- Time-Harmonic Fields
- Complex Power

Interested Readers:

- Textbook 3.8, Balanis 1.7
- Balanis 1.7.3

Time-Harmonic Fields

Euler's formula: $e^{j\alpha} = \cos\alpha + j\sin\alpha$

$$\cos\alpha = Re(e^{j\alpha})$$

If $V(x, y, z, t)$ is oscillating at a single freq.

$$V(x, y, z, t) = V'(x, y, z)\cos(\omega t + \alpha)$$

$$= V'(x, y, z)\, Re\left(e^{j(\omega t+\alpha)}\right)$$

$$= Re\left[V'(x, y, z)\, e^{j\alpha} e^{j\omega t}\right]$$

$$= Re\left[\hat{V}(x, y, z)\, e^{j\omega t}\right]$$

$\hat{V}(x, y, z)$: phasor or complex quantity

Extension to vectors:

$$E_{x,y,z}(x, y, z, t) = Re\left[\hat{E}_{x,y,z}(x, y, z)\, e^{j\omega t}\right]$$

$$\vec{E}(x,y,z,t) = Re[\hat{\vec{E}}(x,y,z)e^{j\omega t}]$$

$$\vec{H}(x,y,z,t) = Re[\hat{\vec{H}}(x,y,z)e^{j\omega t}]$$

Same for other quantities

<u>Consider</u> $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} - \vec{M}$

$$Re[\nabla\times\hat{\vec{E}}e^{j\omega t}] = -\frac{\partial}{\partial t}Re[\hat{\vec{B}}e^{j\omega t}] - Re[\hat{\vec{M}}e^{j\omega t}]$$

$$= Re\{[-j\omega\hat{\vec{B}} - \hat{\vec{M}}]e^{j\omega t}\}$$

Lemma: If $Re(Ae^{j\omega t}) = Re(Be^{j\omega t})$ for any t, then $A = B$

Therefore

$$\nabla \times \hat{\vec{E}} = -j\omega \hat{\vec{B}} - \hat{\vec{M}}$$

$$\nabla \times \hat{\vec{H}} = j\omega \hat{\vec{D}} + \hat{\vec{J}}$$

$$\nabla \cdot \hat{\vec{J}} = -j\omega \hat{\rho}_e$$

$$\nabla \cdot \hat{\vec{M}} = -j\omega \hat{\rho}_m$$

For convenience, use $\vec{E}$ for $\hat{\vec{E}}$

$$\boxed{\begin{aligned} \nabla \times \vec{E} &= -j\omega \vec{B} - \vec{M} \\ \nabla \times \vec{H} &= j\omega \vec{D} + \vec{J} \\ \nabla \cdot \vec{D} &= \rho_e \\ \nabla \cdot \vec{B} &= \rho_m \\ \nabla \cdot \vec{J} &= -j\omega \rho_e \\ \nabla \cdot \vec{M} &= -j\omega \rho_m \end{aligned}}$$

Maxwell's equations for time-harmonic fields

Alternative view:

Fourier transform:

$$F(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(x,y,z,\omega)e^{j\omega t}d\omega$$

$$\hat{F}(x,y,z,\omega) = \int_{-\infty}^{\infty} F(x,y,z,t)e^{-j\omega t}dt$$

Expand:

$$\vec{E}(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\vec{E}}(x,y,z,\omega)e^{j\omega t}d\omega$$

$$\vec{B}(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\vec{B}}(x,y,z,\omega)e^{j\omega t}d\omega$$

$$\vec{M}(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\vec{M}}(x,y,z,\omega)e^{j\omega t}d\omega$$

$$\nabla\times\vec{E} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \nabla\times\hat{\vec{E}}(x,y,z,\omega)e^{j\omega t}d\omega$$

$$= -\frac{\partial\vec{B}}{\partial t} - \vec{M}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-j\omega\hat{\vec{B}}(x,y,z,\omega) - \hat{\vec{M}}(x,y,z,\omega)\right]e^{j\omega t}d\omega$$

Hence,

$$\nabla \times \hat{\vec{E}} = -j\omega \hat{\vec{B}} - \hat{\vec{M}} \qquad \left(\text{How to prove?}\right)$$

or $$\boxed{\nabla \times \vec{E} = -j\omega \vec{B} - \vec{M}}$$

Complex Power

Consider Poynting vector

$$\vec{S} = \vec{E} \times \vec{H} = Re[\hat{\vec{E}} e^{j\omega t}] \times Re[\hat{\vec{H}} e^{j\omega t}]$$

$$= \frac{1}{2}[\hat{\vec{E}} e^{j\omega t} + (\hat{\vec{E}} e^{j\omega t})^*]$$

$$\times \frac{1}{2}[\hat{\vec{H}} e^{j\omega t} + (\hat{\vec{H}} e^{j\omega t})^*]$$

$$= \frac{1}{4}\hat{\vec{E}} \times \hat{\vec{H}} e^{j2\omega t} + \frac{1}{4}\hat{\vec{E}} \times \hat{\vec{H}}^* + \frac{1}{4}\hat{\vec{E}}^* \times \hat{\vec{H}}$$

$$+ \frac{1}{4}\hat{\vec{E}}^* \times \hat{\vec{H}}^* e^{-j2\omega t}$$

$$= \frac{1}{2} Re(\hat{\vec{E}} \times \hat{\vec{H}}^*) + \frac{1}{2} Re(\hat{\vec{E}} \times \hat{\vec{H}} e^{j2\omega t})$$

$$\vec{S}_{av} = \frac{1}{T_0}\int_0^{T_0} \vec{S}\, dt = \frac{1}{2} Re(\hat{\vec{E}} \times \hat{\vec{H}}^*)$$

Let $\hat{\vec{S}} = \frac{1}{2}\hat{\vec{E}} \times \hat{\vec{H}}^*$ $\quad \vec{S}_{av} = Re(\hat{\vec{S}})$

Omit the hat sign:

$$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^* \qquad \vec{S}_{av} = Re(\vec{S})$$

For time-harmonic fields:

$$\nabla \times \vec{E} = -j\omega\mu \vec{H} - \vec{M}_i$$

$$\nabla \times \vec{H} = j\omega\varepsilon \vec{E} + \sigma \vec{E} + \vec{J}_i$$

$$\vec{H}^* \cdot \nabla \times \vec{E} = -j\omega\mu \vec{H}^* \cdot \vec{H} - \vec{H}^* \cdot \vec{M}_i$$

$$\vec{E} \cdot (\nabla \times \vec{H})^* = -j\omega\varepsilon \vec{E} \cdot \vec{E}^* + \sigma \vec{E} \cdot \vec{E}^* + \vec{E} \cdot \vec{J}_i^*$$

$$\nabla \cdot (\vec{E} \times \vec{H}^*) = \vec{H}^* \cdot \nabla \times \vec{E} - \vec{E} \cdot (\nabla \times \vec{H})^*$$

$$= -j\omega\mu |\vec{H}|^2 + j\omega\varepsilon |\vec{E}|^2 - \sigma |\vec{E}|^2 - \vec{H}^* \cdot \vec{M}_i - \vec{E} \cdot \vec{J}_i^* \qquad (1)$$

$$\frac{1}{2}\oiint_s (\vec{E}\times\vec{H}^*)\cdot d\vec{s} + \frac{1}{2}\iiint_V \sigma|\vec{E}|^2 dV$$

$$+ j\frac{\omega}{2}\iiint_V (\mu|\vec{H}|^2 - \varepsilon|\vec{E}|^2)\, dV$$

$$= -\frac{1}{2}\iiint_V (\vec{E}\cdot\vec{J}_i^* + \vec{H}^*\cdot\vec{M}_i)\, dV$$

$$P_s = -\frac{1}{2}\iiint_V (\vec{E}\cdot\vec{J}_i^* + \vec{H}^*\cdot\vec{M}_i)\, dV$$ — Supplied Complex Power

$$P_e = \frac{1}{2}\iiint_S (\vec{E}\times\vec{H}^*)\cdot d\vec{s}$$ — exiting complex power

$$\bar{P}_d = \frac{1}{2}\iiint_V \sigma|\vec{E}|^2 dV$$ — time-averaged dissipated real power

$$\bar{W}_m = \frac{1}{4}\iiint_V \mu|\vec{H}|^2 dV$$ — time-averaged magnetic energy

$$\bar{W}_e = \frac{1}{4}\iiint_V \varepsilon|\vec{E}|^2 dV$$ — time-averaged electric energy

$$\boxed{P_s = P_e + \bar{P}_d + j2\omega(\bar{W}_m - \bar{W}_e)} \qquad (2)$$

Conservation of energy in integral form

Define: $p_s = -\frac{1}{2}(\vec{E}\cdot\vec{J}_i^* + \vec{H}^*\cdot\vec{M}_i)$

$$p_e = \frac{1}{2}\nabla\cdot(\vec{E}\times\vec{H}^*)$$

$$\bar{p}_d = \frac{1}{2}\sigma|\vec{E}|^2$$

$$\bar{w}_m = \frac{1}{4}\mu|\vec{H}|^2$$

$$\bar{w}_e = \frac{1}{4}\varepsilon|\vec{E}|^2$$

From (1)

$$\Rightarrow \boxed{p_s = p_e + \bar{p}_d + j2\omega(\bar{w}_m - \bar{w}_e)} \qquad (3)$$

Conservation of energy in differential form

Take the real part of (2)

$$\boxed{Re(P_s) = Re(P_e) + \bar{P}_d}$$

Take the imaginary part of (2)

$$\boxed{Im(P_s) = Im(P_e) + 2\omega(\bar{W}_m - \bar{W}_e)}$$

(example) Discussion

The electric field inside an infinite length rectangular pipe, with all four vertical sides perfectly electric conducting is given by

$$\vec{E} = \hat{a}_z (1+j) \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{\pi}{b}y\right)$$

Assuming that there are no sources within the box and $a = \lambda_0$, $b = 0.5\lambda_0$, and $\mu = \mu_0$, where λ_0 is free-space wavelength, find the

(a) conductivity

(b) dielectric constant

of the medium within the box.