ECE 604, Lecture 13

October 16, 2018

1 Introduction

In this lecture, we will cover the following topics:

- Terminated Transmission Line
- Smith Chart
- Voltage Standing Wave Ratio (VSWR)

Additional Reading:

- ECE350X lecture notes 5, 6, 7, and 8. https://engineering.purdue.edu/wcchew/ece350.html
- Sections 5.1, 5.2, 5.3, 5.4, 5.6, 5.7, 5.8, and 5.9 Ramo, Whinnery, and Van Duzer. (Reading of the textbook is for supplementary knowledge and not necessary for doing the homework.)

Printed on October 20, 2018 at 11 : 06: W.C. Chew and D. Jiao.

2 Terminated Transmission Lines

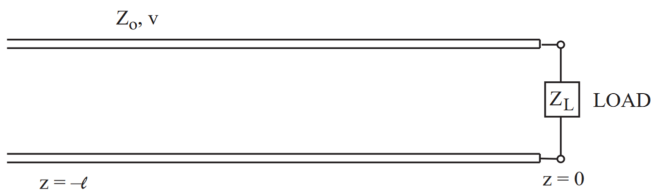

Figure 1:

For an infinitely long transmission line, the solution consists of the linear superposition of a wave traveling to the right plus a wave traveling to the left. If transmission line is terminated by a load as shown in Figure 1, a right-traveling wave will be reflected by the load, and in general, the wave on the transmission line will be a linear superposition of the left and right traveling waves, viz.,

$$V(z) = a_+ e^{\gamma z} + a_- e^{\gamma z} = V_+(z) + V_-(z) \tag{2.1}$$

At $z = 0$, we can define the amplitude of the left-going reflected wave a_- to be related to the amplitude of the right-going or incident wave a_+. In other words, at $z = 0$

$$a_- = \Gamma_L a_+ \tag{2.2}$$

where Γ_L is the reflection coefficient. Hence, (2.1) becomes

$$V(z) = a_+ e^{-\gamma z} + \Gamma_L a_+ e^{\gamma z} = a_+ \left(e^{-\gamma z} + \Gamma_L e^{\gamma z} \right) \tag{2.3}$$

The corresponding current $I(z)$ on the transmission line is given by using the telegrapher's equations from the previous lecture, namely that

$$I(z) = -\frac{1}{Z}\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{a_+}{Z}\gamma(e^{-\gamma z} - \Gamma_L e^{\gamma z}) \tag{2.4}$$

where $\gamma = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)}$, and $Z = j\omega L + R$ and $Y = j\omega C + G$. Hence, $Z/\gamma = \sqrt{Z/Y} = Z_0$, the characteristic impedance of the transmission line. Thus, from (2.4),

$$I(z) = \frac{a_+}{Z_0}\left(e^{-\gamma z} - \Gamma_L e^{\gamma z} \right) \tag{2.5}$$

Notice the sign change in the second term of the above expression.

A general reflection coefficient at z can be defined such that

$$\Gamma(z) = \frac{V_-(z)}{V_+(z)} = \frac{a_- e^{\gamma z}}{a_+ e^{-\gamma z}} = \Gamma_L e^{2\gamma z} \tag{2.6}$$

Of course, $\Gamma(z = 0) = \Gamma_L$. Furthermore, we must have

$$\frac{V(z=0)}{I(z=0)} = Z_L \tag{2.7}$$

or that using (2.3) and (2.4) with $z = 0$, the left-hand side of the above can be rewritten, and we have

$$\frac{1+\Gamma_L}{1-\Gamma_L} Z_0 = Z_L \tag{2.8}$$

From the above, we can solve for Γ_L in terms of Z_L/Z_0 to get

$$\Gamma_L = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.9}$$

Thus, given the termination load Z_L, the reflection coefficient Γ_L can be found, or vice versa. It is seen that $\Gamma_L = 0$ if $Z_L = Z_0$. Thus a right-traveling wave will not be reflected and the left-traveling is absent. This is the case of a matched load. All energy of the right-traveling wave will be absorbed by the load.

In general, we can define a generalized impedance at z to be

$$\begin{aligned} Z(z) = \frac{V(z)}{I(z)} &= \frac{a_+(e^{-\gamma z} + \Gamma_L e^{\gamma z})}{\frac{1}{Z_0} a_+(e^{-\gamma z} - \Gamma_L e^{\gamma z})} \\ &= Z_0 \frac{1+\Gamma_L e^{2\gamma z}}{1-\Gamma_L e^{2\gamma z}} = Z_0 \frac{1+\Gamma(z)}{1-\Gamma(z)} \end{aligned} \tag{2.10}$$

where $\Gamma(z)$ is as defined in (2.6). Conversely, one can write

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} \tag{2.11}$$

Usually, a transmission line is lossless, and $\gamma = j\beta$. In this case, (2.10) becomes

$$Z(z) = Z_0 \frac{1+\Gamma_L e^{2j\beta z}}{1-\Gamma_L e^{2j\beta z}} \tag{2.12}$$

From the above, one can show that by setting $z = -l$, using (2.9), and after some algebra,

$$Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \tag{2.13}$$

2.1 Shorted Terminations

From (2.13) above, when we have a short such that $Z_L = 0$, then

$$Z(-l) = jZ_0 \tan(\beta l) = jX \tag{2.14}$$

When $\beta \ll l$, then $\tan \beta l \approx \beta l$, and (2.14) becomes

$$Z(-l) \cong jZ_0 \beta l \tag{2.15}$$

After using that $Z_0 = \sqrt{L/C}$ and that $\beta = \omega\sqrt{LC}$, (2.15) becomes

$$Z(-l) \cong j\omega Ll \tag{2.16}$$

The above implies that a short length of transmission line connected to a short as a load looks like an inductor with $L_{\text{eff}} = Ll$, since much current will pass through this short producing a strong magnetic field with stored magnetic energy.

2.2 Open terminations

When we have an open circuit such that $Z_L = \infty$, then from (2.13) above

$$Z(-l) = -jZ_0 \cot(\beta l) = jX \tag{2.17}$$

Then, when $\beta l \ll l$, $\cot(\beta l) \approx 1/\beta l$

$$Z(-l) \approx -j\frac{Z_0}{\beta l} \tag{2.18}$$

And then, again using $\beta = \omega\sqrt{LC}$, $Z_0 = \sqrt{L/C}$

$$Z(-l) \approx \frac{1}{j\omega Cl} \tag{2.19}$$

Hence, an open-circuited terminated short length of transmission line appears like an effective capacitor with $C_{\text{eff}} = Cl$.

But the changing length of l, one can make a shorted or an open terminated line look like an inductor or a capacitor depending on its length l. This effect is shown in Figures 2 and 3. Moreover, the reactance X becomes infinite or zero with the proper choice of the length l. These are resonances or anti-resonances of the transmission line, very much like an LC tank circuit. An LC circuit can look like an open or a short circuit at resonances and depending on if they are connected in parallel or in series.

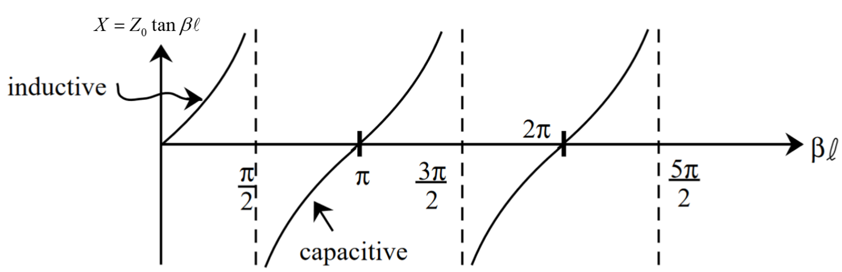

Figure 2:

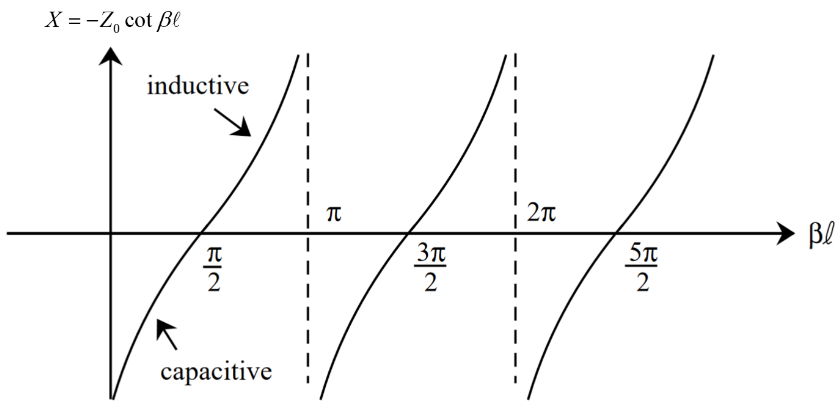

Figure 3:

3 Smith Chart

In general, from (2.12) and (2.13), a length of transmission line can transform a load Z_L to a range of possible values $Z(-l)$. To understand this range of values better, we can use the Smith chart (invented by P.H. Smith 1939 before the advent of the computer). The Smith chart is essentially a graphical calculator for solving transmission line problems. Equation (2.11) indicates that there is a unique map between the impedance $Z(z)$ and reflection coefficient $\Gamma(z)$. In the normalized impedance form, from (2.10) and (2.11)

$$\Gamma = \frac{Z_n - 1}{Z_n + 1}, \qquad Z_n = \frac{1 + \Gamma}{1 - \Gamma} \tag{3.1}$$

Equations in (3.1) are related to a bilinear transform in complex variables. It is a kind of conformal map that maps circles to circles. Such a map is shown in Figure 4, where lines on the right-half of the complex Z_n plane are mapped to the circles on the complex Γ plane. Since straight lines on the complex Z_n plane are circles with infinite radii, they are mapped to circles on the complex Γ plane.

Notice that the imaginary axis on the complex Z_n plane maps to the circle of unit radius on the complex Γ plane. All points on the right-half plane are mapped to within the unit circle. The reason being that the right-half plane of the complex Z_n plane corresponds to passive impedances that will absorb energy. Hence, such an impedance load will have reflection coefficient with amplitude less than one, which are points within the unit circle.

On the other hand, the left-half of the complex Z_n plane corresponds to impedances with negative resistances. These will be active elements that can generate energy, and hence, yielding $|\Gamma| > 1$, and will be outside the unit circle on the complex Γ plane.

Another point to note is that points at infinity on the complex Z_n plane map to the point at $\Gamma = 1$ on the complex Γ plane, while the point zero on the complex Z_n plane maps to $\Gamma = -1$ on the complex Γ plane. These are the reflection coefficients of an open-circuit load and a short-circuit load, respectively. For a matched load, $Z_n = 1$, and it maps to the zero point on the complex Γ plane implying no reflection.

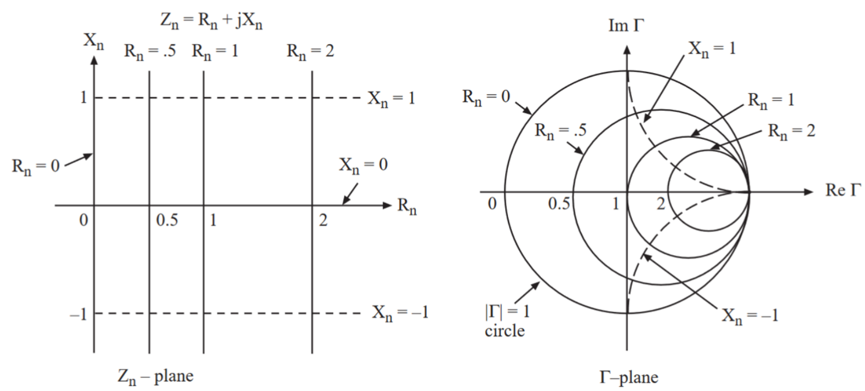

Figure 4:

The Smith chart also allows one to evaluate the expression

$$\Gamma(-l) = \Gamma_L e^{-2j\beta l} \tag{3.2}$$

quickly. Since $\beta = 2\pi/\lambda$, it is more convenient to write $\beta l = 2\pi l/\lambda$, and measure the length of the transmission line in terms of wavelength. To this end, the above becomes

$$\Gamma(-l) = \Gamma_L e^{-4j\pi l/\lambda} \tag{3.3}$$

For increasing l, one moves away from the load to the generator, l increases, and the phase is decreasing because of the negative sign. So given a point for Γ_L on the Smith chart, one has negative phase or decreasing phase by rotating the point clockwise. Also, due to the $\exp(-4j\pi l/\lambda)$ dependence of the phase, when $l = \lambda/4$, the reflection coefficient rotates a half circle around the chart. And when $l = \lambda/2$, the reflection coefficient will rotate a full circle, or back to the original point.

Also, for two points diametrically opposite to each other on the Smith chart, Γ changes sign, and it can be shown easily that the normalized impedances are reciprocal of each other. Hence, the Smith chart can also be used to find the reciprocal of a complex number quickly. A full blown Smith chart is shown in Figure 5.

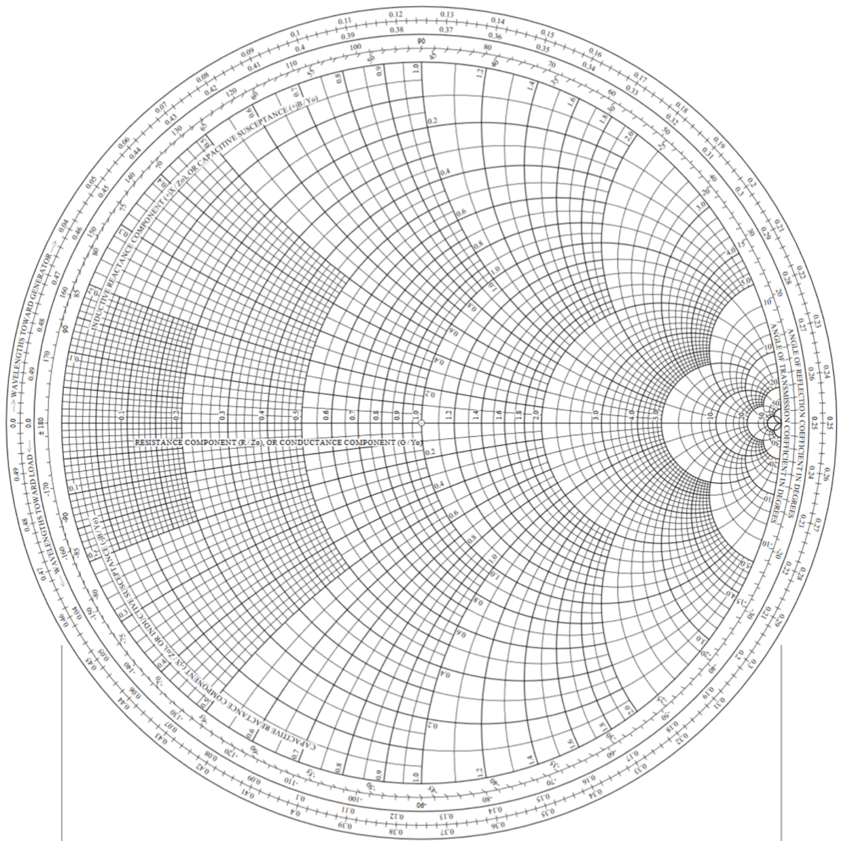

Figure 5:

4 VSWR (Voltage Standing Wave Ratio)

The general reflection coefficient for a lossless transmission line at z is given as

$$\Gamma(z) = \frac{V_{-}(z)}{V_{+}(z)} = \Gamma_L e^{2j\beta z} \tag{4.1}$$

The standing wave on a terminated transmission line is given as

$$\begin{aligned} V(z) &= V_0 e^{-j\beta z} + V_0 e^{j\beta z}\Gamma_L \\ &= V_0 e^{-j\beta z}\left(1 + \Gamma_L e^{2j\beta z}\right) \\ &= V_0 e^{-j\beta z}\left(1 + \Gamma(z)\right) \end{aligned} \tag{4.2}$$

Hence, $V(z)$ is not a constant or independent of z, but

$$|V(z)| = |V_0||1 + \Gamma(z)| \tag{4.3}$$

In Figure 6, the relationship variation of $1 + \Gamma(z)$ as z varies is shown.

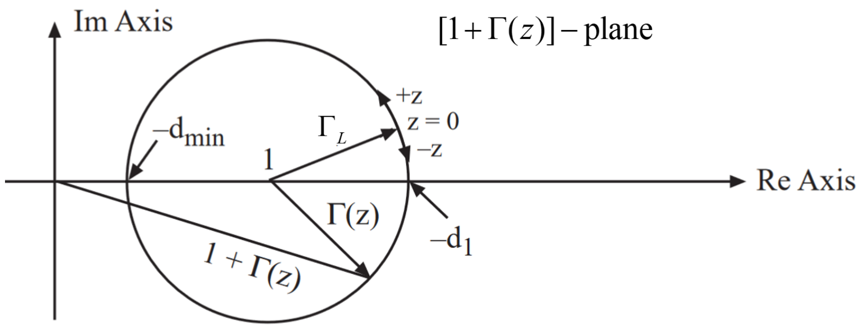

Figure 6:

Using the triangular inequality, one gets

$$|V_0|(1 - |\Gamma(z)|) \le |V(z)| \le |V_0|(1 + |\Gamma(z)|) \tag{4.4}$$

But from (4.1), $|\Gamma(z)| = |\Gamma_L|$; hence

$$V_{\min} = |V_0|(1 - |\Gamma_L|) \le |V(z)| \le |V_0|(1 + |\Gamma_L|) = V_{\max} \tag{4.5}$$

The voltage standing wave ratio, VSWR is defined to be

$$\text{VSWR} = \frac{V_{\max}}{V_{\min}} \tag{4.6}$$

Therefore, from (4.5)

$$\text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \tag{4.7}$$

Conversely,

$$|\Gamma_L| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} \tag{4.8}$$

Hence, the knowledge of voltage standing wave pattern, as shown in Figure 7, yields the knowledge of $|\Gamma_L|$. Notice that the relations between VSWR and $|\Gamma_L|$ are homomorphic to those between Z_n and Γ. Therefore, the Smith chart can also be used to evaluate the above equations.

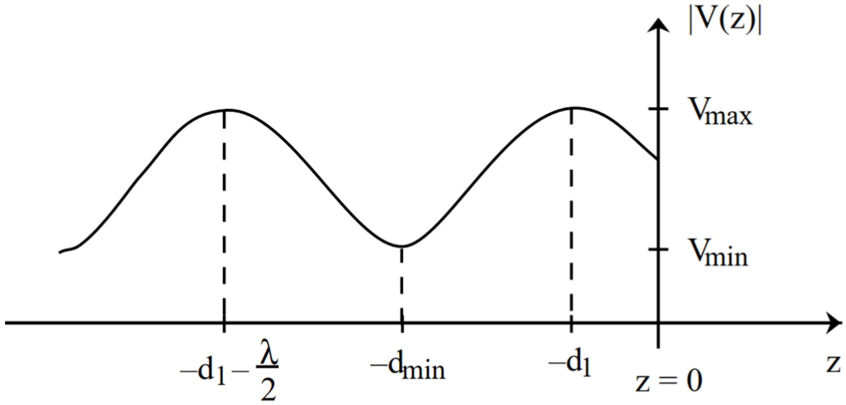

Figure 7:

The phase of Γ_L can also be determined from the measurement of the voltage standing wave pattern. The location of Γ_L in Figure 6 is determined by the phase of Γ_L. Hence, the value of d_1 in Figure 6 is determined by the phase of Γ_L as well. The length of the transmission line waveguide needed to null the original phase of Γ_L to bring the voltage standing wave pattern to a maximum value at $z = -d_1$ is shown in Figure 7. Hence, d_1 is the value where the following equation is satisfied:

$$|\Gamma_L| e^{j\phi_L} e^{-4\pi j(d_1/\lambda)} = |\Gamma_L| \tag{4.9}$$

Thus, by measuring the voltage standing wave pattern, one deduces both the amplitude and phase of Γ_L. From Γ_L, one can determine Z_L, the load impedance. Hence, measuring the impedance of a device at microwave frequency is a tricky business. At low frequency, one can use an ohm meter with two wire probes to do such a measurement. But at microwave frequency, two pieces of wire become inductors, and two pieces of metal become capacitors. More sophisticated ways to measure the impedance need to be designed.

In the old days, the voltage standing wave pattern was measured by a slotted-line equipment which consists of a coaxial waveguide with a slot opening as shown in Figure 8. A field probe can be put into the slotted line to determine the strength of the electric field inside the coax waveguide.

Figure 8: Courtesy of Microwave101.com.

A typical experimental setup for a slotted line measurement is shown in Figure 9. A generator source, with low frequency modulation, feeds microwave energy into the coaxial waveguide. The isolator, allowing only the unidirectional propagation of microwave energy, protects the generator. The attenuator protects the slotted line equipment. The wavemeter is an adjustable resonant cavity. When the wavemeter is tuned to the frequency of the microwave, it siphons off some energy from the source, giving rise to a dip in the signal of the SWR meter. Hence, the wavemeter measures the frequency of the microwave.

The slotted line probe is usually connected to a square law detector that converts the microwave signal to a low-frequency signal. In this manner, the amplitude of the voltage in the slotted line can be measured with some low-frequency equipment, such as the SWR meter. Low-frequency equipment is a lot cheaper to make and maintain. That is also the reason why the source is modulated with a low-frequency signal.

The above describes how the impedance of the device-under-test (DUT) can be measured at microwave frequencies. Nowadays, automated network analyzers make these measurements a lot simpler in a microwave laboratory. Notice that the above is based on the interference of the two traveling wave on a terminated transmission line. Such interference experiments are increasingly difficult in optical frequencies because of the much shorter wavelengths. Hence, many experiments are easier to perform at microwave frequencies rather than at optical frequencies.

Many technologies are first developed at microwave frequency, and later developed at optical frequency. Examples are phase imaging, optical coherence tomography, and beam steering with phase array sources. Another example is that quantum information and quantum computing can be done at optical frequency, but the recent trend is to use artificial atoms working at microwave frequencies. Engineering with longer wavelength and larger component is easier; and hence, microwave engineering.

Another new frontier in the electromagnetic spectrum is in the terahertz range. Due to the dearth of sources, and the added difficulty in having to engineer smaller components, this is an exciting and a largely untapped frontier in electromagnetic technology.

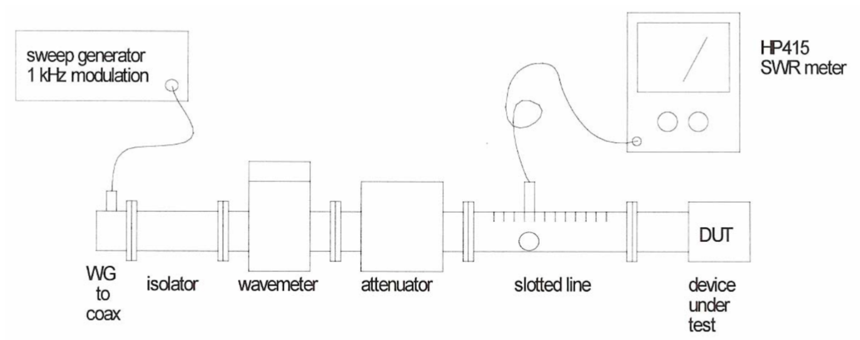

Figure 9: Courtesy of Pozar and Knapp, U. Mass.