ECE 604, Lecture 11

October 2, 2018

1 Introduction

In this lecture, we will cover the following topics:

- More on Complex Power
- Wave Phenomenon in the Frequency Domain
- When is Cicuit Theory Valid?

Additional Reading:

- Sections 3.12, 3.13, 3.9, 3.10, 3.11, 4.1, 4.2 Ramo, Whinnery, and Van Duzer.

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2 More on Complex Power

Now that we realize that complex power is quite different from instantaneous power: The real part of complex power is proportional to the time average of the instantaneous power while the imaginary part is proportional to a time varying power that averages to zero. This imaginary part is termed the reactive power which is proportional to stored energy in the system. Reactive power corresponds to time varying part of the instantaneous power that can be both positive or negative. Hence, when it is positive, it corresponds to power flowing from the source to the load, but when it is negative, it corresponds to power returning to the source from the load.

Because of the above observation, it is prudent to look at the conservative property of complex power for a Maxwellian system. We shall consider a system where conductive loss as well as impressed sources are present. Using phasor technique for time-harmonic fields, Maxwell's equations can be written in the frequency domain as

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{M}_i \tag{2.1}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \sigma\mathbf{E} + \mathbf{J}_i \tag{2.2}$$

where $\mathbf{M}_i$ and $\mathbf{J}_i$ are impressed magnetic current and electric current sources, respectively. From now on, we will neglect to use under tilde whenever convenient and assume that these complex vectors or phasors are understood from the context.

To this end, we use the vector identity that

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^* \tag{2.3}$$

The above is like the product rule for derivatives. Then using (2.1) and (2.2) in (2.3), we have

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = -\mathbf{H}^* \cdot (j\omega\mu\mathbf{H}) - \mathbf{H}^* \cdot \mathbf{M}_i - \mathbf{E} \cdot (-j\omega\varepsilon\mathbf{E}^*) - \sigma\mathbf{E} \cdot \mathbf{E}^* - \mathbf{E} \cdot \mathbf{J}_i^* \tag{2.4}$$

The above can be further rearranged as

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = -j\omega\mu\mathbf{H} \cdot \mathbf{H}^* + j\omega\varepsilon\mathbf{E} \cdot \mathbf{E}^* - \sigma\mathbf{E} \cdot \mathbf{E}^* - \mathbf{H}^* \cdot \mathbf{M}_i - \mathbf{E} \cdot \mathbf{J}_i^* \tag{2.5}$$

Noticing that for time-harmonic signals, the following identity holds, namely,

$$\langle V(t)I(t)\rangle = \frac{1}{2}\Re e[\underset{\sim}{V}(\omega)\underset{\sim}{I}^*(\omega)] \tag{2.6}$$

Thus, using this, we can show that

$$\langle W_H \rangle = \frac{1}{2}\mu\langle \mathbf{H}(\mathbf{r},t) \cdot \mathbf{H}(\mathbf{r},t)\rangle = \frac{1}{2}\mu\langle |\mathbf{H}(\mathbf{r},t)|^2\rangle = \frac{1}{4}\mu|\mathbf{H}(\mathbf{r},\omega)|^2 \tag{2.7}$$

Similarly, we can show that

$$\langle W_E \rangle = \frac{1}{2}\varepsilon\langle |\mathbf{E}(\mathbf{r},t)|^2\rangle = \frac{1}{4}\varepsilon|\mathbf{E}(\mathbf{r},\omega)|^2 \tag{2.8}$$

In the above, $\langle W_H \rangle$ and $\langle W_E \rangle$ are time-average magnetic and electric field stored energy densities, respectively.

First, we take care to rewrite (2.5) as

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = -j\omega\mu|\mathbf{H}|^2 + j\omega\varepsilon|\mathbf{E}|^2 - \sigma|\mathbf{E}|^2 - \mathbf{H}^* \cdot \mathbf{M}_i - \mathbf{E} \cdot \mathbf{J}_i^* \tag{2.9}$$

Since we are dealing with time-harmonic fields, (2.7) and (2.8) imply that

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = -j4\omega[\langle W_H \rangle - \langle W_E \rangle] - \sigma|\mathbf{E}|^2 - \mathbf{H}^* \cdot \mathbf{M}_i - \mathbf{E} \cdot \mathbf{J}_i^* \tag{2.10}$$

If we assume that $\mathbf{M}_i = \mathbf{J}_i = 0$ to begin with, taking half the real part of (2.9) gives

$$\nabla \cdot \frac{1}{2}\Re e(\mathbf{E} \times \mathbf{H}^*) = -\frac{1}{2}\sigma|\mathbf{E}|^2 \tag{2.11}$$

a statement of energy conservation. Taking half the imaginary part of (2.10) gives

$$\nabla \cdot \frac{1}{2}\Im m(\mathbf{E} \times \mathbf{H}^*) = -2\omega[\langle W_H \rangle - \langle W_E \rangle] \tag{2.12}$$

Eq. (2.11) implies that $\nabla \cdot \frac{1}{2}\Re e(\mathbf{E} \times \mathbf{H}^*)$ is the time-average power exuding from a point in space while $\frac{1}{2}\sigma|\mathbf{E}|^2$ is the time-average power dissipating in the lossy conductor at that point in space. Meanwhile, $-2\omega[\langle W_H \rangle - \langle W_E \rangle]$ is the reactive power leaving the point in space. Notice the curious point that the reactive power is proportional to the difference of the stored magnetic field and electric field energies. This can be understood from one of the homework problems.

When we turn the impressed sources $\mathbf{M}_i$ and $\mathbf{J}_i$ back on to be nonzero, then we can think of $-\mathbf{H}^* \cdot \mathbf{M}_i$ and $-\mathbf{E} \cdot \mathbf{J}_i^*$ as complex powers in (2.9) supplied to the system by the impressed sources.

3 Wave Phenomenon in the Frequency Domain

Given that we have seen the emergence of wave phenomenon in the time domain, it will be interesting to ask how this phenomenon presents itself for time-harmonic field or in the frequency domain. In the frequency domain, the source-free Maxwell's equations are

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu_0\mathbf{H}(\mathbf{r}) \tag{3.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\varepsilon_0\mathbf{E}(\mathbf{r}) \tag{3.2}$$

Taking the curl of (3.1) and then substituting (3.2) into its right-hand side, one obtains

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu_0\nabla \times \mathbf{H}(\mathbf{r}) = \omega^2\mu_0\varepsilon_0\mathbf{E}(\mathbf{r}) \tag{3.3}$$

Again, using the identity that

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \cdot \nabla\mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} \tag{3.4}$$

and that $\nabla \cdot \mathbf{E} = 0$ in a source-free medium, (3.3) becomes

$$(\nabla^2 + \omega^2 \mu_0 \varepsilon_0)\mathbf{E}(\mathbf{r}) = 0 \tag{3.5}$$

This is known as the Helmholtz wave equation or just the Helmholtz equation.

For simplicity of seeing the wave phenomenon, we let $\mathbf{E} = \hat{x}E_x(z)$, a field pointing in the x direction, but varies only in the z direction. Evidently, $\nabla \cdot \mathbf{E}(\mathbf{r}) = \partial E_x(z)/\partial x = 0$. Then (3.5) becomes

$$\left(\frac{d^2}{dz^2} + k_0^2\right) E_x(z) = 0 \tag{3.6}$$

where $k_0^2 = \omega^2 \mu_0 \varepsilon_0 = \omega^2/c_0^2$. The general solution to (3.6) is of the form

$$E_x(z) = E_{0+}e^{-jk_0 z} + E_{0-}e^{jk_0 z} \tag{3.7}$$

One can convert the above back to the time domain using phasor technique, or by using that $E_x(z,t) = \Re e[E_x(z,\omega)e^{j\omega t}]$, yielding

$$E_x(z,t) = |E_{0+}|\cos(\omega t - k_0 z + \alpha_+) + |E_{0-}|\cos(\omega t + k_0 z + \alpha_-) \tag{3.8}$$

where we have assumed that

$$E_{0\pm} = |E_{0\pm}|e^{j\alpha_\pm} \tag{3.9}$$

The physical picture of the above expressions can be appreciated by rewriting

$$\cos(\omega t \mp k_0 z + \alpha_\pm) = \cos\left[\frac{\omega}{c_0}(c_0 t \mp z) + \alpha_\pm\right] \tag{3.10}$$

where we have used the fact that $k_0 = \frac{\omega}{c_0}$. One can see that the first term on the right-hand side of (3.8) is a sinusoidal plane wave traveling to the right, while the second term is a sinusoidal plane wave traveling to the left, with velocity c_0.

Moreover, for a fixed t or $t = 0$, the sinusoidal functions are proportional to $\cos(\mp k_0 z + \alpha_\pm)$. From this, we can see that whenever $k_0 z = 2n\pi$, $n \in Q$ where Q is the set of integers, the functions repeat themselves. Calling this repetition length the wavelength λ_0, we deduce that $\lambda_0 = \frac{2\pi}{k_0}$, or that

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega}{c_0} = \frac{2\pi f}{c_0} \tag{3.11}$$

One can see that because c_0 is a humongous number, λ_0 can be very large. You can plug in the frequency of your local AM station to see how big λ_0 is.

4 When is Static Theory Valid?

Historically, static electromagnetic theory, like Ampere's law, Faraday's law, Coulomb's law, and Gauss law, were discovered first. From them came circuit theory. Circuit theory consists of elements like resistors, capacitors, and inductors. Static electromagnetic theory, or quasi-static electromagnetic theory was used to derive the formulas for these elements. Given that we have now seen electromagnetic theory in its full form, we like to ponder when we can use static or quasi-static theory to describe electromagnetic phenomena.

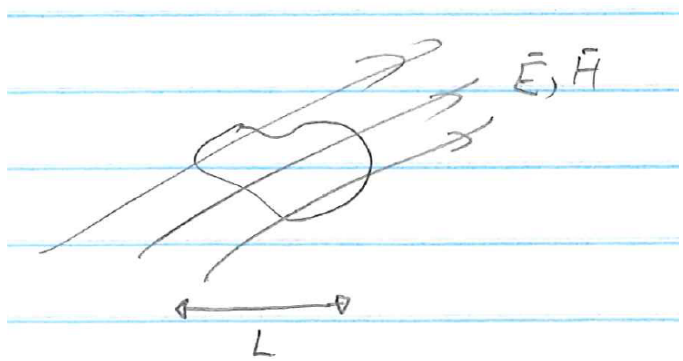

Figure 1:

To see this lucidly, it is best to write Maxwell's equations in dimensionless units or the same units. Say if we want to solve Maxwell's equations for the fields close to an object of size L. This object can be a small particle like the sphere in your take home exam, or it could be a capacitor, or an inductor. It is clear that these $\mathbf{E}$ and $\mathbf{H}$ fields will have to satisfy boundary conditions in the vicinity of the object as shown in Figure 1. They become great contortionist in order to do so. Hence, we do not expect a constant field around the object but that the field will vary on the length scale of L. So we renormalize our length scale by this length L by defining a new dimensionless coordinate system such that.

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{L} \tag{4.1}$$

By so doing, then $Ldx' = dx$, $Ldy' = dy$, and $Ldz' = dz$, and

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{1}{L}\frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{1}{L}\frac{\partial}{\partial z'} \tag{4.2}$$

Then, the first two of Maxwell's equations become

$$\frac{1}{L}\nabla' \times \mathbf{E} = -j\omega\mu_0\mathbf{H} \tag{4.3}$$

$$\frac{1}{L}\nabla' \times \mathbf{H} = j\omega\varepsilon_0\mathbf{E} + \mathbf{J} \tag{4.4}$$

Here, we still have apples and oranges to compare to since **E** and **H** have different units. For instance, the ratio of **E** to the **H** field has a dimension of impedance. To bring them to the same unit, we define a new $\mathbf{E}'$ such that

$$\eta_0 \mathbf{E}' = \mathbf{E} \tag{4.5}$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0} \cong 377$ ohms in vacuum. In this manner, the new $\mathbf{E}'$ has the same unit as the **H** field. Then, (4.3) and (4.4) become

$$\frac{\eta_0}{L}\nabla' \times \mathbf{E}' = -j\omega\mu_0 \mathbf{H} \tag{4.6}$$

$$\frac{1}{L}\nabla' \times \mathbf{H} = j\omega\varepsilon_0\eta_0 \mathbf{E}' + \mathbf{J} \tag{4.7}$$

With this change, the above can be rearranged to become

$$\nabla' \times \mathbf{E}' = -j\omega\mu_0 \frac{L}{\eta_0}\mathbf{H} \tag{4.8}$$

$$\nabla' \times \mathbf{H} = j\omega\varepsilon_0\eta_0 L\mathbf{E}' + L\mathbf{J} \tag{4.9}$$

The above can be further simplified to become

$$\nabla' \times \mathbf{E}' = -j\frac{\omega}{c_0}L\mathbf{H} \tag{4.10}$$

$$\nabla' \times \mathbf{H} = j\frac{\omega}{c_0}L\mathbf{E}' + L\mathbf{J} \tag{4.11}$$

Notice now that in the above, **H**, $\mathbf{E}'$, and $L\mathbf{J}$ have the same unit, and ∇' is dimensionless. Therefore, one can compare terms, and one can ignore the frequency dependent term when

$$\frac{\omega}{c_0}L \ll 1 \tag{4.12}$$

Or when

$$2\pi\frac{L}{\lambda_0} \ll 1 \tag{4.13}$$

Therefore, the above criteria are for the validity of the static or quasi-static approximation. When these criteria are satisfied, then Maxwell's equations can be simplified to and approximated with the following equations

$$\nabla' \times \mathbf{E}' = 0 \tag{4.14}$$

$$\nabla' \times \mathbf{H} = L\mathbf{J} \tag{4.15}$$

which are the static equations of electromagnetic theory. In other words, one can solve an optics problem where ω is humongous or the wavelength very short, with a plasmonic nanoparticle using quasi-static analysis if the particle is small enough compared to wavelength of the light. Also in circuit theory where static analysis prevails, we better check if (4.13) is satisfied before we use circuit theory comfortably.

At 3 GHz, where the wavelength is 10 cm, when the circuit board or the computer chip is much smaller than this dimension, circuit theory can be used. But when the clock rate of the computer switches up to 10 GHz, many circuit analysis do not hold any more. Then one really has to perform electrodynamic analysis of the electromagnetic phenomena inside the circuit board, especially in the computer chassis board. One needs to solve Maxwell's equations in its full glory.

In (4.13), this criterion has been expressed in terms of the dimension of the object L compared to the wavelength λ_0. Alternatively, we can express this criterion in terms of transit time. The transit time for an electromagnetic wave to traverse an object of size L is $\tau = L/c_0$ and $\omega = 2\pi/T$ where T is the period of the time-harmonic oscillation. Hence, (4.12) can be re-expressed as

$$\frac{2\pi\tau}{T} \ll 1 \tag{4.16}$$

The above implies that if the transit time τ needed to traverse the object of length L is much small than the period of oscillation of the electromagnetic field, then static theory can be used. The finite speed of light gives rise to delay or retardation of electromagnetic signal when it propagates through space. When this retardation effect can be ignored, then static theory can be used. In other words, if the speed of light had been infinite, there is no retardation effect, and static theory can always be used. Alternatively, the infinite speed of light will give rise to infinite wavelength, and criterion (4.13) will always be satisfied, and static theory prevails.

In closing, we would like to make one more remark. The right-hand side of (4.8), which is Faraday's law, is essential for capturing the physical mechanism of an inductor and flux linkage. And yet, if we drop it, there will be no inductor in this world. To understand this dilemma, let us rewrite (4.8) in integral form, namely,

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -j\omega\mu_0 \frac{L}{\eta_0} \iint_S d\mathbf{S} \cdot \mathbf{H} \tag{4.17}$$

In the inductor, the right-hand side has been amplified by multi-turns, effectively increasing S, the flux linkage area. Or one can think of an inductor as having a much longer effective length L_{eff} when untwined so as to compensate for decreasing frequency ω. Hence, the importance flux linkage or the inductor in circuit theory is not diminished even when the frequency is low.