# Chapter VII

## Waveguides and Cavities

- TM and TE modes
- Rectangular waveguides

Interested Readers:

- Textbook 8.2, Balanis 6.5.2, 6.5.3
- Textbook 8.7, 8.8, Balanis 8.1, 8.2.1 – 8.2.3

# TM and TE modes

Given $\vec{J}$, $\vec{M}$

Solve: $$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$$

Calculate:

$$\vec{E} = \vec{E}_e + \vec{E}_m$$

$$= -j\omega \vec{A} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot \vec{A}) - \frac{1}{\varepsilon} \nabla \times \vec{F}$$

$$\vec{H} = \vec{H}_e + \vec{H}_m$$

$$= \frac{1}{\mu} \nabla \times \vec{A} - j\omega \vec{F} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot \vec{F})$$

1. Assume $\vec{A} = \hat{z} A_z$ , $\vec{F} = 0$

$$\underline{\vec{E} = -j\omega \hat{z} A_z - \frac{j}{\omega\mu\varepsilon}\left(\hat{x}\frac{\partial^2 A_z}{\partial x \partial z} + \hat{y}\frac{\partial^2 A_z}{\partial y \partial z} + \hat{z}\frac{\partial^2 A_z}{\partial z^2}\right)}$$

$$E_x = -\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x \partial z}$$

$$E_y = -\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y \partial z}$$

$$E_z = -\frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z\right)$$

$$\underline{\vec{H} = \hat{x}\frac{1}{\mu}\frac{\partial A_z}{\partial y} - \hat{y}\frac{1}{\mu}\frac{\partial A_z}{\partial x}}$$

$$H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y} \qquad H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x} \qquad H_z = 0$$

$\Longrightarrow$ $\boxed{\text{Transverse Magnetic } (TM_z) \text{ modes}}$

2. Assume $\vec{A} = 0$, $\vec{F} = \hat{z} F_z$

$$\underline{\vec{E} = -\frac{1}{\varepsilon} \nabla \times \vec{F} = -\hat{x}\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y} + \hat{y}\frac{1}{\varepsilon}\frac{\partial F_z}{\partial x}}$$

$$E_x = -\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y} \qquad E_y = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial x} \qquad E_z = 0$$

$$\underline{\vec{H} = -j\omega F_z \hat{z} - \frac{j}{\omega\mu\varepsilon}\left(\hat{x}\frac{\partial^2 F_z}{\partial x \partial z} + \hat{y}\frac{\partial^2 F_z}{\partial y \partial z} + \hat{z}\frac{\partial^2 F_z}{\partial z^2}\right)}$$

$$H_x = -\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial x \partial z}$$

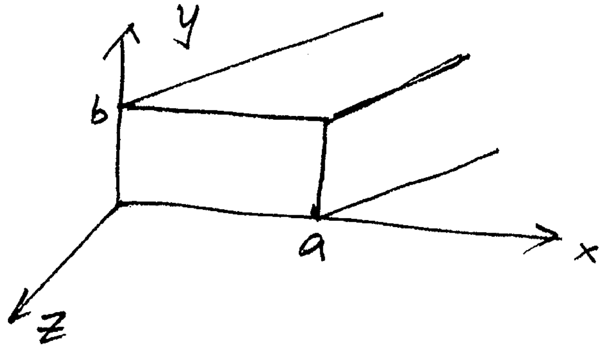
$$H_y = -\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial y \partial z}$$

$$H_z = -\frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_z}{\partial z^2} + k^2 F_z\right)$$

$\Rightarrow$ $\boxed{\text{Transverse Electric } (TE_z) \text{ modes}}$

# Rectangular Waveguides

Empty, infinitely long waveguide




1. $TE_z$ modes

$$\vec{F} = \hat{z} F_z$$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M} \Rightarrow \nabla^2 F_z + k^2 F_z = 0$$

or $$\frac{\partial F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2} + k^2 F_z = 0$$

Let $F_z(x, y, z) = f(x)\, g(y)\, h(z)$

$$gh \frac{\partial^2 f}{\partial x^2} + fh \frac{\partial^2 g}{\partial y^2} + fg \frac{\partial^2 h}{\partial z^2} + k^2 fgh = 0$$

$$\frac{1}{f} \frac{\partial^2 f}{\partial x^2} + \frac{1}{g} \frac{\partial^2 g}{\partial y^2} + \frac{1}{h} \frac{\partial^2 h}{\partial z^2} + k^2 = 0$$

$$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = -k_x^2$$

$$\frac{1}{g}\frac{\partial^2 g}{\partial y^2} = -k_y^2 \qquad k_x^2 + k_y^2 + k_z^2 = k^2$$

$$\frac{1}{h}\frac{\partial^2 h}{\partial z^2} = -k_z^2$$

Consider: $\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = -k_x^2 \Rightarrow \frac{\partial^2 f}{\partial x^2} + k_x^2 f = 0$

Two independent solutions: $e^{jk_x x}$, $e^{-jk_x x}$

or $\sin k_x x$, $\cos k_x x$

$$f(x) = C_1 \cos k_x x + D_1 \sin k_x x$$

Similarly: $g(y) = C_2 \cos k_y y + D_2 \sin k_y y$

$$h(z) = A e^{-jk_z z} + B e^{jk_z z}$$

General solution:

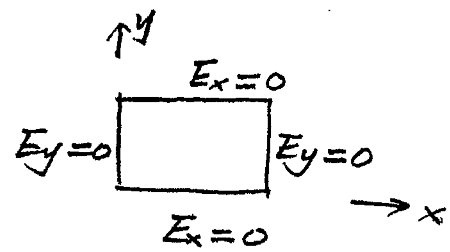
$$F_z(x, y, z) = (C_1 \cos k_x x + D_1 \sin k_x x)(C_2 \cos k_y y + D_2 \sin k_y y)(A e^{-jk_z z} + B e^{jk_z z})$$

For waves traveling in $z$ direction:

$$F_z = (C_1 \cos k_x x + D_1 \sin k_x x)(C_2 \cos k_y y + D_2 \sin k_y y) \times A e^{-jk_z z}$$

$$E_x = -\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y}$$

$$= -\frac{1}{\varepsilon}(C_1 \cos k_x x + D_1 \sin k_x x)(-k_y C_2 \sin k_y y + k_y D_2 \cos k_y y) \cdot A e^{-jk_z z}$$




$$E_x \Big|_{y=0} = 0$$

$$\Rightarrow D_2 = 0$$

$$E_x \Big|_{y=b} = 0$$

$$\Rightarrow E_x \Big|_{y=b} = -\frac{1}{\varepsilon}(C_1 \cos k_x x + D_1 \sin k_x x)(-k_y C_2 \sin k_y b) A e^{-jk_z z} = 0$$

$$\sin k_y b = 0 \Rightarrow k_y b = n\pi \qquad n = 0, 1, 2, \ldots$$

$$k_y = \frac{n\pi}{b}$$

$$E_y = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial x} = \frac{1}{\varepsilon}(-C_1 k_x \sin k_x x + D_1 k_x \cos k_x x)$$
$$\times C_2 \cos k_y y \, A e^{-jk_z z}$$

$$E_y\Big|_{x=0} = 0 \implies D_1 = 0$$

$$E_y\Big|_{x=a} = 0 \implies \sin k_x a = 0$$
$$k_x a = m\pi \qquad m = 0, 1, 2, \ldots$$
$$k_x = \frac{m\pi}{a}$$

Therefore:

$$F_z = C_1 C_2 A \cos k_x x \cos k_y y \, e^{-jk_z z}$$
$$= A_{mn} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-jk_z z}$$

$$E_x = A_{mn}\frac{n\pi}{\varepsilon b} \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-jk_z z}$$

$$E_y = -A_{mn}\frac{m\pi}{\varepsilon a} \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-jk_z z}$$

$$E_z = 0$$

$$H_x = A_{mn} \frac{m\pi k_z}{\omega\mu\varepsilon a} \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$H_y = A_{mn} \frac{n\pi k_z}{\omega\mu\varepsilon b} \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

$$H_z = -\frac{j}{\omega\mu\varepsilon} A_{mn}(k^2 - k_z^2)\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_z z}$$

Consider $k_z$:

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2\mu\varepsilon$$

$$k_z^2 = k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \qquad m, n = 0, 1, 2, \ldots$$

except for $m = n = 0$

$$k_z = \begin{cases} \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} & k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \\ 0 & k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \\ -j\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} & k^2 < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \end{cases}$$

Define $k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ Cutoff wavenumber

$$k_c = \omega_c\sqrt{\mu\varepsilon} = 2\pi f_c\sqrt{\mu\varepsilon}$$

$$f_{cmn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$ Cutoff frequency

$$k_c = \frac{2\pi}{\lambda_c} \qquad \lambda_c = \frac{2\pi}{k_c} = \frac{2\pi}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

Cutoff wavelength

<u>Guide wavelength:</u> $k_z = \frac{2\pi}{\lambda_g}$

$$\lambda_g = \frac{2\pi}{k_z} = \frac{2\pi}{\sqrt{k^2 - k_c^2}} = \frac{2\pi}{k\sqrt{1 - \left(\frac{k_c}{k}\right)^2}}$$

$$= \frac{\lambda}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

<u>Wave impedance,</u>

$$Z_W = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{k_z}$$

$$= \frac{\omega\mu}{\sqrt{k^2 - k_c^2}} = \frac{\omega\mu}{k\sqrt{1 - \left(\frac{k_c}{k}\right)^2}} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}}$$

$$Z_W = \frac{\eta}{\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$= \begin{cases} \text{Real} & f > f_c \text{ or } \lambda < \lambda_c \\ \infty & f = f_c \text{ or } \lambda = \lambda_c \\ \text{imaginary} & f < f_c \text{ or } \lambda > \lambda_c \end{cases}$$

<u>Phase velocity:</u>

$$V_p = \frac{\omega}{k_z} = \frac{\omega}{\sqrt{k^2 - k_c^2}} = \frac{c}{\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}} > c$$

<u>Group velocity:</u>

$$V_g = \frac{1}{\frac{dk_z}{d\omega}} = \frac{1}{\frac{d\sqrt{k^2-k_c^2}}{d\omega}} = \frac{1}{\frac{k}{\sqrt{k^2-k_c^2}}\frac{dk}{d\omega}}$$

$$= \frac{1}{\frac{1}{\sqrt{1-\left(\frac{k_c}{k}\right)^2}}\sqrt{\mu\varepsilon}} = c\sqrt{1-\left(\frac{k_c}{k}\right)^2}$$

$$= c\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2} < c$$

2. $TM_z$ modes

$$\nabla^2 A_z + k^2 A_z = 0$$

$$A_z = (C_1 \cos k_x x + D_1 \sin k_x x)(C_2 \cos k_y y + D_2 \sin k_y y) \times A e^{-jk_z z}$$

$$= \underbrace{D_1 D_2 A}_{B_{mn}} \sin k_x x \sin k_y y \, e^{-jk_z z}$$

$$k_x = \frac{m\pi}{a} \qquad m = 1, 2, ..$$

$$k_y = \frac{n\pi}{b} \qquad n = 1, 2, ..$$

Guide wavelength, phase velocity, group velocity are the same as in the $TE_z$ modes

Wave impedance:

$$\boxed{Z_w = \eta\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

Consider $f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$

$m, n = 0, 1, 2, \ldots$
except for $m = n = 0$

Define the dominant mode as the mode having the lowest cutoff frequency.

Assume $a > b$, the dominant mode: $TE_{z10}$ mode

$m = 1, \; n = 0$

$$f_c = \frac{1}{2a\sqrt{\mu\varepsilon}} \qquad k_c = \frac{\pi}{a} \qquad \lambda_c = 2a$$

$$k_z = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} = k\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}$$

$$= \begin{cases} k\sqrt{1 - \left(\frac{\lambda}{2a}\right)^2} & \text{for } \lambda < 2a \\ -jk\sqrt{\left(\frac{\lambda}{2a}\right)^2 - 1} & \text{for } \lambda > 2a \end{cases}$$

$$E_y = -A_{10}\frac{\pi}{\varepsilon a}\sin\left(\frac{\pi}{a}x\right)e^{-jk_z z}$$

$$E_x = E_z = 0$$