- Circular Waveguides
- Circular Cavity

Interested Readers:

- Textbook 8.9, Balanis 9.1, 9.2
- Textbook 10.5, Balanis 9.3

or

$$A_x = A_r \sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$
$$A_y = A_r \sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$
$$A_z = A_r\cos\theta - A_\theta\sin\theta \tag{II-13b}$$

II.2 VECTOR DIFFERENTIAL OPERATORS

The differential operators of gradient of a scalar ($\nabla\psi$), divergence of a vector ($\nabla \cdot \mathbf{A}$), curl of a vector ($\nabla \times \mathbf{A}$), Laplacian of a scalar ($\nabla^2\psi$), and Laplacian of a vector ($\nabla^2\mathbf{A}$) frequently encountered in electromagnetic field analysis will be listed in the rectangular, cylindrical, and spherical coordinate systems.

II.2.1 Rectangular Coordinates

$$\nabla\psi = \hat{a}_x\frac{\partial\psi}{\partial x} + \hat{a}_y\frac{\partial\psi}{\partial y} + \hat{a}_z\frac{\partial\psi}{\partial z} \tag{II-14}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{II-15}$$

$$\nabla\times\mathbf{A} = \hat{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \tag{II-16}$$

$$\nabla\cdot\nabla\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \tag{II-17}$$

$$\nabla^2\mathbf{A} = \hat{a}_x\nabla^2 A_x + \hat{a}_y\nabla^2 A_y + \hat{a}_z\nabla^2 A_z \tag{II-18}$$

II.2.2 Cylindrical Coordinates

$$\nabla\psi = \hat{a}_\rho\frac{\partial\psi}{\partial\rho} + \hat{a}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{a}_z\frac{\partial\psi}{\partial z} \tag{II-19}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{II-20}$$

$$\nabla\times\mathbf{A} = \hat{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)$$
$$+\hat{a}_z\left(\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right) \tag{II-21}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{II-22}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A} \tag{II-23}$$

or in an expanded form

$$\nabla^2\mathbf{A} = \hat{a}_\rho\left(\frac{\partial^2 A_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_\rho}{\partial\rho} - \frac{A_\rho}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\rho}{\partial\phi^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial^2 A_\rho}{\partial z^2}\right)$$
$$+\hat{a}_\phi\left(\frac{\partial^2 A_\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\rho} - \frac{A_\phi}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi} + \frac{\partial^2 A_\phi}{\partial z^2}\right)$$
$$+\hat{a}_z\left(\frac{\partial^2 A_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial\phi^2} + \frac{\partial^2 A_z}{\partial z^2}\right) \quad \text{(II-23a)}$$

In the cylindrical coordinate system $\nabla^2\mathbf{A} \neq \hat{a}_\rho\nabla^2 A_\rho + \hat{a}_\phi\nabla^2 A_\phi + \hat{a}_z\nabla^2 A_z$ because the orientation of the unit vectors $\hat{a}_\rho$ and $\hat{a}_\phi$ varies with the ρ and ϕ coordinates.

II.2.3 Spherical Coordinates

$$\nabla\psi = \hat{a}_r\frac{\partial\psi}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \quad \text{(II-24)}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \quad \text{(II-25)}$$

$$\nabla\times\mathbf{A} = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right] + \frac{\hat{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]$$
$$+\frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \quad \text{(II-26)}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \quad \text{(II-27)}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A} \quad \text{(II-28)}$$

or in an expanded form

$$\nabla^2\mathbf{A} = \hat{a}_r\left(\frac{\partial^2 A_r}{\partial r^2} + \frac{2}{r}\frac{\partial A_r}{\partial r} - \frac{2}{r^2}A_r + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2}\right.$$
$$\left. - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta}{r^2}A_\theta - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)$$
$$+\hat{a}_\theta\left(\frac{\partial^2 A_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2\sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\theta}{\partial\theta}\right.$$
$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right)$$
$$+\hat{a}_\phi\left(\frac{\partial^2 A_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\phi}{\partial r} - \frac{1}{r^2\sin^2\theta}A_\phi + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial\theta^2}\right.$$
$$+ \frac{\cot\theta}{r^2}\frac{\partial A_\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi}$$
$$\left. + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right) \quad \text{(II-28a)}$$

Circular Waveguides

TE_z modes:

$$\vec{A} = 0 \qquad \vec{F} = \hat{z} F_z(\rho, \phi, z)$$

$$\vec{E} = -\frac{1}{\varepsilon} \nabla \times \vec{F} \Rightarrow E_\rho = -\frac{1}{\varepsilon \rho} \frac{\partial F_z}{\partial \phi}$$

$$E_\phi = +\frac{1}{\varepsilon} \frac{\partial F_z}{\partial \rho}$$

$$E_z = 0$$

$$-j\omega\mu \vec{H} = \nabla \times \vec{E} \Rightarrow H_\rho = -\frac{j}{\omega\mu\varepsilon} \frac{\partial^2 F_z}{\partial \rho \partial z}$$

$$H_\phi = -\frac{j}{\omega\mu\varepsilon} \frac{\partial^2 F_z}{\partial \phi \partial z} \frac{1}{\rho}$$

$$H_z = -\frac{j}{\omega\mu\varepsilon} \left(\frac{\partial^2}{\partial z^2} + k^2\right) F_z$$

$$\nabla^2 F_z + k^2 F_z = 0 \qquad k^2 = \omega^2 \mu \varepsilon$$

$$\frac{\partial^2 F_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F_z}{\partial \phi^2} + \frac{\partial^2 F_z}{\partial z^2} + k^2 F_z = 0$$

Use the method of separation of variables:

$$F_z(\rho, \phi, z) = f(\rho)\, g(\phi)\, h(z)$$

$$\frac{1}{f}\frac{d^2 f}{\partial \rho^2} + \frac{1}{f\rho}\frac{df}{d\rho} + \frac{1}{g\rho^2}\frac{d^2 g}{d\phi^2} + \frac{1}{h}\frac{d^2 h}{dz^2} + k^2 = 0$$

$$\frac{1}{h}\frac{d^2 h}{dz^2} = -k_z^2 \Rightarrow \frac{d^2 h}{dz^2} + k_z^2 h = 0$$

$$h(z) = A e^{-jk_z z} + B e^{jk_z z}$$

$$\frac{1}{f}\frac{d^2 f}{d\rho^2} + \frac{1}{f\rho}\frac{df}{d\rho} + \frac{1}{g\rho^2}\frac{d^2 g}{d\phi^2} + (k^2 - k_z^2) = 0$$

$$\frac{\rho^2}{f}\frac{d^2 f}{d\rho^2} + \frac{\rho}{f}\frac{df}{d\rho} + \frac{1}{g}\frac{d^2 g}{d\phi^2} + (k^2 - k_z^2)\rho^2 = 0$$

$$\frac{1}{g}\frac{d^2g}{d\varphi^2} = -k_\varphi^2 \Rightarrow \frac{d^2g}{d\varphi^2} + k_\varphi^2 g = 0$$

$$g(\varphi) = C\cos k_\varphi \varphi + D\sin k_\varphi \varphi$$

$$g(\varphi + 2\pi) = g(\varphi)$$

$$\Rightarrow \quad \cos k_\varphi(\varphi + 2\pi) = \cos k_\varphi \varphi$$

$$\sin k_\varphi(\varphi + 2\pi) = \sin k_\varphi \varphi$$

$$\Rightarrow \quad k_\varphi = m, \quad m = 0, 1, 2, \dots$$

$$\rho^2 \frac{d^2f}{d\rho^2} + \rho\frac{df}{d\rho} + [(k^2 - k_z^2)\rho^2 - m^2]f = 0$$

$$\rho^2 \frac{d^2f}{d\rho^2} + \rho\frac{df}{d\rho} + [(k_\rho \rho)^2 - m^2]f = 0$$

$$k_\rho^2 = k^2 - k_z^2$$

$$f(\rho) = E J_m(k_\rho \rho) + F Y_m(k_\rho \rho)$$

Bessel functions of the first and second kind

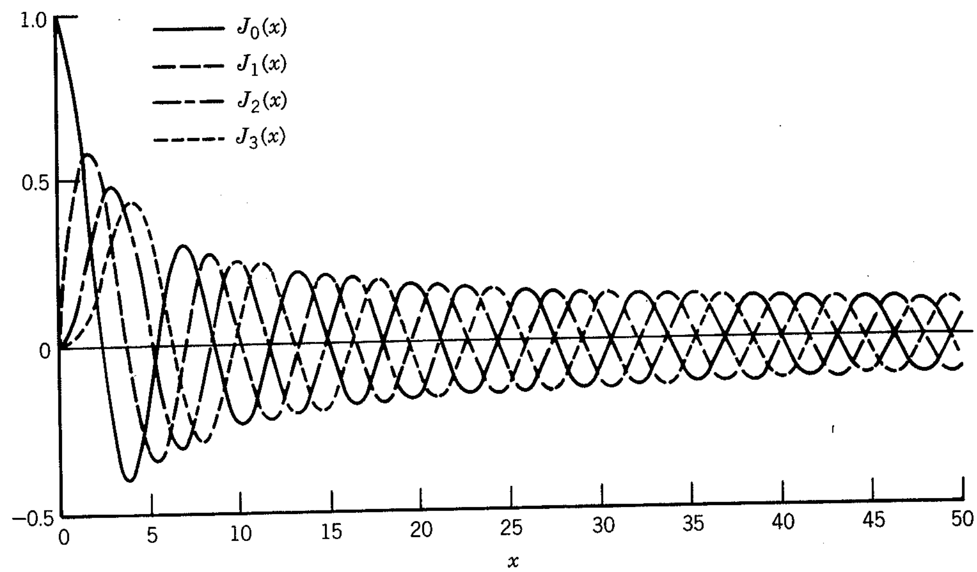


FIGURE IV-1 Bessel functions of the first kind [$J_0(x)$, $J_1(x)$, $J_2(x)$, and $J_3(x)$]. (*Source:* C. A. Balanis, *Antenna Theory: Analysis and Design*, copyright © 1982, John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.)

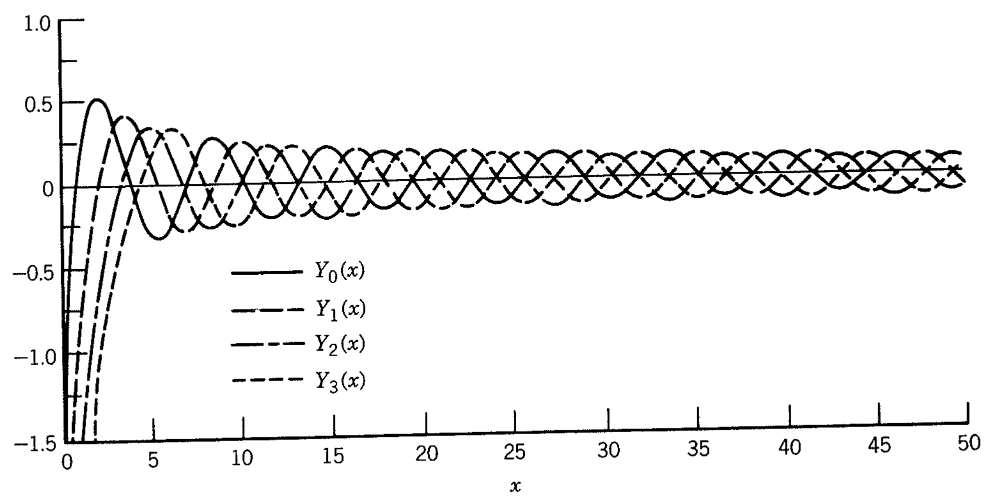


FIGURE IV-2 Bessel functions of the second kind [$Y_0(x)$, $Y_1(x)$, $Y_2(x)$, and $Y_3(x)$]. (*Source:* C. A. Balanis, *Antenna Theory: Analysis and Design*, copyright © 1982, John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.)

Special properties:

1. $J_m(x)$ is finite at $x=0$

   $Y_m(x)$ is infinite at $x=0$

2. $J_m(x) \sim \cos x \quad x \to \infty$

   $Y_m(x) \sim \sin x \quad x \to \infty$

The general solution:

$$F_z(\rho, \phi, z) = f(\rho)\, g(\phi)\, h(z)$$

$$= \left(E J_m(k_\rho \rho) + F Y_m(k_\rho \rho)\right) \times$$

$$(C \cos m\phi + D \sin m\phi) \times$$

$$(A e^{-jk_z z} + B e^{jk_z z})$$

Since F_z should be finite at $\rho=0$

$$\Longrightarrow F = 0$$

Consider the propagation along the z direction

$$F_z = J_m(k_\rho \rho)(C \cos m\phi + D \sin m\phi) e^{-jk_z z}$$

$$k_\rho^2 = k^2 - k_z^2$$

$$E_\phi = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial \rho} = \frac{1}{\varepsilon} k_\rho J_m'(k_\rho \rho)(C\cos m\phi + D\sin m\phi)e^{-jk_z z}$$

$$E_\phi\Big|_{\rho=a} = 0 \implies J_m'(k_\rho a) = 0$$

$J_m'(\chi'_{mn}) = 0$:

	$n=1$	$n=2$	$n=3$
$m=0$	3.8318	7.0156	10.1735
$m=1$	1.8412	5.3315	8.5363

$$k_\rho a = \chi'_{mn} \qquad k_\rho = \frac{\chi'_{mn}}{a}$$

$$k_z = \sqrt{k^2 - k_\rho^2} = \sqrt{k^2 - \left(\frac{\chi'_{mn}}{a}\right)^2} = \begin{cases} \text{real} & k > k_\rho \\ 0 & k = k_\rho \\ \text{imag} & k < k_\rho \end{cases}$$

Cutoff:

$$k_{cmn} = \frac{\chi'_{mn}}{a} = \omega_{cmn}\sqrt{\mu\varepsilon} = 2\pi f_{cmn}\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda_{cmn}}$$

$$f_{cmn} = \frac{\chi'_{mn}}{2\pi a\sqrt{\mu\varepsilon}}$$

Dominant mode: TE_{z11} mode

TM$_z$ modes:

$$\vec{A} = \hat{z} A_z(\rho, \phi, z) \quad \vec{F} = 0$$

$$A_z(\rho, \phi, z) = [E J_m(k_\rho \rho) + F Y_m(k_\rho \rho)] \times (C \cos m\phi + D \sin m\phi) \times (A e^{-jk_z z} + B e^{jk_z z})$$

Since A_z must be finite, $F = 0$

For $+z$ propagation:

$$A_z = J_m(k_\rho \rho)(C \cos m\phi + D \sin m\phi) e^{-jk_z z}$$

$$E_\phi = -\frac{j}{\omega \mu \varepsilon} \frac{1}{\rho} \frac{\partial^2 A_z}{\partial \phi \partial z}$$

$$= \frac{m k_z}{\rho \omega \mu \varepsilon} J_m(k_\rho \rho)(C \sin m\phi - D \cos m\phi) e^{-jk_z z}$$

$$E_\phi \Big|_{\rho = a} = 0 \Longrightarrow J_m(k_\rho a) = 0$$

Denote $J_m(\chi_{mn}) = 0 \quad k_\rho a = \chi_{mn}$

$$E_z^+ = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)A_z^+$$

$$= -jB_{mn}\frac{\beta_\rho^2}{\omega\mu\varepsilon}J_m(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)]e^{-j\beta_z z} \tag{9-23}$$

Application of the boundary condition of (9-20b) using (9-23) gives

$$E_z^+(\rho = a, \phi, z) = -jB_{mn}\frac{\beta_\rho^2}{\omega\mu\varepsilon}J_m(\beta_\rho a)[C_2\cos(m\phi) + D_2\sin(m\phi)]e^{-j\beta_z z} = 0 \tag{9-24}$$

which is only satisfied provided that

$$J_m(\beta_\rho a) = 0 \Rightarrow \beta_\rho a = \chi_{mn} \Rightarrow \beta_\rho = \frac{\chi_{mn}}{a} \tag{9-25}$$

In (9-25) χ_{mn} represents the nth zero ($n = 1, 2, 3, \ldots$) of the Bessel function J_m of the first kind of order m ($m = 0, 1, 2, 3, \ldots$). An abbreviated list of the zeroes χ_{mn} of the Bessel function J_m is found in Table 9-2. The smallest value of χ_{mn} is 2.4049 ($m = 0$, $n = 1$), followed by 3.8318 ($m = 1$, $n = 1$), 5.1357 ($m = 2$, $n = 1$), et cetera.

By using (9-19a) and (9-25), β_z can be written as

$$(\beta_z)_{mn} = \begin{cases} \sqrt{\beta^2 - \beta_\rho^2} = \sqrt{\beta^2 - \left(\dfrac{\chi_{mn}}{a}\right)^2} & \text{(9-26a)} \\ \qquad \text{when } \beta > \beta_\rho = \dfrac{\chi_{mn}}{a} & \\ 0 \qquad \text{when } \beta = \beta_c = \beta_\rho = \dfrac{\chi_{mn}}{a} & \text{(9-26b)} \\ -j\sqrt{\beta_\rho^2 - \beta^2} = -j\sqrt{\left(\dfrac{\chi_{mn}}{a}\right)^2 - \beta^2} & \text{(9-26c)} \\ \qquad \text{when } \beta < \beta_\rho = \dfrac{\chi_{mn}}{a} & \end{cases}$$

By following the same procedure as for the TE^z modes, we can write the expressions for the cutoff frequencies $(f_c)_{mn}$, propagation constant $(\beta_z)_{mn}$, and

TABLE 9-2
Zeroes χ_{mn} of $J_m(\chi_{mn}) = 0$ ($n = 1, 2, 3, \ldots$) of Bessel function $J_m(x)$

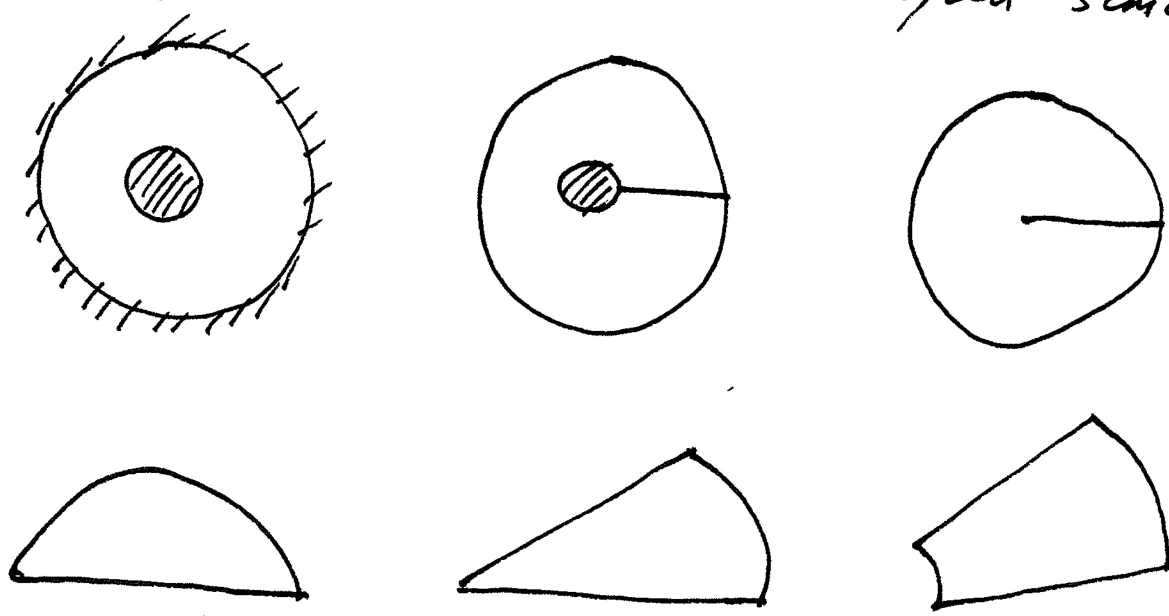
	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$	$m = 11$
$n = 1$	2.4049	3.8318	5.1357	6.3802	7.5884	8.7715	9.9361	11.0864	12.2251	13.3543	14.4755	15.5898
$n = 2$	5.5201	7.0156	8.4173	9.7610	11.0647	12.3386	13.5893	14.8213	16.0378	17.2412	18.4335	19.6160
$n = 3$	8.6537	10.1735	11.6199	13.0152	14.3726	15.7002	17.0038	18.2876	19.5545	20.8071	22.0470	23.2759
$n = 4$	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801	20.3208	21.6415	22.9452	24.2339	25.5095	26.7733
$n = 5$	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178	23.5861	24.9349	26.2668	27.5838	28.8874	30.1791

$$k_\rho = \frac{\chi_{mn}}{a}$$

$$k_z = \sqrt{k^2 - k_\rho^2} = \sqrt{k^2 - \left(\frac{\chi_{mn}}{a}\right)^2}$$

Cutoff: $k_{cmn} = \frac{\chi_{mn}}{a}$

Waveguides that can be analyzed similarly:



Circular Cavity

TE_z modes:

$$\vec{A} = 0, \quad \vec{F} = \hat{z} F_z(\rho, \phi, z)$$

General solution:

$$F_z(\rho, \phi, z) = [E J_m(k_\rho \rho) + F Y_m(k_\rho \rho)] \times$$
$$(C \cos m\phi + D \sin m\phi) \times$$
$$(A \cos k_z z + B \sin k_z z)$$

Applying B.C. to find:

$$F = 0, \quad A = 0, \quad J_m'(k_\rho a) = 0, \quad \sin(k_z h) = 0$$

$$k_\rho a = \chi'_{mn} \qquad k_z h = p\pi \qquad p = 1, 2, \ldots$$

$$k_\rho = \frac{\chi'_{mn}}{a} \qquad k_z = \frac{p\pi}{h}$$

Since $k_\rho^2 + k_z^2 = \omega^2 \mu \varepsilon = k^2$

$$\omega_r = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{\chi'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} \qquad \boxed{TE_{zmnp}}$$

TM_z modes:

$$\vec{F} = 0, \quad \vec{A} = \hat{z} A_z(\rho, \varphi, z)$$

General solution:

$$A_z(\rho, \varphi, z) = [E J_m(k_\rho \rho) + F Y_m(k_\rho \rho)] \times (C \cos m\varphi + D \sin m\varphi) \times (A \cos k_z z + B \sin k_z z)$$

Applying B.C. to find:

$$F = 0, \quad B = 0, \quad J_m(k_\rho a) = 0, \quad \sin k_z h = 0$$

$$k_\rho a = \chi_{mn}, \quad k_z h = p\pi, \quad p = 0, 1, 2, \ldots$$

$$k_\rho = \frac{\chi_{mn}}{a}, \quad k_z = \frac{p\pi}{h}$$

$$\omega_r = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} \qquad \boxed{TM_{zmnp}}$$