ECE 604, Lecture 16

October 25, 2018

In this lecture, we will cover the following topics:

- Phase Velocity and Group Velocity
- Energy Density in Dispersive Media

Additional Reading:

- Section 5.15 of Ramo, Whinnery, and Van Duzer.
- Section 1.7 of H.A. Haus, Electromagnetic Noise and Quantum Optical Measurements.
- Sections 1.8, 2.7 of W.C. Chew, Lectures on Theory of Microwave and Optical Waveguides: http://wcchew.ece.illinois.edu/chew/course/tgwAll20160215.pdf
- Sections 3.1E, 3.1G of J.A. Kong, Electromagnetic Wave Theory

You should be able to do the homework by reading the lecture notes alone. Additional reading is for references.

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1 Phase Velocity and Group Velocity

Now that we know how a medium can be frequency dispersive in the Drude-Lorentz-Sommerfeld (DLS) model, we are ready to distinguish the difference between the phase velocity and the group velocity

1.1 Phase Velocity

The phase velocity is the velocity of the phase of a wave. It is only defined for a mono-chromatic signal (also called time-harmonic, CW (constant wave), or sinusoidal signal) at one given frequency. A sinusoidal wave signal, e.g., the voltage signal on a transmission line, can take the form

$$V(z,t) = V_0 \cos(\omega t - kz + \alpha) \tag{1.1}$$

This sinusoidal signal moves with a velocity

$$v_{ph} = \frac{\omega}{k} \tag{1.2}$$

where, for example, $k = \omega\sqrt{\mu\varepsilon}$, inside a simple coax. Hence,

$$v_{ph} = 1/\sqrt{\mu\varepsilon} \tag{1.3}$$

But a dielectric medium can be frequency dispersive, or $\varepsilon(\omega)$ is not a constant but a function of ω as has been shown with the Drude-Lorentz-Sommerfeld model. Therefore, signals with different ω's will travel with different phase velocity.

More bizarre still, what if the coax is filled with a plasma medium where

$$\varepsilon = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2}\right) \tag{1.4}$$

Then, $\varepsilon < \varepsilon_0$ always meaning that the phase velocity given by (1.3) can be larger than the velocity of light in vacuum if $\mu = \mu_0$. Also, $\varepsilon = 0$ when $\omega = \omega_p$, implying that $k = 0$; then in accordance to (1.2), $v_{ph} = \infty$. These ludicrous observations can be justified or understood only if we can show that information can only be sent by using a wave packet.[1] The same goes for energy which can only be sent by wave packets, but not by CW signal; only in this manner can a finite amount of energy be sent. These wave packets can only travel at the group velocity as shall be shown, which is always less than the velocity of light.

[1]In information theory, according to Shannon, the basic unit of information is a bit, which can only be sent by a digital signal, or a wave packet.

1.2 Group Velocity

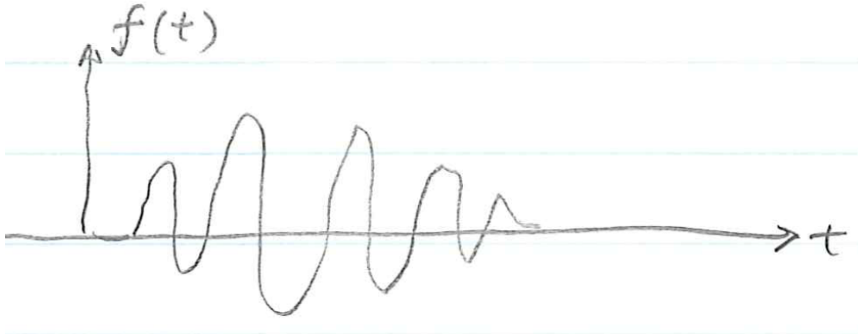

Figure 1:

Now, consider a narrow band wave packet as shown in Figure 1. It cannot be mono-chromatic, but can be written as a linear superposition of many frequencies. One way to express this is to write this wave packet as an integral summation over many frequencies, namely

$$V(z,t) = \int_{-\infty}^{\infty} d\omega V(z,\omega) e^{j\omega t} \tag{1.5}$$

Assume that $V(z,t)$ is the solution, say to the dispersive transmission line equations with $\varepsilon(\omega)$, then it can be shown that $V(z,\omega)$ is the solution to the one-dimensional Helmholtz equation

$$\frac{d^2}{dz^2} V(z,\omega) + k^2(\omega) V(z,\omega) = 0 \tag{1.6}$$

as in the dispersive transmission line filled with dispersive material so that $k^2 = \omega^2 \mu_0 \varepsilon(\omega)$. Thus, upon solving the above equation, one obtains that $V(z,\omega) = V_0(\omega) e^{-jkz}$, and

$$V(z,t) = \int_{-\infty}^{\infty} d\omega V_0(\omega) e^{j(\omega t - kz)} \tag{1.7}$$

In the general case, k is a complicated function of ω as shown in Figure 2.

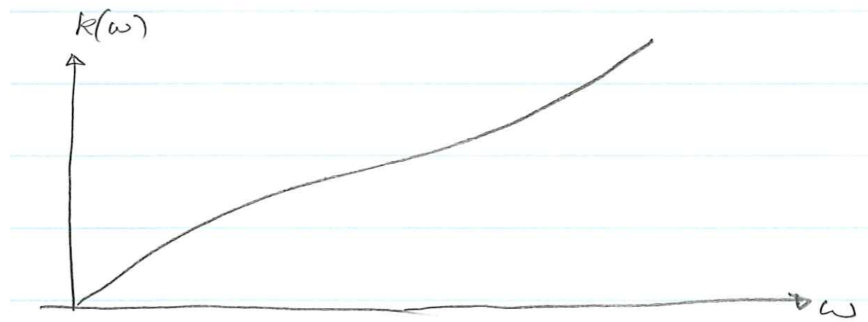

Figure 2:

Since this is a wave packet, we assume that $V_0(\omega)$ is narrow band centered about a frequency ω_0, the carrier frequency as shown in Figure 3. Therefore, when the integral in (1.7) is performed, it needs only be summed over a narrow range of frequencies in the vicinity of ω_0.

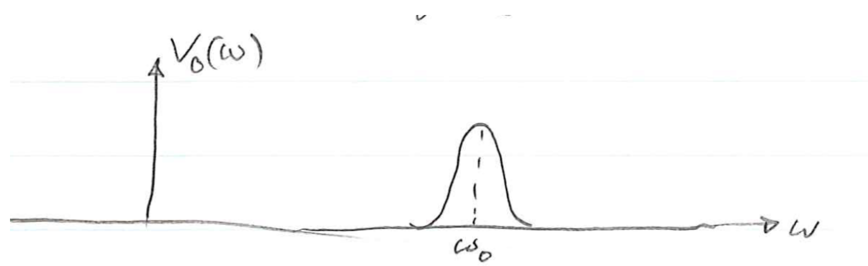

Figure 3:

Thus, we can approximate the integrand in the vicinity of $\omega = \omega_0$, and let

$$k(\omega) \cong k(\omega_0) + (\omega - \omega_0)\frac{dk(\omega_0)}{d\omega} + \frac{1}{2}(\omega - \omega_0)^2 \frac{d^2 k(\omega_0)}{d\omega^2} + \cdots \tag{1.8}$$

To ensure the real-valuedness of (1.5), one can ensure that $-\omega$ part of the integrand is exactly the complex conjugate of the $+\omega$ part. Another way is to sum over only the $+\omega$ part of the integral and take twice the real part of the integral. So, for simplicity, we write (1.5) as

$$V(z,t) = 2\Re e \int_0^\infty d\omega V_0(\omega) e^{j(\omega t - kz)} \tag{1.9}$$

Since we need to integrate over $\omega \approx \omega_0$, we can substitute (1.8) into (1.9) and rewrite it as

$$V(z,t) \cong 2\Re e \left[e^{j[\omega_0 t - k(\omega_0)z]} \underbrace{\int_0^\infty d\omega V_0(\omega) e^{j(\omega-\omega_0)t} e^{-j(\omega-\omega_0)\frac{dk}{d\omega}z}}_{F\left(t-\frac{dk}{d\omega}z\right)} \right] \tag{1.10}$$

where more specifically,

$$F\left(t - \frac{dk}{d\omega}z\right) = \int_0^\infty d\omega V_0(\omega) e^{j(\omega-\omega_0)t} e^{-j(\omega-\omega_0)\frac{dk}{d\omega}z} \tag{1.11}$$

It can be seen that the above integral now involves the integral summation over a small range of ω in the vicinity of ω_0. By a change of variable by letting $\Omega = \omega - \omega_0$, it becomes

$$F\left(t - \frac{dk}{d\omega}z\right) = \int_{-\Delta}^{+\Delta} d\Omega V_0(\Omega + \omega_0) e^{j\Omega\left(t - \frac{dk}{d\omega}z\right)} \tag{1.12}$$

The above itself is a Fourier transform integral that involves only the low frequencies of the Fourier spectrum. Hence, it is a slowly varying function. Moreover, this function F moves with a velocity

$$v_g = \frac{d\omega}{dk} \tag{1.13}$$

Here, $F(t - \frac{z}{v_g})$ in fact is the velocity of the envelope in Figure 1. In (1.10), the envelope function $F(t - \frac{z}{v_g})$ is multiplied by the rapidly varying function

$$e^{j[\omega_0 t - k(\omega_0)z]} \tag{1.14}$$

before one takes the real part of the entire function. Hence, this rapidly varying part represents the rapidly varying carrier frequency shown in Figure 1. More importantly, this carrier, the rapidly varying part of the signal, moves with the velocity

$$v_{ph} = \frac{\omega_0}{k(\omega_0)} \tag{1.15}$$

which is the phase velocity.

2 Energy Density in Dispersive Media

A dispersive medium alters our understanding of phase and group velocity, but it also changes our concept of what energy density is. To this end, we assume that the field has complex ω dependence rather than real ω dependence of $e^{j\omega t}$, where $\omega = \omega' - j\omega''$. We take the divergence of the complex power for fields with such time dependence, and let $e^{j\omega t}$ be attached to the field. So $\mathbf{E}(t)$ and $\mathbf{H}(t)$ are complex field but not exactly like phasors since they are not truly time harmonic. Hence,

$$\begin{aligned}\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] &= \mathbf{H}^*(t) \cdot \nabla \times \mathbf{E}(t) - \mathbf{E}(t) \cdot \nabla \times \mathbf{H}^*(t) \\ &= -\mathbf{H}^*(t) \cdot j\omega\mu\mathbf{H}(t) + \mathbf{E}(t) \cdot j\omega^*\varepsilon^*\mathbf{E}^* \end{aligned} \tag{2.1}$$

where Maxwell's equations have been used to substitute for $\nabla \times \mathbf{E}(t)$ and $\nabla \times \mathbf{H}^*(t)$. The space dependence of the field is implied, and we assure a source-free medium so that $\mathbf{J} = 0$.

If $\mathbf{E}(t) \sim e^{j\omega t}$, then $\mathbf{H}^*(t) \sim e^{-j\omega^* t}$, and the term like

$$\mathbf{E}(t) \times \mathbf{H}^*(t) \sim e^{j(\omega - \omega^*)t} = e^{2\omega'' t} \tag{2.2}$$

And each of the term above will have similar time dependence. Writing (2.1) more explicitly, by letting $\omega = \omega' - j\omega''$, we have

$$\begin{aligned}\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] = -j(\omega' - j\omega'')\mu(\omega' - j\omega'')|\mathbf{H}(t)|^2 + j(\omega' \\ +j\omega'')\varepsilon^*(\omega' - j\omega'')|\mathbf{E}(t)|^2 \end{aligned} \tag{2.3}$$

Assuming that $\omega'' \ll \omega'$, we can let, after using Taylor series approximation, that

$$\mu(\omega' - j\omega'') \cong \mu(\omega') - j\omega'' \frac{\partial \mu(\omega')}{\partial \omega'} \tag{2.4}$$

$$\varepsilon(\omega' - j\omega'') \cong \varepsilon(\omega') - j\omega'' \frac{\partial \varepsilon(\omega')}{\partial \omega'} \tag{2.5}$$

Using (2.4) and (2.5) in (2.3), and collecting terms of the same order gives

$$\begin{aligned}\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] = &-j\omega'\mu(\omega')|\mathbf{H}(t)|^2 + j\omega'\varepsilon^*(\omega')|\mathbf{E}(t)|^2 \\ &- \omega''\mu(\omega')|\mathbf{H}(t)|^2 - \omega'\omega'' \frac{\partial \mu}{\partial \omega'}|\mathbf{H}(t)|^2 \\ &- \omega''\varepsilon^*(\omega)|\mathbf{E}(t)|^2 - \omega'\omega'' \frac{\partial \varepsilon^*}{\partial \omega'}|\mathbf{E}(t)|^2 \end{aligned} \tag{2.6}$$

The above can be rewritten as

$$\begin{aligned}\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] = &-j\omega' \left[\mu(\omega')|\mathbf{H}(t)|^2 + \varepsilon^*(\omega')|\mathbf{E}(t)|^2\right] \\ &- \omega'' \left[\frac{\partial \omega`\mu(\omega`)}{\partial \omega`}|\mathbf{H}(t)|^2 + \frac{\partial \omega`\varepsilon^*(\omega`)}{\partial \omega`}|\mathbf{E}(t)|^2\right] \end{aligned} \tag{2.7}$$

For a lossless medium, $\varepsilon(\omega')$ is purely real, and the first term of the right-hand side is purely imaginary while the second term is purely real. In the limit when $\omega'' \to 0$, when we take half the imaginary part of the above equation, we have

$$\nabla \cdot \frac{1}{2}\Im m\left[\mathbf{E} \times \mathbf{H}^*\right] = -\omega' \left[\frac{1}{2}\mu|\mathbf{H}|^2 - \frac{1}{2}\varepsilon|\mathbf{E}|^2\right] \tag{2.8}$$

which has the physical interpretation of reactive power as has been previously discussed. When we take half the real part of (2.7), we obtain

$$\nabla \cdot \frac{1}{2}\Re e[\mathbf{E} \times \mathbf{H}^*] = -\frac{\omega''}{2}\left[\frac{\partial \omega' \mu}{\partial \omega'}|\mathbf{H}|^2 + \frac{\partial \omega' \varepsilon}{\partial \omega'}|\mathbf{E}|^2\right] \tag{2.9}$$

Since the right-hand side has time dependence of $e^{2\omega'' t}$, it can be written as

$$\nabla \cdot \frac{1}{2}\Re e[\mathbf{E} \times \mathbf{H}^*] = -\frac{\partial}{\partial t}\frac{1}{4}\left[\frac{\partial \omega' \mu}{\partial \omega'}|\mathbf{H}|^2 + \frac{\partial \omega' \varepsilon}{\partial \omega'}|\mathbf{E}|^2\right] = -\frac{\partial}{\partial t}\langle W_T \rangle \tag{2.10}$$

Therefore, the time-average stored energy density is

$$\langle W_T \rangle = \frac{1}{4}\left[\frac{\partial \omega' \mu}{\partial \omega'}|\mathbf{H}|^2 + \frac{\partial \omega' \varepsilon}{\partial \omega'}|\mathbf{E}|^2\right] \tag{2.11}$$

For a non-dispersive medium, the above reduces to

$$\langle W_T \rangle = \frac{1}{4}\left[\mu|\mathbf{H}|^2 + \varepsilon|\mathbf{E}|^2\right] \tag{2.12}$$

which is what we have derived before. In the above analysis, we have used a quasi-time-harmonic signal with $\exp(j\omega t)$ dependence. In the limit when $\omega'' \to 0$, this signal reverts back to a time-harmonic signal, and our interpretation of complex power. However, by assuming the frequency ω to have a small imaginary part ω'', it forces to the stored energy to grow very slightly, and hence, power has to be supplied to maintain the growth of this stored energy. By so doing, it allows us to identify the expression for energy density for a dispersive medium.