$\boxed{\text{Chapter III}}$ Maxwell's Equations

- Maxwell's Equations
  - Integral Form
  - Differential Form
- Boundary Conditions

Interested Readers:

- Textbook 3.1 — 3.7, 3.14
- Balanis 1.2, 1.5

Maxwell's Equations

Fundamental postulates:

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot d\vec{s} - K \qquad \text{Faraday's law}$$

$$\oint_C \vec{H} \cdot d\vec{l} = \frac{\partial}{\partial t} \iint_S \vec{D} \cdot d\vec{s} + I \qquad \text{Ampère's law}$$

$$\oiint_S \vec{D} \cdot d\vec{s} = Q \qquad \text{Gauss's law}$$

$$\oiint_S \vec{B} \cdot d\vec{s} = Q_m \qquad \text{Gauss's law-magnetic}$$

$\boxed{\text{Maxwell's equations in integral form}}$

Basic quantities:

$\vec{E}$: V/m $\vec{H}$: A/m

$\vec{D}$: C/m^2 $\vec{B}$: webers/m^2

K: V I: A

Q: coulumbs Q_m: webers

For continuous media,

$$\because \oint_C \vec{E} \cdot d\vec{l} = \iint_S (\nabla \times \vec{E}) \cdot d\vec{s}$$ Stokes's theorem

$$\therefore \iint_S (\nabla \times \vec{E}) \cdot d\vec{s} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot d\vec{s} - \iint_S \vec{M} \cdot d\vec{s}$$

$$= \iint_S \left[-\frac{\partial \vec{B}}{\partial t} - \vec{M} \right] \cdot d\vec{s}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}}$$ Faraday's law

Similarly

$$\boxed{\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}}$$ Ampère's law

③

$\because \oiint_S \vec{D} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{D} \, dV$ Gauss's theorem

$\because \iiint_V \nabla \cdot \vec{D} \, dV = \iiint_V \rho \, dV$

$$\boxed{\nabla \cdot \vec{D} = \rho}$$ Gauss's Law

Similarly

$$\boxed{\nabla \cdot \vec{B} = \rho_m}$$ Gauss's Law — magnetic

④

Maxwell's equations in differential form

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = \rho_m$$

⑤

$$\nabla\cdot\left(\nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t}+\vec{J}\right)$$

$$\Rightarrow \quad 0=\frac{\partial\rho}{\partial t}+\nabla\cdot\vec{J}$$

or $\boxed{\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t}}$ continuity equation

$$\iiint_V \nabla\cdot\vec{J}\,dV=-\iiint_V\frac{\partial\rho}{\partial t}\,dV$$

$$\Rightarrow \quad \boxed{\oint\!\!\!\oint_S \vec{J}\cdot d\vec{s}=-\frac{dQ}{dt}}$$

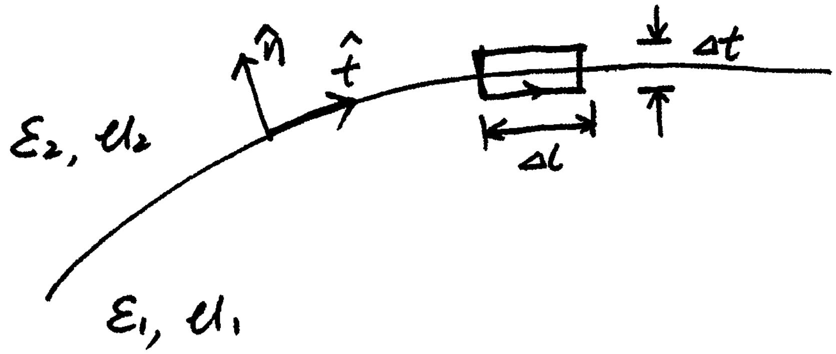
Similarly

$$\boxed{\begin{aligned}\nabla\cdot\vec{M}&=-\frac{\partial\rho_m}{\partial t}\\ \oint\!\!\!\oint_S\vec{M}\cdot d\vec{s}&=-\frac{dQ_m}{dt}\end{aligned}}$$

Continuity equation
— magnetic

$\boxed{\text{Boundary Conditions}}$

At a discontinuous interface



$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot d\vec{s} - K$$

$$\vec{E}_1 \cdot \hat{t}\,\Delta l + \vec{E}_1 \cdot \hat{n}\,\frac{\Delta t}{2} + \vec{E}_2 \cdot \hat{n}\,\frac{\Delta t}{2} - \vec{E}_2 \cdot \hat{t}\,\Delta l$$

$$- \vec{E}_2 \cdot \hat{n}\,\frac{\Delta t}{2} - \vec{E}_1 \cdot \hat{n}\,\frac{\Delta t}{2}$$

$$= -\frac{\partial}{\partial t} \vec{B} \cdot (\hat{t} \times \hat{n})\,\Delta t\,\Delta l - \vec{M}_s \cdot (\hat{t} \times \hat{n})\,\Delta l$$

⑦ Let $\Delta t \to 0$

$$\vec{E}_1 \cdot \hat{t}\,\Delta l - \vec{E}_2 \cdot \hat{t}\,\Delta l = -\vec{M}_s \cdot (\hat{t} \times \hat{n})\,\Delta l$$

or
$$\vec{E}_1 \cdot \hat{t} - \vec{E}_2 \cdot \hat{t} = -\vec{M}_s \cdot (\hat{t} \times \hat{n})$$

Vector identity:

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}}$$

$$\therefore \quad \hat{n} \times (\hat{t} \times \hat{n}) = \hat{t} - (\hat{n} \cdot \hat{t})\hat{n}$$

$$= \hat{t}$$

$$\vec{E}_1 \cdot [\hat{n} \times (\hat{t} \times \hat{n})] - \vec{E}_2 \cdot [\hat{n} \times (\hat{t} \times \hat{n})] = -\vec{M}_s \cdot (\hat{t} \times \hat{n})$$

Vector identity: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

$$\therefore \quad [\vec{E}_1 \times \hat{n} - \vec{E}_2 \times \hat{n}] \cdot (\hat{t} \times \hat{n}) = -\vec{M}_s \cdot (\hat{t} \times \hat{n})$$

$$\therefore \quad \hat{n} \times (\vec{E}_1 - \vec{E}_2) = \vec{M}_s$$

$$\boxed{\hat{n} \times (\vec{E}_2 - \vec{E}_1) = -\vec{M}_s}$$

In reality, $\vec{M}_s = 0$

$$\therefore \quad \hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$$

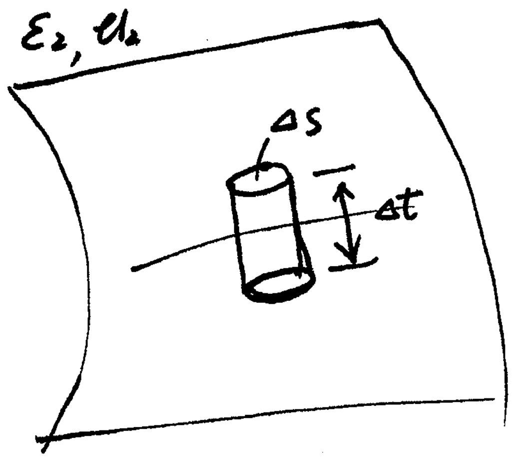
tangential electric field is always continuous

(8) Similarly

$$\boxed{\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s}$$

If there is a surface electric current, the tangential component of the magnetic field is discontinuous, otherwise, it is continuous

How about $\vec{D}$ and $\vec{B}$?



$$\oiint_s \vec{D} \cdot d\vec{s} = Q$$

$$\vec{D}_2 \cdot \hat{n} \Delta s + \vec{D}_1 \cdot (-\hat{n}) \Delta s + \vec{D} \cdot \hat{\rho}\, 2\pi \rho\, \Delta t = \rho_s \Delta s$$

Let $\Delta t \to 0$

$$\boxed{\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \rho_s}$$

Similarly:

$$\boxed{\hat{n} \cdot (\vec{B}_2 - \vec{B}_1) = \rho_{ms}}$$

In reality $\rho_{ms} = 0$

$$\therefore \quad \hat{n} \cdot \left(\vec{B}_2 - \vec{B}_1\right) = 0$$

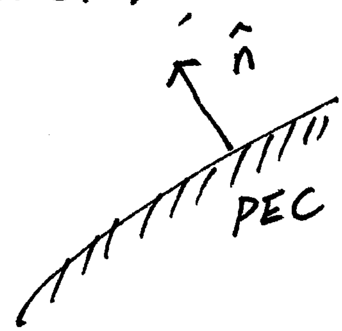
Boundary conditions:

$$\boxed{\begin{aligned} \hat{n} \times \left(\vec{E}_2 - \vec{E}_1\right) &= 0 \quad (\text{or } -\vec{M}_s) \\ \hat{n} \times (\vec{H}_2 - \vec{H}_1) &= \vec{J}_s \\ \hat{n} \cdot (\vec{D}_2 - \vec{D}_1) &= \rho_s \\ \hat{n} \cdot (\vec{B}_2 - \vec{B}_1) &= 0 \quad (\text{or } \rho_{ms}) \end{aligned}}$$

If medium 1 is a perfect conductor,

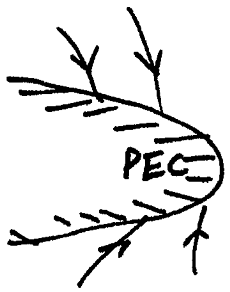
then $\vec{E}_1 = \vec{H}_1 = \vec{D}_1 = \vec{B}_1 = 0$

$$\boxed{\begin{aligned} \hat{n} \times \vec{E} &= 0 & \hat{n} \cdot \vec{D} &= \rho_s \\ \hat{n} \times \vec{H} &= \vec{J}_s & \hat{n} \cdot \vec{B} &= 0 \end{aligned}}$$



10

electric field line



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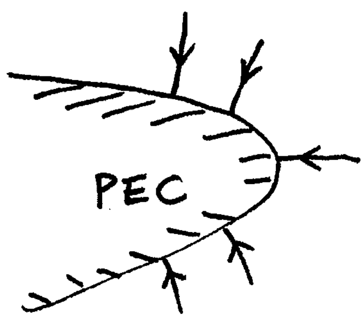

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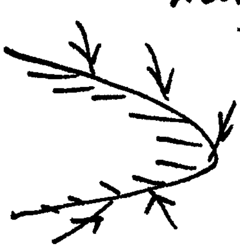
magnetic field line



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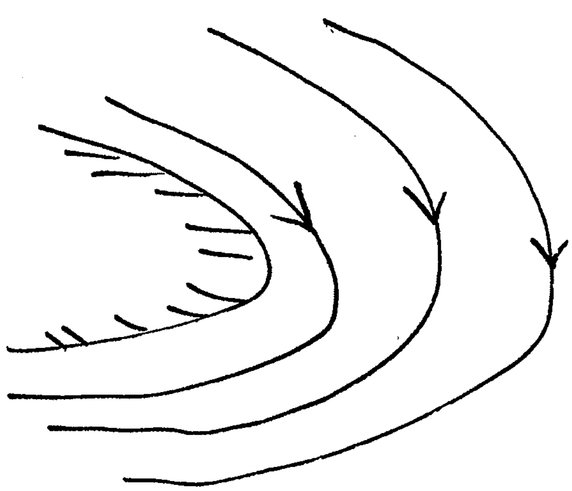
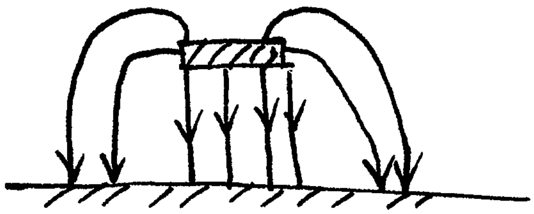
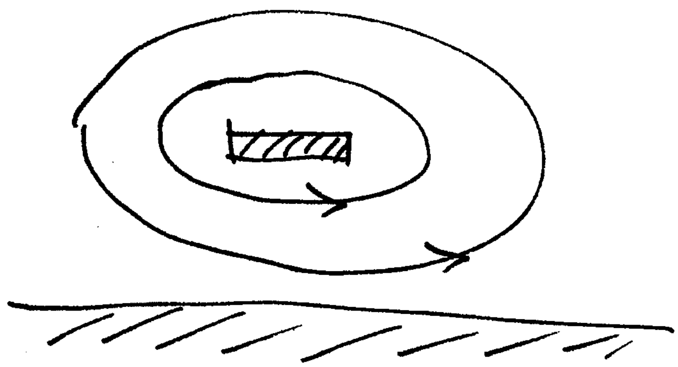


(11)

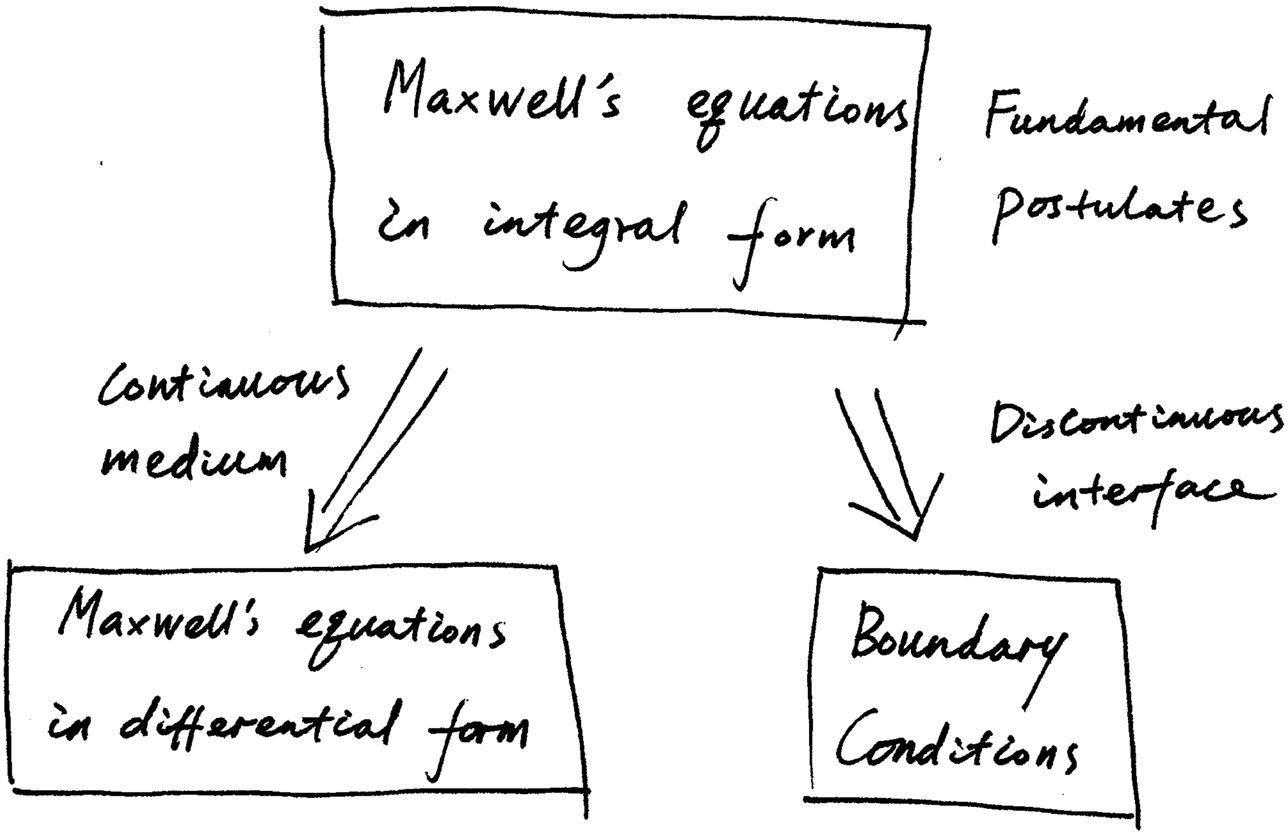


$\vec{E}$ line



$\vec{H}$ line

Discussion:


Maxwell's equations in integral form
Fundamental postulates
Continuous medium
Discontinuous interface
Maxwell's equations in differential form
Boundary Conditions

Another approach:

Maxwell's equations in differential form

Gauss's theorem $\Downarrow$ Stokes's theorem

Maxwell's equations in integral form

$\Downarrow$

Boundary conditions

Elegant or illogical?

14

Another correct approach

Maxwell's equations in differential form

Boundary conditions

Maxwell's equations in integral form