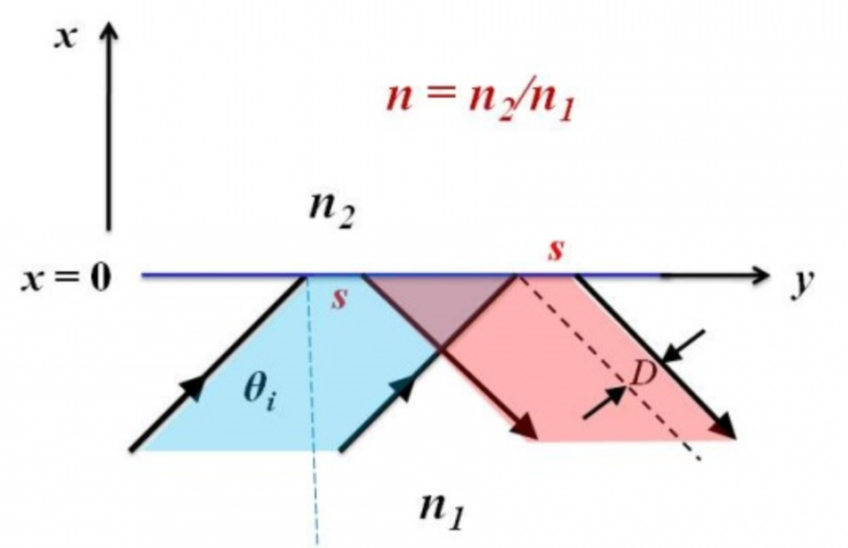

Figure 5: Goos-Hanschen Shift. Courtesy of Paul R. Berman (2012), Scholarpedia, 7(3):11584.

The guidance of a wave in a dielectric slab is due to total internal reflection at the dielectric-to-air interface. The wave bounces between the two interfaces of the slab, and creates evanescent waves outside, as shown in Figure 6. The guidance of waves in an optical fiber works by similar mechanism, as shown in Figure 7.

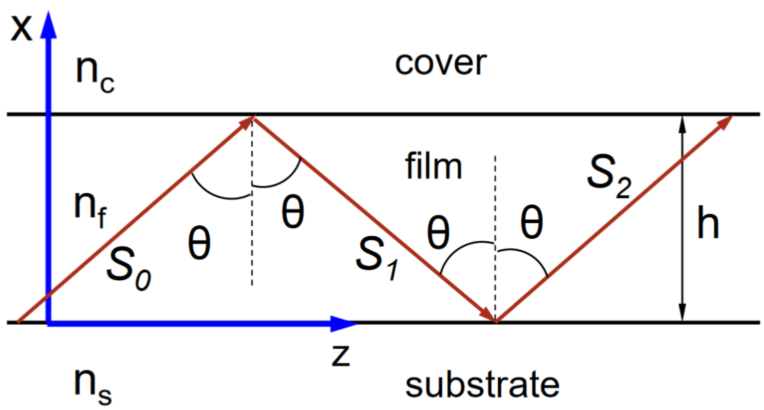

Figure 6: Courtesy of E.N. Glytsis, NTUA, Greece.

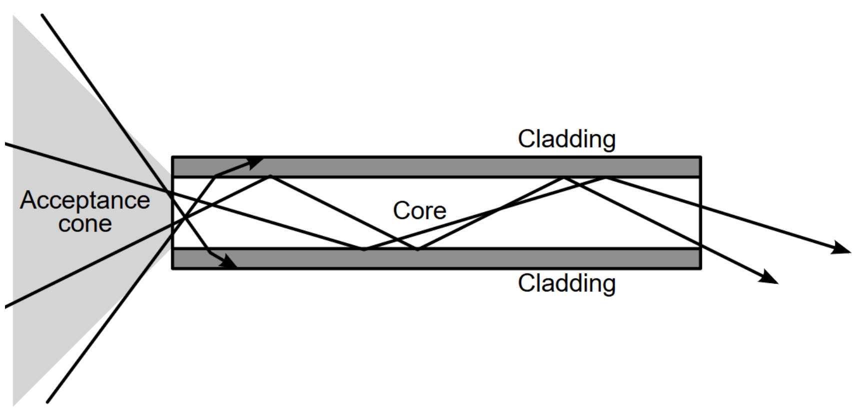

Figure 7: Courtesy of Wikepedia.

3.2 Brewster Angle

Since most materials at optical frequencies have $\varepsilon_2 \neq \varepsilon_1$, but $\mu_2 \approx \mu_1$, the TM polarization for light behaves differently from TE polarization. For R^{TM}, it is possible that $R^{TM} = 0$ if

$$\varepsilon_2 \beta_{1z} = \varepsilon_1 \beta_{2z} \tag{3.4}$$

Squaring the above assuming $\mu_1 = \mu_2$, one gets

$$\varepsilon_2{}^2(\beta_1{}^2 - \beta_x{}^2) = \varepsilon_1{}^2(\beta_2{}^2 - \beta_x{}^2) \tag{3.5}$$

Solving the above gives

$$\beta_x = \omega\sqrt{\mu}\sqrt{\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1 + \varepsilon_2}} = \beta_1 \sin\theta_1 = \beta_2 \sin\theta_2 \tag{3.6}$$

The latter two equations come from phase matching at the interface. Therefore,

$$\sin\theta_1 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}, \quad \sin\theta_2 = \sqrt{\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}} \tag{3.7}$$

or that

$$\sin^2\theta_1 + \sin^2\theta_2 = 1, \tag{3.8}$$

Then

$$\sin\theta_2 = \cos\theta_1 \tag{3.9}$$

or that

$$\theta_1 + \theta_2 = \pi/2 \tag{3.10}$$

This is used to explain why at Brewster angle, no light is reflected back to Region 1. Figure 8 shows that the induced polarization dipoles in Region 2 always have their axes aligned in the direction of reflected wave. A dipole does not radiate along its axis, which can be verified heuristically by field sketch and looking at the Poynting vector. Therefore, these induced dipoles in Region 2 do not radiate in the direction of the reflected wave.

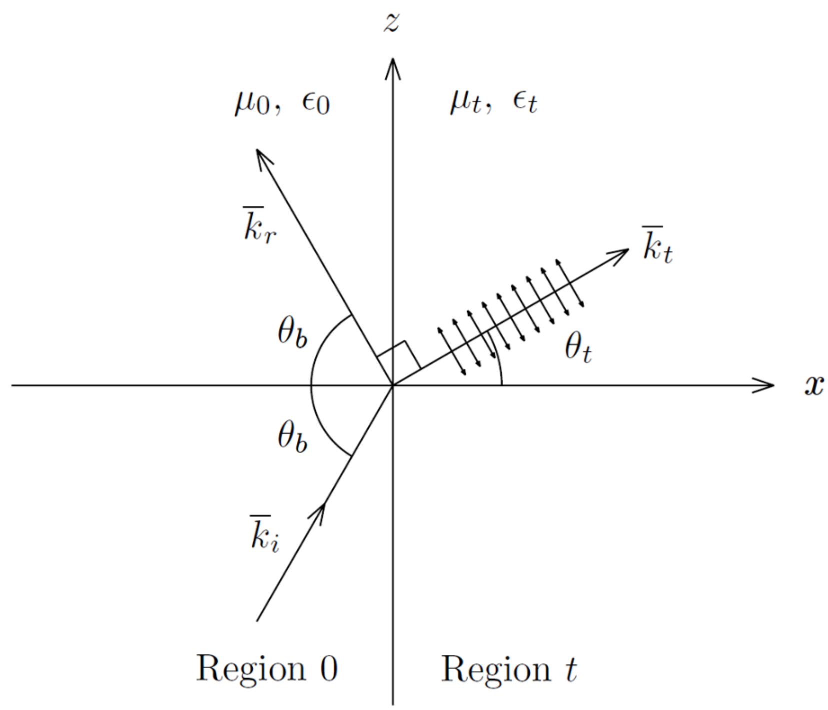

Figure 8: Courtesy of J.A. Kong, EM Wave Theory.

Because of the Brewster angle effect, and that $\varepsilon_2 \neq \varepsilon_1$, $|R^{TM}| \leq |R^{TE}|$ as shown in Fig. 9. This phenomenon is used to design sun glasses to reduce road glare for drivers. For light reflected off a road surface, they are predominantly horizontally polarized. When sun glasses are made with vertical polarizers, they will filter out and mitigate the reflected rays from the road surface.

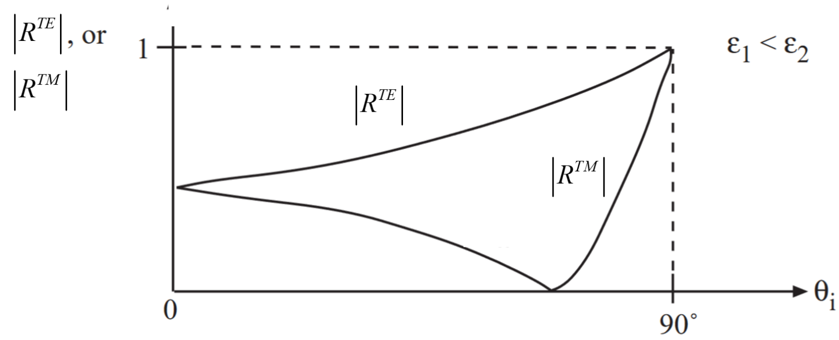

Figure 9:

3.3 Surface Plasmon Polariton

Surface plasmon polariton occurs for the same mathematical reason for the Brewster angle effect but the physical mechanism is quite different. The reflection coefficient R^{TM} can become infinite if, say, $\varepsilon_2 < 0$, as in a plasma medium.

In this case, the criterion is

$$-\varepsilon_2\beta_{1z} = \varepsilon_1\beta_{2z} \tag{3.11}$$

When the above is satisfied, R^{TM} becomes infinite. This implies a reflected wave exists when there is no incident wave. This is often encountered in a resonance system. Here, there is a plasmonic resonance or propagating mode generated at the interface.

Solving (3.11) after squaring it yields

$$\beta_x = \omega\sqrt{\mu}\sqrt{\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}} \tag{3.12}$$

This is the same equation for the Brewster angle except now that ε_2 is negative. Even if $\varepsilon_2 < 0$, but $\varepsilon_1 + \varepsilon_2 < 0$ is still possible so that the expression under the square root sign (3.12) is positive. Thus, β_x can be pure real. This corresponds to a guided wave propagating in the x direction. When this happens,

$$\beta_{1z} = \sqrt{{\beta_1}^2 - {\beta_x}^2} = \omega\sqrt{\mu}\left[\varepsilon_1\left(1 - \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)\right]^{1/2} \tag{3.13}$$

Since $\varepsilon_2 < 0$, $\varepsilon_2/(\varepsilon_1+\varepsilon_2) > 1$, then β_{1z} becomes pure imaginary. Moreover, $\beta_{2z} = \sqrt{{\beta_2}^2 - {\beta_x}^2}$ and ${\beta_2}^2 < 0$ making β_{2z} becomes even a larger imaginary number. This corresponds to a trapped wave at the interface. The wave decays exponentially in both directions away from the interface and they are evanescent waves. This mode is shown in Figure 10, and is the only case in electromagnetics where a single interface can guide a surface wave, while such phenomena abounds for elastic waves.

When one operates close to the resonance of the mode so that the denominator in (3.12) is almost zero, then β_x can be very large. The wavelength becomes very short in this case, and β_{1z} and β_{2z} become even larger imaginary numbers. Hence, the mode becomes tightly confined to the surface, making the confinement of the mode very tight. It portends use in tightly packed optical components, and has caused some excitement in the optics community. But loss is still an issue to be overcome here!

https://en.wikipedia.org/wiki/Surface_plasmon

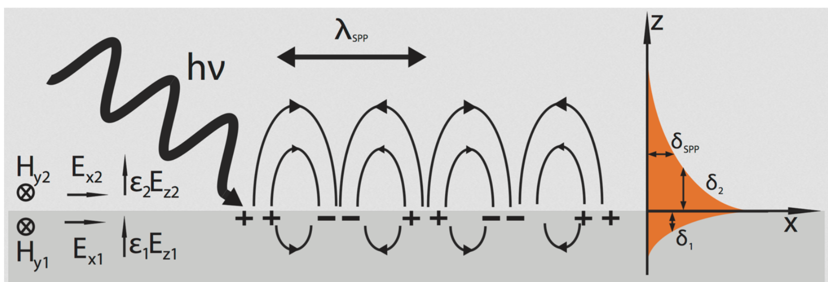

Figure 10: Courtesy of Wikipedia.