- Constitutive Relations
- Energy and Power

Interested Readers:

- Textbook chapter 13. Balanis's book 1.3
- Textbook 3.12. Balanis 1.6

Constitutive Relations

Relations between $\vec{D}$, $\vec{B}$ and $\vec{E}$, $\vec{H}$

1. Free space (vacuum)

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\vec{B} = \mu_0 \vec{H}$$

permittivity: $\varepsilon_0 = \frac{1}{36\pi} \times 10^{-9}$ F/m

$= 8.85 \times 10^{-12}$ F/m

permeability: $\mu_0 = 4\pi \times 10^{-7}$ H/m

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s}$$

↑ speed of light in vacuum

2. Material

$$\vec{D} = \varepsilon \vec{E}$$

$\varepsilon_r = \dfrac{\varepsilon}{\varepsilon_0}$ relative permittivity or. dilectric constant

$$\vec{B} = \mu \vec{H}$$

$\mu_r = \dfrac{\mu}{\mu_0}$ relative permeability

3. Conductors

electric conduction current

$$\vec{J_c} = \sigma \vec{E}$$

σ: conductivity (s/m)

4. Classification of materials

(a) According to space variables:

Homogeneous media

Inhomogeneous media

(b) According to time variable:

Stationary media

Nonstationary media

(c) According to the value of σ:

Perfect dielectric $(\sigma = 0)$

Perfect conductor $(\sigma \to \infty)$

(d) According to the direction of $\vec{D}$ and $\vec{B}$

* If $\vec{D} \parallel \vec{E}$, $\vec{B} \parallel \vec{H}$ $\Rightarrow$ isotropic media

$\vec{D} = \varepsilon \vec{E}$: $D_x = \varepsilon E_x$, $D_y = \varepsilon E_y$, $D_z = \varepsilon E_z$

* If $\vec{D} \nparallel \vec{E}$, $\vec{B} \nparallel \vec{H}$ $\Rightarrow$ anisotropic media

$$D_x = \varepsilon_{xx} E_x + \varepsilon_{xy} E_y + \varepsilon_{xz} E_z$$

$$D_y = \varepsilon_{yx} E_x + \varepsilon_{yy} E_y + \varepsilon_{yz} E_z$$

$$D_z = \varepsilon_{zx} E_x + \varepsilon_{zy} E_y + \varepsilon_{zz} E_z$$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

or $\vec{D} = \overline{\overline{\varepsilon}} \cdot \vec{E}$

$\vec{B} = \overline{\overline{\mu}} \cdot \vec{H}$ tensors

For crystals, $\overline{\overline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$

If $\varepsilon_{xx} \neq \varepsilon_{yy} \neq \varepsilon_{zz}$: biaxial crystal

If two of them are equal: uniaxial crystal

Isotropic media: ε, μ are scalars

Anisotropic media: $\overline{\overline{\varepsilon}}, \overline{\overline{\mu}}$ are tensors

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* Bianisotropic media:

$$\vec{D} = \bar{\bar{\varepsilon}} \cdot \vec{E} + \bar{\bar{\zeta}} \cdot \vec{H}$$

$$\vec{B} = \bar{\bar{\mu}} \cdot \vec{H} + \bar{\bar{\xi}} \cdot \vec{E}$$

(e) According to the form of ε, μ, σ

Linear media

Nonlinear media:

$$\varepsilon = \varepsilon(\vec{E}, \vec{H})$$
$$\mu = \mu(\vec{E}, \vec{H})$$
$$\sigma = \sigma(\vec{E}, \vec{H})$$

(f) According to the relation of ε and μ with frequency:

Dispersive media

Nondispersive media

Energy and Power

Consider a medium with ε, μ, σ:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}_i = -\mu \frac{\partial \vec{H}}{\partial t} - \vec{M}_i \qquad (1)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{J}_i + \sigma \vec{E} \qquad (2)$$

$$\vec{H} \cdot (1) \Rightarrow \quad \vec{H} \cdot (\nabla \times \vec{E}) = -\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} - \vec{H} \cdot \vec{M}_i$$

$$\vec{E} \cdot (2) \Rightarrow \quad \vec{E} \cdot (\nabla \times \vec{H}) = \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{E} \cdot \vec{J}_i + \sigma \vec{E} \cdot \vec{E}$$

$$\vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) = -\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} - \vec{H} \cdot \vec{M}_i - \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} - \vec{E} \cdot \vec{J}_i - \sigma \vec{E} \cdot \vec{E}$$

Vector identity:

$$\boxed{\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H})}$$

$$\therefore \quad \nabla \cdot (\vec{E} \times \vec{H}) = -\left(\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} \cdot \vec{E} + \vec{H} \cdot \vec{M}_i + \vec{E} \cdot \vec{J}_i\right)$$

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$$\oint\!\!\!\oint_S (\vec{E}\times\vec{H})\cdot d\vec{s} + \iiint_V \left(\mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} + \varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \sigma\vec{E}\cdot\vec{E} + \vec{H}\cdot\vec{M_i} + \vec{E}\cdot\vec{J_i}\right)dV = 0$$

* $\vec{E}\times\vec{H}$: $V/m \cdot A/m = W/m^2$ Poynting's vector

$P_e = \oint\!\!\!\oint_s (\vec{E}\times\vec{H})\cdot d\vec{s}$ — power leaving V bounded by S

* $\mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = \frac{1}{2}\mu\frac{\partial H^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) = \frac{\partial W_m}{\partial t}$

$w_m = \frac{1}{2}\mu H^2$: $\frac{H}{m}\left(\frac{A}{m}\right)^2 = \frac{\text{Joule}}{m^3}$ — energy density

$W_m = \iiint_V w_m\, dV$ magnetic energy

* $\varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\varepsilon\frac{\partial E^2}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right) = \frac{\partial W_e}{\partial t}$

$w_e = \frac{1}{2}\varepsilon E^2$, $\frac{F}{m}\left(\frac{V}{m}\right)^2 = \frac{\text{Joule}}{m^3}$ — energy density

$W_e = \iiint_V w_e\, dV$ electric energy

* $\sigma \vec{E} \cdot \vec{E} = \sigma E^2 \qquad \frac{S}{m}\left(\frac{V}{m}\right)^2 = \frac{W}{m^3}$

$$P_d = \iiint_V \sigma \vec{E} \cdot \vec{E} \, dV \quad \text{— power dissipated}$$

* $P_s = -\iiint_V (\vec{H} \cdot \vec{M}_i + \vec{E} \cdot \vec{J}_i) \, dV$ — power supplied

Hence:

$$P_e + P_d - P_s + \frac{\partial}{\partial t}(W_m + W_e) = 0$$

or

$$\boxed{P_s = P_e + P_d + \frac{\partial}{\partial t}(W_m + W_e)}$$ Poynting's theorem

Conservation of Power

Denote: $p_e = \nabla \cdot (\vec{E} \times \vec{H})$, $p_d = \sigma E^2$

$$p_s = -\vec{H} \cdot \vec{M}_i - \vec{E} \cdot \vec{J}_i$$

$$\boxed{p_s = p_e + p_d + \frac{\partial}{\partial t}(w_m + w_e)}$$