- Spherical Wave Functions
- Spherical Cavity

Interested Readers:

- Textbook 10.7, 10.8
- Balanis 10.2, 10.4

Spherical Wave Functions

Consider Helmholtz's equation

$$(\nabla^2 + k^2)\psi = 0$$

In spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right)$$
$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} + k^2 \psi = 0$$

Separation of variables:

$$\psi(r, \theta, \phi) = f(r)\, g(\theta)\, h(\phi)$$

$$\frac{\sin^2\theta}{f}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \frac{\sin\theta}{g}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right)$$
$$+ \frac{1}{h}\frac{d^2 h}{d\phi^2} + k^2 r^2 \sin^2\theta = 0$$

$$\boxed{\frac{1}{h}\frac{d^2h}{d\varphi^2} = -m^2} \Longrightarrow \boxed{\frac{d^2h}{d\varphi^2} + m^2h = 0}$$

$$\frac{1}{f}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \frac{1}{g\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) + k^2r^2 - \frac{m^2}{\sin^2\theta} = 0$$

$$\boxed{\frac{1}{g\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -n(n+1)}$$

$$\Downarrow$$

$$\boxed{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dg}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]g = 0}$$

$$\boxed{\frac{1}{f}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + k^2r^2 - n(n+1) = 0}$$

$$\Downarrow$$

$$\boxed{\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \left[k^2r^2 - n(n+1)\right]f = 0}$$

Solutions: $h(\varphi) = A_1 e^{jm\varphi} + B_1 e^{-jm\varphi}$

or $h(\varphi) = A_2\cos m\varphi + B_2\sin m\varphi$

$$g(\theta) = C_1 P_n^m(\cos\theta) + D_1 Q_n^m(\cos\theta)$$

$$f(r) = E_1 j_n(kr) + F_1 y_n(kr)$$

or $$f(r) = E_2 h_n^{(2)}(kr) + F_2 h_n^{(1)}(kr)$$

General solution:

$$\psi(r,\theta,\phi) = [E_1 j_n(kr) + F_1 y_n(kr)] \times [C_1 P_n^m(\cos\theta) + D_1 Q_n^m(\cos\theta)] \times [A_2 \cos m\phi + B_2 \sin m\phi]$$

To derive the fields for TM_z waves

$$\vec{A} = \hat{z} A_z = (\hat{r}\cos\theta - \hat{\theta}\sin\theta) A_z$$

$$\vec{F} = 0$$

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0 \Rightarrow \nabla^2 A_z + k^2 A_z = 0$$

For the TE_z waves:

$$\vec{F} = \hat{z} F_z = (\hat{r}\cos\theta - \hat{\theta}\sin\theta) F_z$$

$$\vec{A} = 0$$

$$\nabla^2 \vec{F} + k^2 \vec{F} = 0 \implies \nabla^2 F_z + k^2 F_z = 0$$

More important: $\underline{TM_r \text{ or } TE_r \text{ waves}}$

For TM_r waves:

$$\vec{A} = \hat{r} A_r, \quad \vec{F} = 0$$

Since $\nabla^2 \vec{A} = \nabla^2 (\hat{r} A_r)$

$$\neq \hat{r} \nabla^2 A_r$$

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0 \implies \begin{cases} \nabla^2 A_r + \left(k^2 - \frac{2}{r^2}\right) A_r = 0 \\ \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} = 0 \\ \frac{2}{r^2 \sin\theta} \frac{\partial A_r}{\partial \phi} = 0 \end{cases}$$

Reconsider the vector and scalar potentials

$$\vec{A} \quad \text{and} \quad \phi_e$$

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If we choose

$$\nabla \cdot \vec{A} = -j\omega\mu\varepsilon\phi_e \qquad \text{Lorentz gauge condition}$$

then

$$(\nabla^2 + k^2)\vec{A} = -\mu\vec{J}$$

However, we do not have to make such a choice!

Consider

$$\nabla \times \nabla \times \vec{A} - k^2\vec{A} = -j\omega\mu\varepsilon\nabla\phi_e$$

$$\vec{A} = \hat{r}A_r$$

$$\left.\begin{aligned}
&\frac{1}{r\sin\theta}\left[-\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial A_r}{\partial\theta}\right) - \frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}\right)\right] - k^2 A_r = -j\omega\mu\varepsilon\frac{\partial\phi_e}{\partial r} \\
&\frac{1}{r}\frac{\partial^2 A_r}{\partial r\partial\theta} = -j\frac{\omega\mu\varepsilon}{r}\frac{\partial\phi_e}{\partial\theta} \\
&\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} = -j\frac{\omega\mu\varepsilon}{r\sin\theta}\frac{\partial\phi_e}{\partial\phi}
\end{aligned}\right\}$$

Choose

$$\boxed{\frac{\partial A_r}{\partial r} = -j\omega\mu\varepsilon\,\phi_e}$$

$$\frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2} + k^2 A_r = 0$$

$$\Rightarrow \boxed{(\nabla^2 + k^2)\frac{A_r}{r} = 0}$$

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)A_r \qquad H_r = 0$$

$$E_\theta = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r}\frac{\partial^2 A_r}{\partial r\partial\theta} \qquad H_\theta = \frac{1}{\mu}\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} \qquad H_\phi = -\frac{1}{\mu}\frac{1}{r}\frac{\partial A_r}{\partial\theta}$$

Similarly, for TE_r waves

$$\vec{A} = 0 \qquad \vec{F} = \hat{r} F_r \qquad (\nabla^2 + k^2)\frac{F_r}{r} = 0$$

$$E_r = 0 \qquad H_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + k^2\right)F_r$$

$$E_\theta = -\frac{1}{\varepsilon}\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial \phi} \qquad H_\theta = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r}\frac{\partial^2 F_r}{\partial r \partial \theta}$$

$$E_\phi = \frac{1}{\varepsilon}\frac{1}{r}\frac{\partial F_r}{\partial \theta} \qquad H_\phi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 F_r}{\partial r \partial \phi}$$

Spherical Cavity

TMr modes:

$$\vec{A} = \hat{r} A_r$$

$$(\nabla^2 + k^2)\frac{A_r}{r} = 0$$

$$A_r = [E_1 \hat{J}_n(kr) + F_1 \hat{Y}_n(kr)][C_1 P_n^m(\cos\theta) + D_1 Q_n^m(\cos\theta)][A_2 \cos m\phi + B_2 \sin m\phi]$$

$$\hat{J}_n(kr) = r j_n(kr) \qquad \hat{Y}_n(kr) = r y_n(kr)$$

Since $\hat{Y}_n(kr) \to \infty$ when $r \to 0$, $F_1 = 0$

Since $Q_n^m(\cos\theta) \to \infty$ when $\theta = 0, \pi$, $D_1 = 0$

$$A_r = \hat{J}_n(kr) P_n^m(\cos\theta)[A_2 \cos m\phi + B_2 \sin m\phi]$$

$$E_\theta = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r}\frac{\partial^2 A_r}{\partial r \partial\theta}$$

$$= \frac{k}{j\omega\mu\varepsilon}\frac{\hat{J}_n'(kr)}{r}\frac{\partial P_n^m(\cos\theta)}{\partial\theta}[A_2 \cos m\phi + B_2 \sin m\phi]$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi}$$

$$= \frac{k}{j\omega\mu\varepsilon}\frac{m}{r\sin\theta}\hat{J}_n'(kr)P_n^m(\cos\theta)\left[-A_2\sin m\phi + B_2\cos m\phi\right]$$

Boundary condition: $E_\theta\big|_{r=a} = 0 \quad E_\phi\big|_{r=a} = 0$

$$\Rightarrow \hat{J}_n'(ka) = 0$$

If $\hat{J}_n'(\zeta_{np}') = 0 \quad n = 1, 2, \cdots; \; p = 1, 2, \cdots$

$$k_r a = \zeta_{np}' \Rightarrow \boxed{(f_r)_{mnp} = \frac{\zeta_{np}'}{2\pi a\sqrt{\mu\varepsilon}}}$$

Example: $n=1, \; p=1, \; \zeta_{np}' = 2.744$

$$f_r = \frac{2.744}{2\pi a\sqrt{\mu\varepsilon}}$$

TE_r modes:

$$F_r = \hat{J}_n(kr)P_n^m(\cos\theta)\left(A_2\cos m\phi + B_2\sin m\phi\right)$$

$$E_\theta = -\frac{1}{\varepsilon}\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi}$$

$$= -\frac{1}{\varepsilon}\frac{m}{r\sin\theta}\hat{J}_n(kr)P_n^m(\cos\theta)\left[-A_2\sin m\phi + B_2\cos m\phi\right]$$

$$E_\phi = \frac{1}{\varepsilon}\frac{1}{r}\frac{\partial F_r}{\partial\theta}$$

$$= \frac{1}{\varepsilon}\frac{1}{r}\hat{J}_n(kr)\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\left[A_2 \overset{\cos m\phi}{\cancel{\cos\phi}} + B_2\sin m\phi\right]$$

Boundary condition: $E_\theta\big|_{r=a} = 0 \quad E_\phi\big|_{r=a} = 0$

$$\Longrightarrow \quad \hat{J}_n(ka) = 0$$

Let $\hat{J}_n(\xi_{np}) = 0 \quad p = 1, 2, \ldots$

$$k_r a = \xi_{np} \qquad \boxed{f_{rmnp} = \frac{\xi_{np}}{2\pi a\sqrt{\mu\varepsilon}}}$$