

# ECE 604, Lecture 14

October 18, 2018

In this lecture, we will cover the following topics:

- Multi-Junction Transmission Lines
- Uniform Plane Waves
- Plane Waves in Lossy Conductive Media

Additional Reading:

- Lecture Notes 13.
- Sections 5.8, 5.9, 5.10, 6.1, and 6.2 Ramo, Whinnery, and Van Duzer.  
(Reading of the textbook is for supplementary knowledge and not necessary for doing the homework.)

# 1 Multi-Junction Transmission Lines

By concatenating sections of transmission lines of different characteristic impedances, a large variety of devices such as resonators, filters, radiators, and matching networks can be formed. We will start with a single junction transmission line.

## 1.1 Single-Junction Transmission Lines

Consider two transmission line connected at a single junction as shown in Figure 1. For simplicity, we assume that the transmission line to the right is infinitely long so that there is no reflected wave. And that the two transmission lines have different characteristic impedances,  $Z_{01}$  and  $Z_{02}$ .

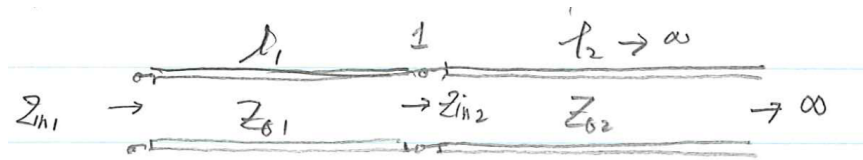


Figure 1:

The impedance of the transmission line at junction 1 looking to the right is

$$Z_{in2} = Z_{02} \quad (1.1)$$

since no reflected wave exists. Transmission line 1 sees a load of  $Z_L = Z_{in2} = Z_{02}$  hooked to its end. Hence, we deduce that the reflection coefficient at junction 1 between line 1 and line 2, using the knowledge from the previous lecture, is  $\Gamma_{12}$ , and is given by

$$\Gamma_{12} = \frac{Z_L - Z_{01}}{Z_L + Z_{01}} = \frac{Z_{in2} - Z_{01}}{Z_{in2} + Z_{01}} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \quad (1.2)$$

## 1.2 Two-Junction Transmission Lines

Now, we look at the two-junction case. To this end, we first look at when line 2 is terminated by a load  $Z_L$  at its end as shown in Figure 2

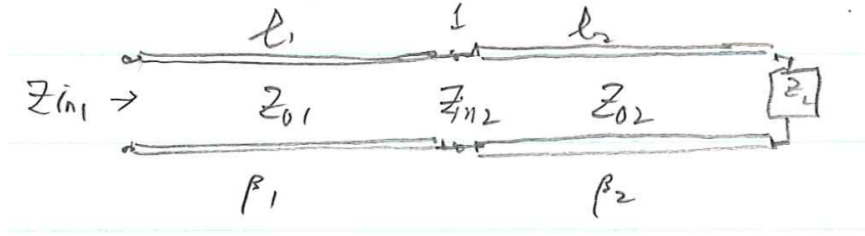


Figure 2:

Then, using the formula derived in the previous lecture,

$$Z_{in2} = Z_{02} \frac{1 + \Gamma(-l_2)}{1 - \Gamma(-l_2)} = Z_{02} \frac{1 + \Gamma_{L2} e^{-2j\beta_2 l_2}}{1 - \Gamma_{L2} e^{-2j\beta_2 l_2}} \quad (1.3)$$

where we have used the fact that  $\Gamma(-l_2) = \Gamma_{L2} e^{-2j\beta_2 l_2}$ . It is to be noted that here, using knowledge from the previous lecture, that

$$\Gamma_{L2} = \frac{Z_L - Z_{02}}{Z_L + Z_{02}} \quad (1.4)$$

Now, line 1 sees a load of  $Z_{in2}$  hooked at its end. The generalized reflection coefficient at junction 1, which includes all the reflection of waves from its right, is now

$$\tilde{\Gamma}_{12} = \frac{Z_{in2} - Z_{01}}{Z_{in2} + Z_{01}} \quad (1.5)$$

Substituting (1.3) into (1.5), we have

$$\tilde{\Gamma}_{12} = \frac{Z_{02} \left( \frac{1+\Gamma}{1-\Gamma} \right) - Z_{01}}{Z_{02} \left( \frac{1+\Gamma}{1-\Gamma} \right) + Z_{01}} \quad (1.6)$$

where  $\Gamma = \Gamma_{L2} e^{-2j\beta_2 l_2}$ . The above can be rearranged to give

$$\tilde{\Gamma}_{12} = \frac{Z_{02}(1 + \Gamma) - Z_{01}(1 - \Gamma)}{Z_{02}(1 + \Gamma) + Z_{01}(1 - \Gamma)} \quad (1.7)$$

Finally, by further rearranging terms, it can be shown that the above becomes

$$\tilde{\Gamma}_{12} = \frac{\Gamma_{12} + \Gamma}{1 + \Gamma_{12}\Gamma} = \frac{\Gamma_{12} + \Gamma_{L2} e^{-2j\beta_2 l_2}}{1 + \Gamma_{12}\Gamma_{L2} e^{-2j\beta_2 l_2}} \quad (1.8)$$

where  $\Gamma_{12}$ , the local reflection coefficient, is given by (1.2), and  $\Gamma = \Gamma_{L2} e^{-2j\beta_2 l_2}$  is the general reflection coefficient at  $z = -l_2$  due to the load  $Z_L$ . In other words,

$$\Gamma_{L2} = \frac{Z_L - Z_{02}}{Z_L + Z_{02}} \quad (1.9)$$

Equation (1.8) is a powerful formula for multi-junction transmission lines. Imagine now that we add another section of transmission line as shown in Figure 3.

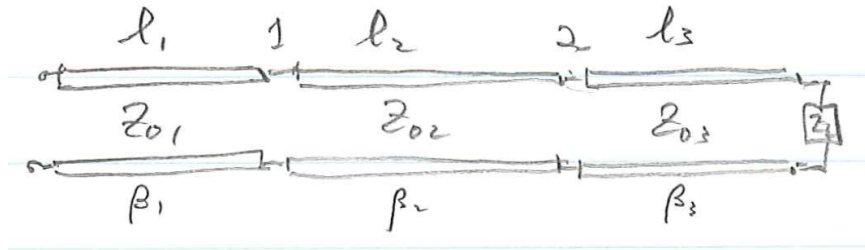


Figure 3:

We can use the aforementioned method to first find  $\tilde{\Gamma}_{23}$ , the generalized reflection coefficient at junction 2. Using formula (1.8), it is given by

$$\tilde{\Gamma}_{23} = \frac{\Gamma_{L3} + \Gamma_{L3}e^{-2j\beta_3l_3}}{1 + \Gamma_{23}\Gamma_{L3}e^{-2j\beta_3l_3}} \quad (1.10)$$

where  $\Gamma_{L3}$  is the load reflection coefficient due to the load  $Z_L$  hooked to the end of transmission line 3 as shown in Figure 3. Here, it is given as

$$\Gamma_{L3} = \frac{Z_L - Z_{03}}{Z_L + Z_{03}} \quad (1.11)$$

Given the knowledge of  $\tilde{\Gamma}_{23}$ , we can use (1.8) again to find the new  $\tilde{\Gamma}_{12}$  at junction 1. It is now

$$\tilde{\Gamma}_{12} = \frac{\Gamma_{12} + \tilde{\Gamma}_{23}e^{-2j\beta_2l_2}}{1 + \Gamma_{12}\tilde{\Gamma}_{23}e^{-2j\beta_2l_2}} \quad (1.12)$$

Therefore, we can use (1.8) recursively to find the generalized reflection coefficient for a multi-junction transmission line. Once the reflection coefficient is known, the impedance at that location can also be found. For instance, at junction 1, the impedance is now given by

$$Z_{in2} = Z_{01} \frac{1 + \tilde{\Gamma}_{12}}{1 - \tilde{\Gamma}_{12}} \quad (1.13)$$

instead of (1.3). In the above,  $Z_{01}$  is used because the generalized reflection coefficient  $\tilde{\Gamma}_{12}$  is the total reflection coefficient for an incident wave from transmission line 1 that is sent toward the junction 1. Previously,  $Z_{02}$  was used in (1.3) because the reflection coefficients in that equation was for an incident wave sent from transmission line 2.

If the incident wave were to have come from line 2, then one can write  $Z_{in2}$  as

$$Z_{in2} = Z_{02} \frac{1 + \tilde{\Gamma}_{23} e^{-2j\beta_2 l_2}}{1 - \tilde{\Gamma}_{23} e^{-2j\beta_2 l_2}} \quad (1.14)$$

With some algebraic manipulation, it can be shown that (1.13) are (1.14) identical. But (1.13) is closer to an experimental scenario where one measures the reflection coefficient by sending a wave from line 1 with no knowledge of what is to the right of junction 1.

## 2 Uniform Plane Waves

By first writing the first two Maxwell's equations in the frequency domain in a source-free medium, namely,

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad (2.1)$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} \quad (2.2)$$

and taking the curl of the first equation, and then substituting the second equation to its right-hand side, we have the vector Helmholtz equation for a source-free homogenous medium as given by

$$\nabla \times \nabla \times \mathbf{E} - \omega^2\mu\epsilon\mathbf{E} = 0 \quad (2.3)$$

Taking the divergence of the above equation, we have

$$\nabla \cdot (\nabla \times \nabla \times \mathbf{E}) - \omega^2\mu\epsilon\nabla \cdot \mathbf{E} = 0 \quad (2.4)$$

Since the first term is zero because  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ , and if  $\omega \neq 0$ , then  $\nabla \cdot \mathbf{E} = 0$ . Hence, the solution to (2.3) is consistent with  $\nabla \cdot \mathbf{E} = 0$  when  $\omega \neq 0$ . Therefore, using the fact that  $\nabla \times \nabla \times \mathbf{E} = \nabla\nabla \cdot \mathbf{E} - \nabla \cdot \nabla\mathbf{E}$ , one can see that the above equation is equivalent to solving

$$\nabla^2\mathbf{E} + \omega^2\mu\epsilon\mathbf{E} = 0 \quad (2.5)$$

if  $\nabla \cdot \mathbf{E} = 0$ .

The general solution to (2.5) is hence

$$\mathbf{E} = \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z} = \mathbf{E}_0 e^{-j\mathbf{k} \cdot \mathbf{r}} \quad (2.6)$$

where  $\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$ ,  $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$ . And upon substituting (2.6) into (2.5), it is seen that

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\epsilon \quad (2.7)$$

This is called the dispersion relation for a plane wave.

In general,  $k_x$ ,  $k_y$ , and  $k_z$  can be arbitrary as long as this relation is satisfied. To simplify the discussion, we will focus on the case where  $k_x$ ,  $k_y$ , and  $k_z$  are

all real. When this is the case, the vector function represents a uniform plane wave propagating in the  $\mathbf{k}$  direction. As can be seen, when  $\mathbf{k} \cdot \mathbf{r} = \text{constant}$ , it is represented by all points of  $\mathbf{r}$  that represents a flat plane. This flat plane represents the constant phase wave front. By increasing the constant, we obtain different planes for progressively changing phase fronts.<sup>1</sup>

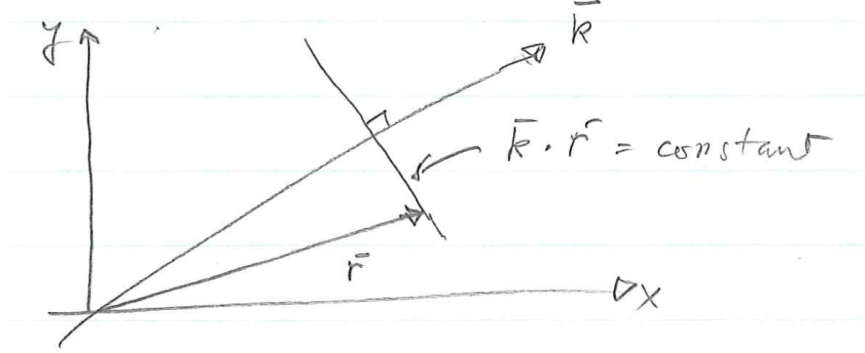


Figure 4:

Further, since  $\nabla \cdot \mathbf{E} = 0$ , we have

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \nabla \cdot \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z} = \nabla \cdot \mathbf{E}_0 e^{-j\mathbf{k} \cdot \mathbf{r}} \\ &= (-\hat{x}jk_x - \hat{y}jk_y - \hat{z}jk_z) \cdot \mathbf{E} e^{-j\mathbf{k} \cdot \mathbf{r}} \\ &= -j(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z) \cdot \mathbf{E} e^{-j\mathbf{k} \cdot \mathbf{r}} = 0\end{aligned}\quad (2.8)$$

or

$$\mathbf{k} \cdot \mathbf{E}_0 = \mathbf{k} \cdot \mathbf{E} = 0 \quad (2.9)$$

Thus,  $\mathbf{E}$  is orthogonal to  $\mathbf{k}$  for a uniform plane wave.

The above exercise shows that whenever  $\mathbf{E}$  is a plane wave, and when the  $\nabla$  operator operates on such a vector function, one can do the substitution that  $\nabla \rightarrow -j\mathbf{k}$ .

Hence, in a source-free homogenous medium,

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad (2.10)$$

the above equation becomes

$$-j\mathbf{k} \times \mathbf{E} = -j\omega\mu\mathbf{H} \quad (2.11)$$

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<sup>1</sup>In the  $\exp(j\omega t)$  time convention, this phase front is decreasing, whereas in the  $\exp(-i\omega t)$  time convention, this phase front is increasing.

or that

$$\mathbf{H} = \frac{\mathbf{k} \times \mathbf{E}}{\omega\mu} \quad (2.12)$$

Also, from

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \quad (2.13)$$

we get that

$$\mathbf{E} = -\frac{\mathbf{k} \times \mathbf{H}}{\omega\varepsilon} = -\frac{\mathbf{k} \times (\mathbf{k} \times \mathbf{E})}{\omega^2\mu\varepsilon} \quad (2.14)$$

Again, using the vector identity, the above simplifies to

$$\mathbf{E} = -\frac{\mathbf{k}(\mathbf{k} \cdot \mathbf{E}) - (\mathbf{k} \cdot \mathbf{k})\mathbf{E}}{\omega^2\mu\varepsilon} = \frac{\mathbf{k} \cdot \mathbf{k}}{\omega^2\mu\varepsilon}\mathbf{E} \quad (2.15)$$

where  $\mathbf{k} \cdot \mathbf{E} = 0$  has been used. For the above equation to be consistent, it is necessary that

$$\mathbf{k} \cdot \mathbf{k} = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon \quad (2.16)$$

or that  $\mathbf{k}$  has to satisfy the dispersion relation previously derived in (2.7).

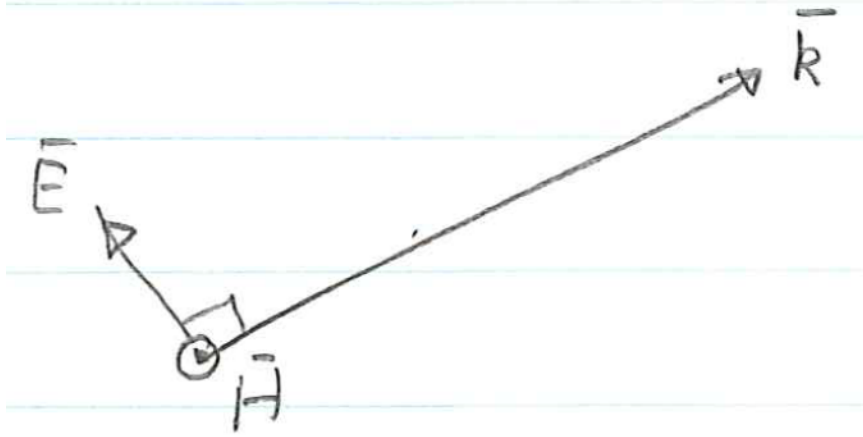


Figure 5:

Figure 5 shows that  $\mathbf{k} \cdot \mathbf{E} = 0$ , and that  $\mathbf{k} \times \mathbf{E}$  points in the direction of  $\mathbf{H}$  as shown in (2.12). Figure 5 also shows, as  $\mathbf{k}$ ,  $\mathbf{E}$ , and  $\mathbf{H}$  are orthogonal to each other. Hence, taking the magnitude of (2.12), then

$$|\mathbf{H}| = \frac{|\mathbf{k}||\mathbf{E}|}{\omega\mu} = \sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}| = \frac{1}{\eta}|\mathbf{E}| \quad (2.17)$$

where the quantity

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (2.18)$$

is call the intrinsic impedance. For vacuum or free-space, it is about  $377\Omega$ . It is also noted that  $\mathbf{E} \times \mathbf{H}^*$  points in the direction of the vector  $\mathbf{k}$ . This is also required by the Poynting's theorem.

In the above, when  $k_x$ ,  $k_y$ , and  $k_z$  are not all real, the wave is known as an inhomogeneous wave.<sup>2</sup>

### 3 Plane Waves in Lossy Conductive Media

The above can be generalized to a lossy conductive medium by invoking mathematical homomorphism. When conductive loss is present,  $\sigma \neq 0$ , and  $\mathbf{J} = \sigma\mathbf{E}$ . Then generalized Ampere's law becomes

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \sigma\mathbf{E} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\mathbf{E} \quad (3.1)$$

A complex permittivity can be defined as  $\underline{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega}$ . Eq. (3.1) can be rewritten as

$$\nabla \times \mathbf{H} = j\omega\underline{\varepsilon}\mathbf{E} \quad (3.2)$$

This equation is of the same form as (2.2). Using the same method as before, a plane-wave solution  $\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$  will have the dispersion relation which is now given by

$$k_y^2 + k_y^2 + k_z^2 = \omega^2 \mu \underline{\varepsilon} \quad (3.3)$$

Since  $\underline{\varepsilon}$  is complex now,  $k_x$ ,  $k_y$ , and  $k_z$  need not be all real. Equation (2.17) is derived by assuming that  $\mathbf{k}$  is a real vector. When  $\mathbf{k}$  is a complex vector, the derivation that leads to (2.17) may not be correct. It is also difficult to visualize a  $\mathbf{k}$  vector that the wave is propagating in. So again, we can look at the simplified case where

$$\mathbf{E} = \hat{x}E_x(z) \quad (3.4)$$

so that  $\nabla \cdot \mathbf{E} = \partial_x E_x(z) = 0$ , And let  $\mathbf{k} = \hat{z}k = \hat{z}\omega\sqrt{\mu\underline{\varepsilon}}$ . In this manner, we are requiring that the wave decays only in the  $z$  direction. For such a simple plane wave,

$$\mathbf{E} = \hat{x}\mathbf{E}_x(z) = \hat{x}E_0 e^{-jkz} \quad (3.5)$$

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<sup>2</sup>The term inhomogeneous plane wave is used sometimes, but it is a misnomer since there is no more a planar wave front in this case.



where  $k = \omega\sqrt{\mu\tilde{\varepsilon}}$ , since  $\mathbf{k} \cdot \mathbf{k} = k^2 = \omega^2\mu\tilde{\varepsilon}$  is still true.

Then (2.12) gives rise to

$$\mathbf{H} = \hat{y} \frac{kE_x(z)}{\omega\mu} = \hat{y} \sqrt{\frac{\varepsilon}{\mu}} E_x \quad (3.6)$$

or by letting  $k = \omega\sqrt{\mu\tilde{\varepsilon}}$ , then

$$E_x/H_y = \sqrt{\frac{\mu}{\tilde{\varepsilon}}} \quad (3.7)$$

When the medium is highly conductive,  $\sigma \rightarrow \infty$ ,

$$k = \omega\sqrt{\mu\tilde{\varepsilon}} \simeq \omega\sqrt{-\mu\frac{j\sigma}{\omega}} = \sqrt{-j\omega\mu\sigma} \quad (3.8)$$

Taking  $\sqrt{-j} = \frac{1}{\sqrt{2}}(1-j)$ , we have

$$k = (1-j)\sqrt{\frac{\omega\mu\sigma}{2}} = k' - jk'' \quad (3.9)$$

For a plane wave,  $e^{-jkz}$ , it becomes

$$e^{-jkz} = e^{-jk'z - k''z} \quad (3.10)$$

This plane wave decays exponentially in the  $z$  direction. The penetration depth of this wave is then

$$\delta = \frac{1}{k''} = \sqrt{\frac{2}{\omega\mu\sigma}} \quad (3.11)$$

this distance  $\delta$ , the penetration depth, is called the skin depth of a plane wave propagating in a highly lossy conductive medium. This happens for radio wave propagating in the saline solution of the ocean, the earth, or wave propagating in highly conductive metal.

When the conductivity is low, then  $\frac{\sigma}{\omega\varepsilon} \ll 1$ , we have

$$\begin{aligned} k &= \omega\sqrt{\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)} = \omega\sqrt{\mu\varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right)} \\ &\approx \omega\sqrt{\mu\varepsilon}\left(1 - j\frac{1}{2}\frac{\sigma}{\omega\varepsilon}\right) = k' - jk'' \end{aligned} \quad (3.12)$$

The term  $\frac{\sigma}{\omega\varepsilon}$  is called the loss tangent of a lossy medium.

In general, in a lossy medium  $\varepsilon = \varepsilon' - j\varepsilon''$ ,  $\varepsilon''/\varepsilon'$  is called the loss tangent of the medium. It is to be noted that in the optics and physics community,  $e^{-i\omega t}$  time convention is preferred. In that case, we need to do the switch  $j \rightarrow -i$ , and a loss medium is denoted by  $\varepsilon = \varepsilon' + i\varepsilon''$ .