Chapter VIII

- Antenna and Radiation

Interested Readers:

- Textbook 12.1-12.7

Solution of Vector Potential

Vector wave equation:

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

Scalar wave equations

$$\nabla^2 A_x + k^2 A_x = -\mu J_x$$

$$\nabla^2 A_y + k^2 A_y = -\mu J_y$$

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

Green's function G

$$\nabla^2 G(\vec{r}, \vec{r}') + k^2 G(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}')$$

Principle of linear superposition:

$$J_x(\vec{r}) = \iiint_V J_x(\vec{r}')\, \delta(\vec{r} - \vec{r}')\, dV'$$

$$A_x(\vec{r}) = \mu \iiint_V J_x(\vec{r}')\, G(\vec{r}, \vec{r}')\, dV'$$

Place $\vec{r}'$ at $(0, 0, 0)$

$$\nabla^2 G(\vec{r}-\vec{r}', 0) + k^2 G(\vec{r}-\vec{r}', 0) = -\delta(\vec{r}-\vec{r}')$$

Denote $\vec{r}_1 = \vec{r} - \vec{r}'$

$$\nabla^2 G(\vec{r}_1) + k^2 G(\vec{r}_1) = -\delta(\vec{r}_1)$$

For $\vec{r}_1 \neq 0$

$$\nabla^2 G(\vec{r}_1) + k^2 G(\vec{r}_1) = 0$$

$$\nabla^2 G = \frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left(r_1^2 \frac{\partial G}{\partial r_1}\right) + \frac{1}{r_1^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial G}{\partial \theta}\right)$$
$$+ \frac{1}{r_1^2 \sin\theta}\frac{\partial^2 G}{\partial \varphi^2}$$

$$= \frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left(r_1^2 \frac{\partial G}{\partial r_1}\right)$$

$$\therefore \frac{1}{r_1^2}\frac{\partial}{\partial r_1}\left(r_1^2 \frac{\partial G}{\partial r_1}\right) + k^2 G = 0$$

$$\frac{\partial^2 G}{\partial r_1^2} + \frac{2}{r_1}\frac{\partial G}{\partial r_1} + k^2 G = 0$$

Since $$\frac{\partial (r_1 G)}{\partial r_1} = G + r_1 \frac{\partial G}{\partial r_1}$$

$$\frac{\partial^2 (r_1 G)}{\partial r_1^2} = \frac{\partial G}{\partial r_1} + \frac{\partial G}{\partial r_1} + r_1 \frac{\partial^2 G}{\partial r_1^2}$$

$$= r_1 \frac{\partial^2 G}{\partial r_1^2} + 2 \frac{\partial G}{\partial r_1}$$

$$\therefore \quad \frac{\partial^2 (r_1 G)}{\partial r_1^2} + k^2 (r_1 G) = 0$$

Two solutions:

$$r_1 G = C e^{jkr_1} \qquad\qquad r_1 G = C e^{-jkr_1}$$

Non-physical $\qquad\qquad$ physical

Hence,

$$G = C \frac{e^{-jkr_1}}{r_1}$$

To determine C,

$$\iiint_{V_\varepsilon} \left[\nabla^2 G(\vec{r}_1) + k^2 G(\vec{r}_1)\right] dV \qquad \varepsilon \to 0$$

$$= -\iiint_{V_\varepsilon} \delta(\vec{r}_1)\, dV = -1$$

$$\nabla^2 G = \nabla \cdot (\nabla G)$$

$$\underbrace{\oint\!\!\oint_{S_\varepsilon} \nabla G \cdot d\vec{s}}_{= -4\pi C} + k^2 \underbrace{\iiint_{V_\varepsilon} G\, dV}_{= 0} = -1$$

$$\Longrightarrow \quad C = \frac{1}{4\pi}$$

$$\therefore \quad G(\vec{r}_1) = \frac{e^{-jkr_1}}{4\pi r_1}$$

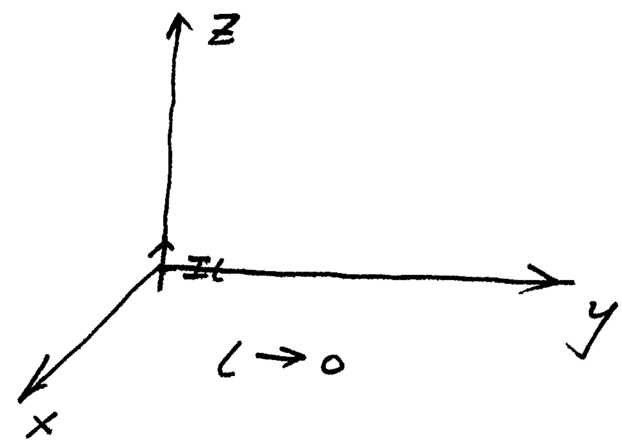
or. $$G(\vec{r}, \vec{r}\,') = \frac{e^{-jk|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}$$

$$\therefore \quad \vec{A}_x(\vec{r}) = \frac{\mu}{4\pi} \iiint_V J_x(\vec{r}\,') \frac{e^{-jk|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\, dV'$$

Similarly for $\vec{A}_y$, $\vec{A}_z$

$$\therefore \quad \vec{A} = \frac{\mu}{4\pi} \iiint_V \vec{J}(\vec{r}\,') \frac{e^{-jk|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}\, dV'$$

Electric dipole antenna



$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

$$\vec{A} = \frac{\mu}{4\pi} \iiint_V \vec{J}(\vec{r}') \frac{e^{-jkR}}{R} dV'$$

$$= \hat{z} \frac{\mu}{4\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{I e^{-jkR}}{R} dz'$$

$$= \hat{z} \frac{\mu I l}{4\pi r} e^{-jkr}$$

In spherical coordinate system:

$$A_r = A_z \cos\theta = \frac{\mu I l}{4\pi r} e^{-jkr} \cos\theta$$

$$A_\theta = -A_z \sin\theta = -\frac{\mu I l}{4\pi r} e^{-jkr} \sin\theta$$

$$A_\phi = 0$$

$$\vec{H} = \frac{1}{u}\nabla\times\vec{A} = \hat{\varphi}\frac{1}{ur}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

$$\boxed{\vec{H} = \hat{\varphi}\,\frac{jkIl\sin\theta}{4\pi r}\left(1+\frac{1}{jkr}\right)e^{-jkr}}$$

$$\vec{E} = \frac{1}{j\omega\varepsilon}\nabla\times\vec{H}$$

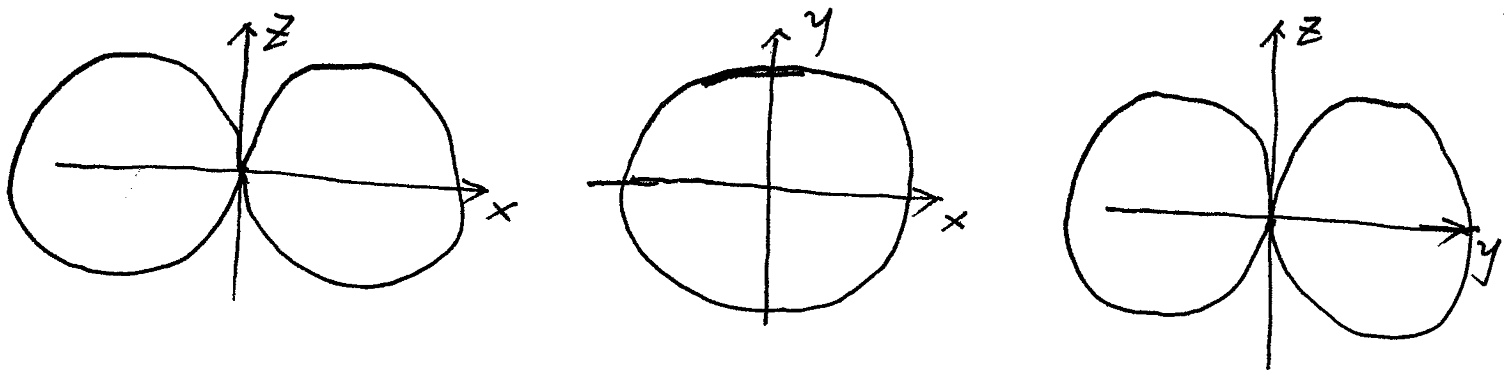
$$\boxed{\begin{aligned} E_r &= \eta\frac{Il\cos\theta}{2\pi r^2}\left(1+\frac{1}{jkr}\right)e^{-jkr} \\ E_\theta &= \eta\,\frac{jkIl\sin\theta}{4\pi r}\left(1+\frac{1}{jkr}\right)e^{-jkr} \\ E_\varphi &= 0 \end{aligned}}$$

When $r \gg \lambda$ (far field zone)

$$E_\theta = \eta\,\frac{jkIl\sin\theta}{4\pi r}e^{-jkr} = \eta H_\varphi$$

$$H_\varphi = \frac{jkIl\sin\theta}{4\pi r}e^{-jkr} = \frac{E_\theta}{\eta}$$

Radiation Pattern



$$P_e = \oiint_S \vec{S} \cdot d\vec{s}$$

$$= \frac{1}{2} \oiint_S \left(\vec{E} \times \vec{H}^* \right) \cdot d\vec{s}$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} E_\theta H_\phi^* r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} \eta \left| \frac{kIl}{4\pi} \right|^2 \sin^3\theta \, d\theta \, d\phi$$

$$= \eta \pi \left| \frac{kIl}{4\pi} \right|^2 \int_0^{\pi} (\cos^2\theta - 1) \, d\cos\theta$$

$$= \eta \frac{4\pi}{3} \left| \frac{kIl}{4\pi} \right|^2 = \eta \frac{\pi}{3} \left| \frac{Il}{\lambda} \right|^2$$

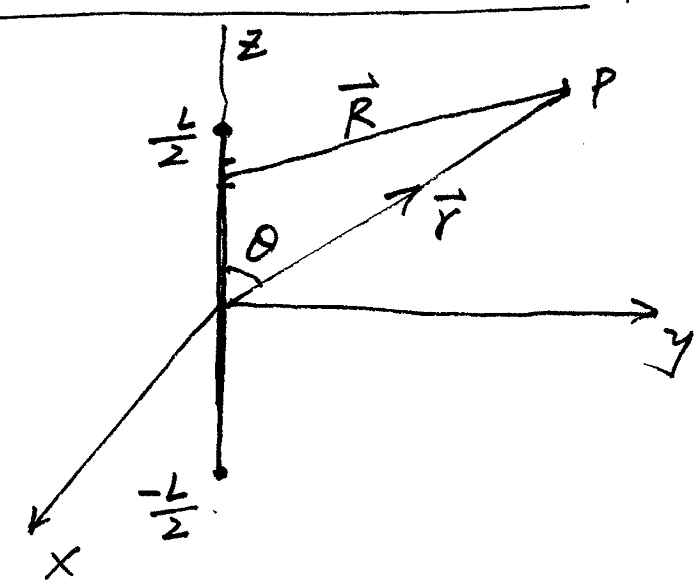
Time average radiated power

$$= Re(P_e) = \eta \frac{\pi}{3} \left| \frac{Il}{\lambda} \right|^2$$

Radiation Resistance

$$R_r = \frac{2 Re(P_e)}{I^2} = 80\pi^2 \left(\frac{l}{\lambda} \right)^2 \Omega$$

Long Straight Wire Antenna



$$I(z) = I_m \sin\left[k\left(\frac{L}{2} - |z|\right)\right]$$

$$\vec{A} = \frac{\mu}{4\pi}\int_C \vec{I}\,\frac{e^{-jkR}}{R}\,dl'$$

$$= \hat{z}\,\frac{\mu}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} I_m \sin\left[k\left(\frac{L}{2} - |z'|\right)\right]\frac{e^{-jkR}}{R}\,dz'$$

$$\simeq \hat{z}\,\frac{\mu}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} I_m \sin\left[k\left(\frac{L}{2} - |z'|\right)\right]\frac{e^{-jk(r - z'\cos\theta)}}{r}\,dz'$$

$$= \hat{z}\,\frac{\mu I_m}{4\pi r}\,e^{-jkr}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin\left[k\left(\frac{L}{2} - |z'|\right)\right]e^{jkz'\cos\theta}\,dz'$$

Consider

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin[k(\tfrac{L}{2} - |z'|)] e^{jkz'\cos\theta} dz'$$

$$= \int_{0}^{\frac{L}{2}} \sin[k(\tfrac{L}{2} - z')] e^{jkz'\cos\theta} dz'$$

$$+ \int_{-\frac{L}{2}}^{0} \sin[k(\tfrac{L}{2} + z')] e^{jkz'\cos\theta} dz'$$

$$= \int_{0}^{\frac{L}{2}} \sin[k(\tfrac{L}{2} - z')] e^{jkz'\cos\theta} dz'$$

$$+ \int_{0}^{\frac{L}{2}} \sin[k(\tfrac{L}{2} - z')] e^{-jkz'\cos\theta} dz'$$

$$= 2\int_{0}^{\frac{L}{2}} \sin[k(\tfrac{L}{2} - z')] \cos(kz'\cos\theta) dz'$$

$$= \frac{2}{k\cos\theta} \int_{0}^{\frac{L}{2}} \sin[k(\tfrac{L}{2} - z')] d\sin(kz'\cos\theta)$$

$$= \frac{2}{k\cos\theta}\left\{ \sin\left[k\left(\frac{L}{2} - z'\right)\right] \sin(kz'\cos\theta)\Big|_0^{\frac{L}{2}} + k\int_0^{\frac{L}{2}} \sin(kz'\cos\theta)\cos\left[k\left(\frac{L}{2} - z'\right)\right] dz' \right\}$$

$$= \frac{2}{\cos\theta}\int_0^{\frac{L}{2}} \sin(kz'\cos\theta)\cos\left[k\left(\frac{L}{2} - z'\right)\right] dz'$$

$$= -\frac{2}{k\cos^2\theta}\int_0^{\frac{L}{2}} \cos\left[k\left(\frac{L}{2} - z'\right)\right] d\cos(kz'\cos\theta)$$

$$= -\frac{2}{k\cos^2\theta}\left\{ \cos\left[k\left(\frac{L}{2} - z'\right)\right]\cos(kz'\cos\theta)\Big|_0^{\frac{L}{2}} - k\int_0^{\frac{L}{2}} \cos(kz'\cos\theta)\sin\left[k\left(\frac{L}{2} - z'\right)\right] dz' \right\}$$

$$= -\frac{2}{k\cos^2\theta}\left[\cos\left(k\frac{L}{2}\cos\theta\right) - \cos\left(k\frac{L}{2}\right)\right] + \frac{I}{\cos^2\theta}$$

$$I = \frac{2\left[\cos\left(k\frac{L}{2}\cos\theta\right) - \cos\left(k\frac{L}{2}\right)\right]}{k\sin^2\theta}$$

$$A_z = \frac{\mu I_m}{4\pi r} e^{-jkr} \frac{2\left[\cos\left(k\frac{L}{2}\cos\theta\right) - \cos\left(k\frac{L}{2}\right)\right]}{k\sin^2\theta}$$

For field:

$$\begin{cases} E_\theta = \dfrac{j\eta I_m e^{-jkr}}{2\pi r}\left[\dfrac{\cos\left(k\frac{L}{2}\cos\theta\right) - \cos\left(k\frac{L}{2}\right)}{\sin\theta}\right] \\ H_\varphi = \dfrac{E_\theta}{\eta} \end{cases}$$

For half-wave dipole antenna

$$L = \frac{\lambda}{2}$$

$$\begin{cases} E_\theta = \dfrac{j\eta I_m e^{-jkr}}{2\pi r} \dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \\ H_\varphi = \dfrac{E_\theta}{\eta} \end{cases}$$