

ECE 604, Lecture 12

October 4, 2018

1 Introduction

In this lecture, we will cover the following topics:

- Transmission Line Theory
- Lossy Transmission Lines

Additional Reading:

- ECE350X lecture notes 5. <https://engineering.purdue.edu/wcchew/ece350.html>
- Sections 5.1, 5.2, 5.3, Ramo, Whinnery, and Van Duzer.

2 Transmission Line Theory

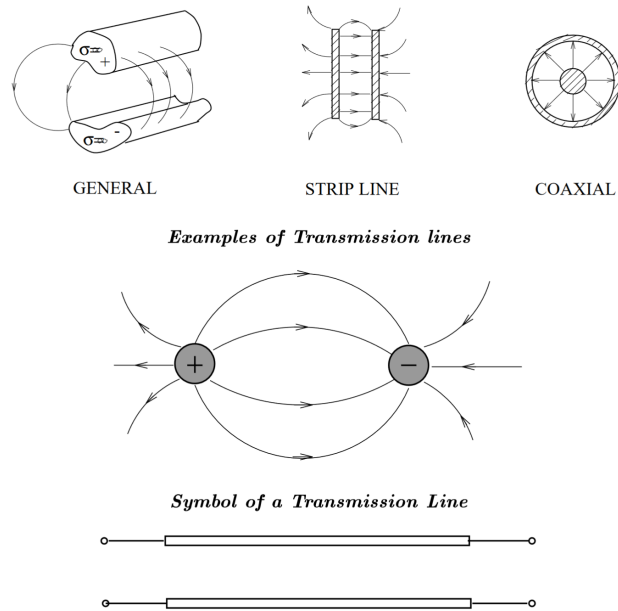


Figure 1:

Transmission lines were the first electromagnetic waveguides ever invented. They were driven by the need in telegraphy technology. It is best to introduce transmission line theory from the viewpoint of circuit theory.

Circuit theory is robust and is not sensitive to the detail shapes of the components involved such as capacitors or inductors. Moreover, many transmission line problems cannot be analyzed with the full form of Maxwell's equations,¹ but approximate solutions can be obtained using circuit theory in the long-wavelength limit.

Examples of transmission lines are shown in Figure 1. The symbol for a transmission line is usually represented by two pieces of parallel wires, but in practice, these wires need not be parallel.

¹Usually called full-wave analysis.



Figure 2: Courtesy of slides by A. Wadhwa, A.L. Dal, N. Malhotra.

Circuit theory also explains why waveguides can be made sloppily when wavelength is long or the frequency low. For instance, in the long-wavelength limit, we can make twisted-pair waveguides with abandon, and they still work well. Hence, we shall first explain the propagation of electromagnetic signal on a transmission line using circuit analysis.

Remember that two pieces of metal can accumulate attractive charges between them, giving rise to capacitive coupling, electric field, and hence stored energy in the electric field. Moreover, a piece of wire carrying a current generates a magnetic field, and hence, yielding stored energy in the magnetic field. These stored energies are the sources of the capacitive and inductive effects. But these capacitive and inductive effects are distributed over the spatial dimension of the transmission line. Therefore, it is helpful to think of the two pieces of metal as consisting of small segments of metal connected together. Each of this segment will have a small inductance, as well as a small capacitive coupling between them. Hence, we can model two pieces of metal with a distributed lumped element model as shown in Figure 3. For simplicity, we assume the other conductor to be a ground plane, so that it need not be approximated with lumped elements.

In the transmission line, the voltage $V(z, t)$ and $I(z, t)$ are functions of both space z and time t , but we will model the space variation of the voltage and current with discrete step approximation. The voltage varies from node to node while the current varies from branch to branch of the lump-element model.

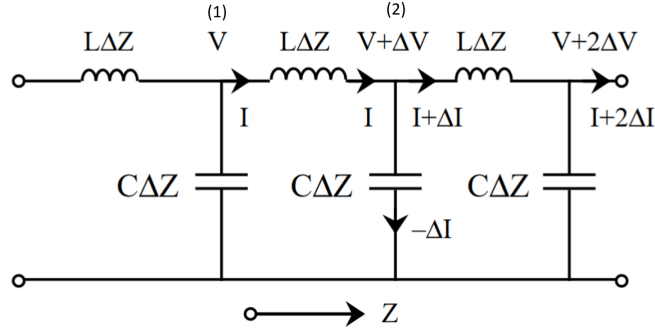


Figure 3:

First, we recall that the I-V relation of an inductor is

$$V_0 = L_0 \frac{dI_0}{dt} \quad (2.1)$$

where L_0 is the inductor, V_0 is the time-varying voltage drop across the inductor, and I_0 is the current through the inductor. Then using this relation between node 1 and node 2, we have

$$V - (V + \Delta V) = L\Delta z \frac{\partial I}{\partial t} \quad (2.2)$$

The left-hand side is the voltage drop across the inductor, while the right-hand side follows from the aforementioned V-I relation of an inductor, but we have replaced $L_0 = L\Delta z$. Here, L is the inductance per unit length (line inductance) of the transmission line. Here, $L\Delta z$ is the incremental inductance due to the small segment of metal of length Δz . Then the above can be simplified to

$$\Delta V = -L\Delta z \frac{\partial I}{\partial t} \quad (2.3)$$

Next, we make use of the V-I relation for a capacitor, which is

$$I_0 = C_0 \frac{dV_0}{dt} \quad (2.4)$$

where C_0 is the capacitor, I_0 is the current through the capacitor, and V_0 is a time-varying voltage drop across the capacitor. Thus, applying this relation at node 2, one gets

$$-\Delta I = C\Delta z \frac{\partial}{\partial t}(V + \Delta V) \approx C\Delta z \frac{\partial V}{\partial t} \quad (2.5)$$

where C is the capacitance per unit length, and $C\Delta z$ is the incremental capacitance between the small piece of metal and the ground plane. In the above,

we have used Kirchhoff current law to surmise that the current through the capacitor is $-\Delta I$, where $\Delta I = I(z + \Delta z, t) - I(z, t)$. In the last approximation in (2.5), we have dropped a term involving the product of Δz and ΔV , since it will be very small or second order in magnitude.

In the limit when $\Delta z \rightarrow 0$, one gets from (2.3) and (2.5) that

$$\frac{\partial V(z, t)}{\partial z} = -L \frac{\partial I(z, t)}{\partial t} \quad (2.6)$$

$$\frac{\partial I(z, t)}{\partial z} = -C \frac{\partial V(z, t)}{\partial t} \quad (2.7)$$

The above are the telegrapher's equations. They are two coupled first-order equations, and can be converted into second-order equations easily. Therefore,

$$\frac{\partial^2 V}{\partial z^2} - LC \frac{\partial^2 V}{\partial t^2} = 0 \quad (2.8)$$

$$\frac{\partial^2 I}{\partial z^2} - LC \frac{\partial^2 I}{\partial t^2} = 0 \quad (2.9)$$

The above are wave equations that we have previously studied, where the velocity of the wave is given by

$$v = \frac{1}{\sqrt{LC}} \quad (2.10)$$

Furthermore, if we assume that

$$V(z, t) = f_+(z - vt) \quad (2.11)$$

a right-traveling wave, and substituting it into (2.6) yields

$$-L \frac{\partial I}{\partial t} = f'_+(z - vt) \quad (2.12)$$

or that

$$I = \frac{1}{Lv} f_+(z - vt) = \sqrt{\frac{C}{L}} f_+(z - vt) \quad (2.13)$$

Consequently,

$$\frac{V(z, t)}{I(z, t)} = \sqrt{\frac{L}{C}} = Z_0 \quad (2.14)$$

where Z_0 is the characteristic impedance of the transmission line. The above ratio is only true for one-way traveling wave, in this case, one that propagates in the $+z$ direction.

For a wave that travels in the negative z direction, i.e.,

$$V(z, t) = f_-(z + vt) \quad (2.15)$$

with the corresponding $I(z, t)$ derived, one can show that

$$\frac{V(z, t)}{I(z, t)} = -\sqrt{\frac{L}{C}} = -Z_0 \quad (2.16)$$

2.1 The Time-Harmonic Case

For a time-harmonic signal on a transmission line, one can analyze the problem in the frequency domain using phasor technique. The telegrapher's equations (2.6) and (2.7) then become

$$\frac{d}{dz}V(z, \omega) = -j\omega LI(z, \omega) \quad (2.17)$$

$$\frac{d}{dz}I(z, \omega) = -j\omega CV(z, \omega) \quad (2.18)$$

The corresponding Helmholtz equations are then

$$\frac{d^2V}{dz^2} + \omega^2LCV = 0 \quad (2.19)$$

$$\frac{d^2I}{dz^2} + \omega^2LCI = 0 \quad (2.20)$$

The general solutions to the above are

$$V(z) = V_+e^{-j\beta z} + V_-e^{j\beta z} \quad (2.21)$$

$$I(z) = I_+e^{-j\beta z} + I_-e^{j\beta z} \quad (2.22)$$

where $\beta = \omega\sqrt{LC}$. This is similar to what we have seen previously for plane waves in the one-dimensional wave equation in free space, where

$$E_x(z) = E_{0+}e^{-jk_0z} + E_{0-}e^{jk_0z} \quad (2.23)$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$. We see a much similarity between (2.21), (2.22), and (2.23).

To see the solution in the time domain, we let $V_{\pm} = |V_{\pm}|e^{j\phi_{\pm}}$, and the voltage signal above can be converted back to the time domain as

$$V(z, t) = \Re\{V(z, \omega)e^{j\omega t}\} \quad (2.24)$$

$$= |V_+| \cos(\omega t - \beta z + \phi_+) + |V_-| \cos(\omega t + \beta z + \phi_-) \quad (2.25)$$

As can be seen, the first term corresponds to a right-traveling wave, while the second term is a left-traveling wave.

Furthermore, if we assume only a one-way traveling wave to the right by letting $V_- = I_- = 0$, then it can be shown that, for a right-traveling wave

$$\frac{V(z)}{I(z)} = \frac{V_+}{I_+} = \sqrt{\frac{L}{C}} = Z_0 \quad (2.26)$$

For a left-traveling wave only, by letting $V_+ = I_+ = 0$, then

$$\frac{V(z)}{I(z)} = \frac{V_-}{I_-} = -\sqrt{\frac{L}{C}} = -Z_0 \quad (2.27)$$

3 Lossy Transmission Line

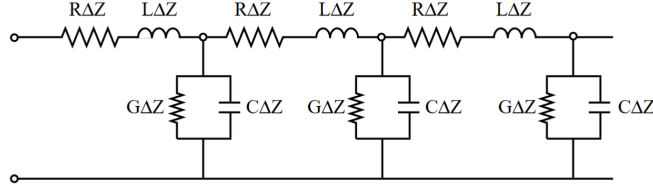


Figure 4:

The above analysis, which is valid for lossless transmission line, can be easily generalized to the lossy case. It is best to use frequency domain and phasor technique, since impedances and complex numbers will be involved.

To include loss, we use the lumped-element model as shown in Figure 4. One thing to note is that $j\omega L$ is actually the series line impedance of the transmission line, while $j\omega C$ is the shunt line admittance of the line.

First, we can rewrite the expressions for the telegrapher's equations in (2.17) and (2.18) in terms of series line impedance and shunt line admittance to arrive at

$$\frac{d}{dz}V = -ZI \quad (3.1)$$

$$\frac{d}{dz}I = -YV \quad (3.2)$$

where $Z = j\omega L$ and $Y = j\omega C$.

The geometry in Figure 4 is homomorphic² to the lossless case in Figure 3. Hence, when lossy elements are added in the geometry, we can surmise that the corresponding telegrapher's equations are similar to those above. But to include loss, we generalize the series line impedance to

$$Z = j\omega L + R \quad (3.3)$$

and the shunt line admittance to

$$Y = j\omega C + G \quad (3.4)$$

where R is the series line resistance, and G is the shunt line conductance. Then, the corresponding Helmholtz equations are

$$\frac{d^2V}{dz^2} - ZYV = 0 \quad (3.5)$$

$$\frac{d^2I}{dz^2} - ZYI = 0 \quad (3.6)$$

²A math term for "similar in structure". The term is even used in computer science describing a emerging field of homomorphic computing.

or

$$\frac{d^2V}{dz^2} - \gamma^2V = 0 \quad (3.7)$$

$$\frac{d^2I}{dz^2} - \gamma^2I = 0 \quad (3.8)$$

where $\gamma^2 = ZY$. The above are second order one-dimensional Helmholtz equations where the general solutions are

$$V(z) = V_+e^{-\gamma z} + V_-e^{\gamma z} \quad (3.9)$$

$$I(z) = I_+e^{-\gamma z} + I_-e^{\gamma z} \quad (3.10)$$

where

$$\gamma = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)} = \alpha + j\beta \quad (3.11)$$

Or focusing on the voltage case,

$$V(z) = V_+e^{-\alpha z - j\beta z} + V_-e^{-\alpha z + j\beta z} \quad (3.12)$$

Again, letting $V_{\pm} = |V_{\pm}|e^{j\phi_{\pm}}$, the above can be converted back to the time domain as

$$V(z, t) = \Re\{V(z, \omega)e^{j\omega t}\} \quad (3.13)$$

$$= |V_+|e^{-\alpha z} \cos(\omega t - \beta z + \phi_+) + |V_-|e^{\alpha z} \cos(\omega t + \beta z + \phi_-) \quad (3.14)$$

The first term corresponds to a decaying wave moving to the right while the second term is also a decaying wave moving to the left. When there is no loss, or $R = G = 0$, and from (3.11), we retrieve the lossless case where $\alpha = 0$ and $\gamma = j\beta = j\omega\sqrt{LC}$. Notice that for the lossy case, the characteristic impedance, which is the ratio of the voltage to the current for a one-way wave, can be derived to be

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega L + R}{j\omega C + G}} \quad (3.15)$$

In the absence of loss, the above becomes

$$Z_0 = \sqrt{\frac{L}{C}} \quad (3.16)$$

the characteristic impedance for the lossless case previously derived. In general, even for the lossy case,

$$\frac{V_+}{I_+} = -\frac{V_-}{I_-} = Z_0 = \sqrt{\frac{Z}{Y}} \quad (3.17)$$