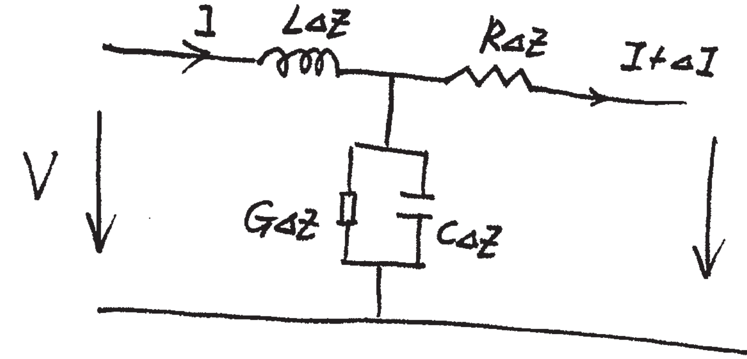
Chapter IV

Review of Transmission Line



kirchhoff's voltage law:

$$V = j\omega L \Delta z\, I + R\Delta z\,(I+\Delta I) + V + \Delta V$$

$$\rightarrow \boxed{\frac{dV}{dz} + (j\omega L + R)\, I = 0}$$

$$I = G\Delta z\, V + j\omega C \Delta z\, V + I + \Delta I$$

$$\rightarrow \boxed{\frac{dI}{dz} + (j\omega C + G)\, V = 0}$$

$$\frac{d^2V}{dz^2} + (j\omega L + R)\frac{dI}{dz} = 0$$

$$\frac{d^2V}{dz^2} - (j\omega L + R)(j\omega C + G)\, V = 0$$

$$\boxed{\frac{d^2V}{dz^2} - \gamma^2 V = 0} \qquad \gamma^2 = (j\omega L + R)(j\omega C + G)$$

Consider a lossless line,

$$R = 0,\ G = 0 \implies \gamma = j\omega\sqrt{LC}$$

or $\gamma = j\beta \qquad \beta = \omega\sqrt{LC}$

$$\frac{d^2V}{dz^2} + \beta^2 V = 0$$

$$\implies V(z) = a_+ e^{-j\beta z} + a_- e^{j\beta z}$$

Consider $V_+(z) = a_+ e^{-j\beta z}$

$$V_+(z,t) = Re[V_+(z)e^{j\omega t}] = Re[a_+ e^{j\omega t - j\beta z}]$$

$$= |a_+| \cos(\omega t - \beta z + \angle a_+)$$

Consider phase

$$\omega t - \beta z_p = \text{constant}$$

$$\omega - \beta \frac{dz_p}{dt} = 0 \implies \underbrace{\frac{dz_p}{dt}}_{\text{phase velocity}} = \frac{\omega}{\beta} \leftarrow \text{Propagation constant}$$

Hence. $\boxed{e^{-j\beta z}: \text{ wave traveling toward } +z \text{ direction}}$

Similarly, $\boxed{e^{j\beta z} : \text{wave traveling toward } -z \text{ direction}}$

Consider a lossy line:

$$\gamma = \sqrt{(j\omega L+R)(j\omega C+G)} = \alpha + j\beta$$

$$\frac{d^2V}{dz^2} - \gamma^2 V = 0$$

$$\Rightarrow \quad V(z) = a_+ e^{-\gamma z} + a_- e^{\gamma z}$$

Consider $V_+(z) = a_+ e^{-\gamma z} = a_+ e^{-(\alpha + j\beta)z}$

$$V_+(z,t) = \mathrm{Re}\left[V_+(z) e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[a_+ e^{j\omega t - j\beta z - \alpha z}\right]$$

$$= |a_+| e^{-\alpha z} \cos(\omega t - \beta z + \angle a_+)$$

α: attenuation constant, β: propagation constant

$$V(z) = a_+ e^{-\gamma z} + a_- e^{\gamma z}$$

Define $\Gamma = \dfrac{V_-(0)}{V_+(0)} = \dfrac{a_-}{a_+}$

$$V(z) = a_+ e^{-\gamma z} + a_+ \Gamma e^{\gamma z}$$

$$\frac{dV}{dz} = -\gamma a_+ e^{-\gamma z} + \gamma a_+ \Gamma e^{\gamma z}$$

$$= -(j\omega L + R) I$$

$$I(z) = \frac{\gamma}{j\omega L + R} a_+ e^{-\gamma z} - \frac{\gamma}{j\omega L + R} a_+ \Gamma e^{\gamma z}$$

$$= \frac{\gamma}{j\omega L + R} a_+ (e^{-\gamma z} - \Gamma e^{\gamma z})$$

Define the characteristic impedance

$$\boxed{Z_0 = \frac{V_+}{I_+} = \frac{j\omega L + R}{\gamma} = \sqrt{\frac{j\omega L + R}{j\omega C + G}}}$$

$$I(z) = \frac{1}{Z_0} a_+ (e^{-\gamma z} - \Gamma e^{\gamma z})$$

Reflection coefficient:

$$\boxed{\Gamma(z) = \frac{V_-(z)}{V_+(z)} = \frac{a_- e^{\gamma z}}{a_+ e^{-\gamma z}} = \Gamma e^{2\gamma z}}$$

$$\Gamma(z_1) = \Gamma e^{2\gamma z_1}, \quad \Gamma(z_2) = \Gamma e^{2\gamma z_2}$$

$$\Longrightarrow \boxed{\Gamma(z_1) = \Gamma(z_2) e^{2\gamma (z_1 - z_2)}}$$

- Review of Transmission Line (Contined)

Interested Readers

- Chapter 5, textbook

Impedance at z:

$$Z(z) = \frac{V(z)}{I(z)} = \frac{a_+ e^{-\gamma z} + a_+ \Gamma e^{\gamma z}}{\frac{1}{Z_0} a_+ (e^{-\gamma z} - \Gamma e^{\gamma z})}$$

$$= Z_0 \frac{1 + \Gamma e^{2\gamma z}}{1 - \Gamma e^{2\gamma z}} = Z_0 \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

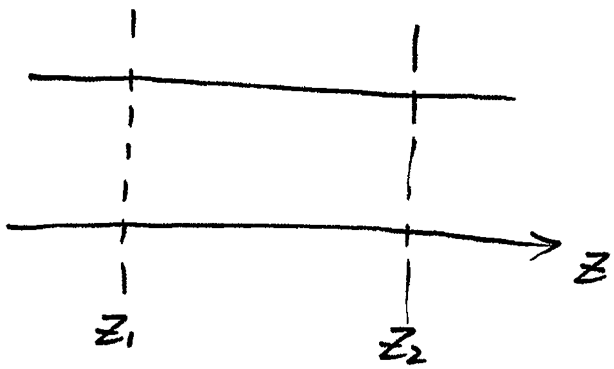
$$\boxed{\begin{aligned} \Gamma(z) &= \frac{Z(z) - Z_0}{Z(z) + Z_0} \\ Z(z) &= Z_0 \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \end{aligned}}$$

Note that:

1. Z_0 is independent of z
2. $Z(z)$ depends on z
3. $\Gamma(z)$ depends on z
4. For a lossless line, $\gamma = j\beta$

   $|\Gamma(z)| = |\Gamma e^{2j\beta z}| = |\Gamma|$ is invariant along z.

Given information at z_1, find information at z_2:



$Z(z_1), \Gamma(z_1) \qquad Z(z_2), \Gamma(z_2)$

$$\Gamma(z) = \Gamma e^{2\gamma z} \Rightarrow \Gamma(z_2) = \Gamma(z_1) e^{2\gamma(z_2 - z_1)}$$

$$\begin{aligned} V(z_2) &= a_+ e^{-\gamma z_2} + \Gamma a_+ e^{\gamma z_2} \\ &= a_+ e^{-\gamma z_2}(1 + \Gamma e^{2\gamma z_2}) \\ &= a_+ e^{-\gamma z_2}(1 + \Gamma(z_2)) \end{aligned}$$

$$\begin{aligned} I(z_2) &= \frac{1}{Z_0} a_+ (e^{-\gamma z_2} - \Gamma e^{\gamma z_2}) = \frac{1}{Z_0} a_+ e^{-\gamma z_2}(1 - \Gamma e^{2\gamma z_2}) \\ &= \frac{1}{Z_0} a_+ e^{-\gamma z_2}(1 - \Gamma(z_2)) \end{aligned}$$

$$\begin{aligned} Z(z_2) = \frac{V(z_2)}{I(z_2)} &= Z_0 \frac{1 + \Gamma(z_2)}{1 - \Gamma(z_2)} \\ &= Z_0 \frac{1 + \Gamma(z_1) e^{2\gamma(z_2 - z_1)}}{1 - \Gamma(z_1) e^{2\gamma(z_2 - z_1)}} \end{aligned}$$

Since

$$Z(z_1) = Z_0 \frac{1+\Gamma(z_1)}{1-\Gamma(z_1)} \Rightarrow \Gamma(z_1) = \frac{Z(z_1)-Z_0}{Z(z_1)+Z_0}$$

$$Z(z_2) = Z_0 \frac{1+\frac{Z(z_1)-Z_0}{Z(z_1)+Z_0} e^{2\gamma(z_2-z_1)}}{1-\frac{Z(z_1)-Z_0}{Z(z_1)+Z_0} e^{2\gamma(z_2-z_1)}}$$

$$= Z_0 \frac{Z_1+Z_0+(Z_1-Z_0)e^{2\gamma(z_2-z_1)}}{Z_1+Z_0-(Z_1-Z_0)e^{2\gamma(z_2-z_1)}}$$

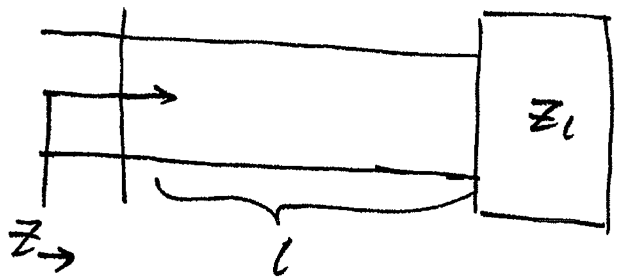
$$= Z_0 \frac{Z_1(e^{\gamma l}+e^{-\gamma l}) - Z_0(e^{\gamma l}-e^{-\gamma l})}{Z_1(e^{-\gamma l}-e^{\gamma l}) + Z_0(e^{\gamma l}+e^{-\gamma l})}$$

$$\boxed{Z(z_2) = Z_0 \frac{Z_1 \cosh\gamma l - Z_0 \sinh\gamma l}{Z_0 \cosh\gamma l - Z_1 \sinh\gamma l}}$$

For a lossless line, $\gamma = j\beta$

$$\boxed{Z_2 = Z_0 \frac{Z_1 \cos\beta l - jZ_0 \sin\beta l}{Z_0 \cos\beta l - jZ_1 \sin\beta l}}$$

$$\boxed{Z_1 = Z_0 \frac{Z_2 \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_2 \sin\beta l}}$$



$$Z_{\rightarrow} = Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l}$$

If $Z_L = 0$, $\quad Z_{\rightarrow} = jZ_0 \tan\beta l$

If $Z_L = \infty$, $\quad Z_{\rightarrow} = -jZ_0 \cot\beta l$