- Auxiliary Potential Functions

- ~~Chapter IV~~

  ~~Review of Transmission Line~~

Interested Readers

- Balanis 6.1, 6.2
- ~~Textbook chapter 5~~

Auxiliary Potential Functions

Maxwell's equations:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{M}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho_e$$

$$\nabla \cdot \vec{B} = \rho_m$$

Field due to $\vec{J}$:

$$\nabla \times \vec{E}_e = -j\omega\mu\vec{H}_e$$

$$\nabla \times \vec{H}_e = j\omega\varepsilon\vec{E}_e + \vec{J}$$

$$\nabla \cdot \vec{D}_e = \rho_e$$

$$\nabla \cdot \vec{B}_e = 0$$

Field due to $\vec{M}$:

$$\nabla \times \vec{E}_m = -j\omega\mu\vec{H}_m - \vec{M}$$

$$\nabla \times \vec{H}_m = j\omega\varepsilon\vec{E}_m$$

$$\nabla \cdot \vec{D}_m = 0$$

$$\nabla \cdot \vec{B}_m = \rho_m$$

Total Field:

$$\boxed{\begin{array}{ll} \vec{E} = \vec{E}_e + \vec{E}_m & \vec{H} = \vec{H}_e + \vec{H}_m \\ \vec{D} = \vec{D}_e + \vec{D}_m & \vec{B} = \vec{B}_e + \vec{B}_m \end{array}}$$

Consider the case of electric Source

Maxwell's equations without $\vec{M}$

$$\begin{cases} \nabla \times \vec{E}_e = -j\omega\mu \vec{H}_e \\ \nabla \times \vec{H}_e = j\omega\varepsilon \vec{E}_e + \vec{J} \\ \nabla \cdot \vec{D}_e = \rho_e \\ \nabla \cdot \vec{B}_e = 0 \end{cases}$$

$\because \nabla \cdot \vec{B}_e = 0$

$$\vec{B}_e = \nabla \times \vec{A}$$

$$\vec{H}_e = \frac{\vec{B}_e}{\mu} = \frac{1}{\mu} \nabla \times \vec{A}$$

$$\nabla \times \vec{E}_e = -j\omega\mu \vec{H}_e = -j\omega \nabla \times \vec{A}$$

$$\nabla \times (\vec{E}_e + j\omega \vec{A}) = 0$$

$$\vec{E}_e + j\omega \vec{A} = -\nabla \phi_e$$

$$\Rightarrow \quad \vec{E}_e = -\nabla \phi_e - j\omega \vec{A}$$

$$\nabla \times \vec{H}_e = \nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A}\right) = j\omega\varepsilon \vec{E}_e + \vec{J}$$

$$= j\omega\varepsilon\left(-\nabla\phi_e - j\omega\vec{A}\right) + \vec{J}$$

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = +\mu\vec{J} - j\omega\mu\varepsilon\nabla\phi_e + k^2\vec{A}$$

$$\nabla^2\vec{A} + k^2\vec{A} = -\mu\vec{J} + \nabla(\nabla\cdot\vec{A} + j\omega\mu\varepsilon\phi_e)$$

Choose

$$\nabla\cdot\vec{A} + j\omega\mu\varepsilon\phi_e = 0$$ Lorentz gauge Condition

$$\boxed{\begin{aligned} &\nabla^2\vec{A} + k^2\vec{A} = -\mu\vec{J} \\ &\vec{E}_e = -j\omega\vec{A} + \frac{1}{j\omega\mu\varepsilon}\nabla(\nabla\cdot\vec{A}) \\ &\vec{H}_e = \frac{1}{\mu}\nabla\times\vec{A} \end{aligned}}$$

$\boxed{\text{Consider the case of magnetic source}}$

$$\nabla \cdot \vec{D}_m = 0 \implies \vec{D}_m = -\nabla \times \vec{F}$$ ← Electric Vector Potential

$$\vec{E}_m = -\frac{1}{\varepsilon} \nabla \times \vec{F}$$

① $$\nabla \times \vec{H}_m = j\omega\varepsilon \vec{E}_m = -j\omega \nabla \times \vec{F}$$

magnetic scalar potential

$$\nabla \times (\vec{H}_m + j\omega \vec{F}) = 0 \implies \vec{H}_m + j\omega \vec{F} = -\nabla \phi_m$$

$$\vec{H}_m = -j\omega \vec{F} - \nabla \phi_m$$

② $$\nabla \times \vec{E}_m = \nabla \times \left(-\frac{1}{\varepsilon} \nabla \times \vec{F}\right) = -j\omega\mu \vec{H}_m - \vec{M}$$

$$= -j\omega\mu(-j\omega \vec{F} - \nabla\phi_m) - \vec{M}$$

$$\nabla \times (\nabla \times \vec{F}) = k^2 \vec{F} - j\omega\mu\varepsilon \nabla \phi_m + \varepsilon \vec{M}$$

$$\nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F} = k^2 \vec{F} - j\omega\mu\varepsilon \nabla\phi_m + \varepsilon \vec{M}$$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M} + \nabla(\nabla \cdot \vec{F} + j\omega\mu\varepsilon \phi_m)$$

Choose $\nabla \cdot \vec{F} = -j\omega\mu\varepsilon \phi_m$ Gauge condition

$$\boxed{\begin{aligned} &\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M} \\ &\vec{E}_m = -\frac{1}{\varepsilon} \nabla \times \vec{F} \\ &\vec{H}_m = -j\omega \vec{F} - \frac{j}{\omega \mu \varepsilon} \nabla (\nabla \cdot \vec{F}) \end{aligned}}$$

Summary

Given $\vec{J}$, $\vec{M}$

Solve: $\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$$

Calculate:

$$\vec{E} = \vec{E}_e + \vec{E}_m$$

$$= -j\omega \vec{A} - \frac{j}{\omega \mu \varepsilon} \nabla (\nabla \cdot \vec{A}) - \frac{1}{\varepsilon} \nabla \times \vec{F}$$

$$\vec{H} = \vec{H}_e + \vec{H}_m$$

$$= \frac{1}{\mu} \nabla \times \vec{A} - j\omega \vec{F} - \frac{j}{\omega \mu \varepsilon} \nabla (\nabla \cdot \vec{F})$$