Chapter VI

Fundamental Theorems and Principles of Electromagnetics

- Duality Theorem
- Reciprocity Theorem

Interested Readers:

- Balanis 7.2
- Balanis Chapter 7

Duality

If equations describing two different phenomena are of same mathematical form, solutions to them will take the Same mathematical form

Dual quantities

Variables occupying same positions in dual equations.

Dual Equations for $\vec{J}$ and $\vec{M}$

Electric sources ($\vec{J} \neq 0, \vec{M} = 0$)

$$\nabla \times \vec{E}_e = -j\omega\mu \vec{H}_e$$

$$\nabla \times \vec{H}_e = j\omega\varepsilon \vec{E}_e + \vec{J}$$

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

$$\vec{H}_e = \frac{1}{\mu} \nabla \times \vec{A}$$

$$\vec{E}_e = -j\omega \vec{A} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot \vec{A})$$

Magnetic sources ($\vec{J} = 0, \vec{M} \neq 0$)

$$\nabla \times \vec{H}_m = j\omega\varepsilon \vec{E}_m$$

$$-\nabla \times \vec{E}_m = +j\omega\mu \vec{H}_m + \vec{M}$$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$$

$$\vec{E}_m = -\frac{1}{\varepsilon} \nabla \times \vec{F}$$

$$\vec{H}_m = -j\omega \vec{F} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot \vec{F})$$

Dual Quantities for $\vec{J}$ and $\vec{M}$

Electric sources ($\vec{J} \neq 0$, $\vec{M} = 0$)	Magnetic sources ($\vec{J} = 0$, $\vec{M} \neq 0$)
$\vec{E}_e$	$\vec{H}_m$
$\vec{H}_e$	$-\vec{E}_m$
$\vec{J}$	$\vec{M}$
$\vec{A}$	$\vec{F}$
ε	μ
μ	ε
k	k
η	$1/\eta$
$1/\eta$	η

Reciprocity Theorem

Consider two sets of sources:

$\vec{J}_1, \vec{M}_1$ $\qquad\qquad$ $\vec{J}_2, \vec{M}_2$

$$\nabla \times \vec{E}_1 = -j\omega\mu \vec{H}_1 - \vec{M}_1 \qquad \nabla \times \vec{E}_2 = -j\omega\mu \vec{H}_2 - \vec{M}_2$$

$$\nabla \times \vec{H}_1 = j\omega\varepsilon \vec{E}_1 + \vec{J}_1 \qquad \nabla \times \vec{H}_2 = j\omega\varepsilon \vec{E}_2 + \vec{J}_2$$

$$\vec{H}_2 \cdot \nabla \times \vec{E}_1 = -j\omega\mu \vec{H}_2 \cdot \vec{H}_1 - \vec{H}_2 \cdot \vec{M}_1$$

$$\vec{E}_1 \cdot \nabla \times \vec{H}_2 = j\omega\varepsilon \vec{E}_1 \cdot \vec{E}_2 + \vec{E}_1 \cdot \vec{J}_2$$

$$\vec{E}_1 \cdot \nabla \times \vec{H}_2 - \vec{H}_2 \cdot \nabla \times \vec{E}_1 = \nabla \cdot (\vec{H}_2 \times \vec{E}_1)$$

$$= j\omega\varepsilon \vec{E}_1 \cdot \vec{E}_2 + j\omega\mu \vec{H}_1 \cdot \vec{H}_2 + \vec{E}_1 \cdot \vec{J}_2 + \vec{H}_2 \cdot \vec{M}_1$$

Similarly,

$$\vec{E}_2 \cdot \nabla \times \vec{H}_1 - \vec{H}_1 \cdot \nabla \times \vec{E}_2 = \nabla \cdot (\vec{H}_1 \times \vec{E}_2)$$

$$= j\omega\varepsilon \vec{E}_1 \cdot \vec{E}_2 + j\omega\mu \vec{H}_1 \cdot \vec{H}_2 + \vec{E}_2 \cdot \vec{J}_1 + \vec{H}_1 \cdot \vec{M}_2$$

$$\nabla\cdot(\vec{H}_2\times\vec{E}_1-\vec{H}_1\times\vec{E}_2)=\vec{E}_1\cdot\vec{J}_2+\vec{H}_2\cdot\vec{M}_1-\vec{E}_2\cdot\vec{J}_1-\vec{H}_1\cdot\vec{M}_2$$

$$\oiint_S(\vec{H}_2\times\vec{E}_1-\vec{H}_1\times\vec{E}_2)\cdot d\vec{s}$$

$$=\iiint_V(\vec{E}_1\cdot\vec{J}_2+\vec{H}_2\cdot\vec{M}_1-\vec{E}_2\cdot\vec{J}_1-\vec{H}_1\cdot\vec{M}_2)\,dV$$

1. If no sources in V, then

$$\oiint_S(\vec{H}_2\times\vec{E}_1-\vec{H}_1\times\vec{E}_2)\cdot d\vec{s}=0$$

or

$$\boxed{\oiint_S(\vec{E}_1\times\vec{H}_2)\cdot d\vec{s}=\oiint_S(\vec{E}_2\times\vec{H}_1)\cdot d\vec{s}}$$

2. In open space, $s\to\infty$: $E_\theta=\eta H_\varphi$, $E_\varphi=-\eta H_\theta$

$$\oiint_S(\vec{H}_2\times\vec{E}_1-\vec{H}_1\times\vec{E}_2)\cdot d\vec{s}$$

$$=\oiint_S(E_{1\varphi}H_{2\theta}-E_{1\theta}H_{2\varphi}-E_{2\varphi}H_{1\theta}+E_{2\theta}H_{1\varphi})\,ds$$

$$=\oiint_S(-\eta H_{1\theta}H_{2\theta}-\eta H_{1\varphi}H_{2\varphi}+\eta H_{2\theta}H_{1\theta}+\eta H_{2\varphi}H_{1\varphi})\,ds$$

$$=0$$

$$\iiint_V \left(\vec{E}_1\cdot\vec{J}_2 + \vec{H}_2\cdot\vec{M}_1 - \vec{E}_2\cdot\vec{J}_1 - \vec{H}_1\cdot\vec{M}_2\right) dV = 0$$

$$\boxed{\iiint_V \left(\vec{E}_1\cdot\vec{J}_2 - \vec{H}_1\cdot\vec{M}_2\right) dV = \iiint_V \left(\vec{E}_2\cdot\vec{J}_1 - \vec{H}_2\cdot\vec{M}_1\right) dV}$$

reciprocity theorem

3. The result above is also valid when there are electric and/or magnetic conductors

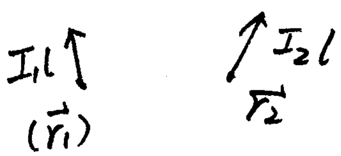
Define reaction:

$$\langle 1,2\rangle = \iiint_V \left(\vec{E}_1\cdot\vec{J}_2 - \vec{H}_1\cdot\vec{M}_2\right) dV$$

$$\langle 2,1\rangle = \iiint_V \left(\vec{E}_2\cdot\vec{J}_1 - \vec{H}_2\cdot\vec{M}_1\right) dV$$

Reciprocity theorem:

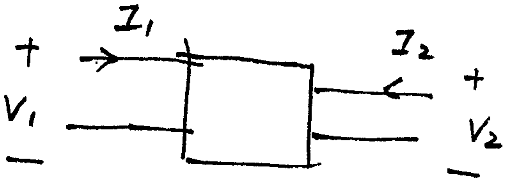
$$\boxed{\langle 1,2\rangle = \langle 2,1\rangle}$$



$$\langle 1,2\rangle = \iiint_V \vec{E}_1\cdot\vec{J}_2\, dV = \vec{E}_1(\vec{r}_2)\cdot\hat{I}_2\, I_2 l$$

$$\langle 2,1\rangle = \iiint_V \vec{E}_2\cdot\vec{J}_1\, dV = \vec{E}_2(\vec{r}_1)\cdot\hat{I}_1\, I_1 l$$

$$\therefore\ \vec{E}_2(\vec{r}_1)\cdot\hat{I}_1\, I_1 = \vec{E}_1(\vec{r}_2)\cdot\hat{I}_2\, I_2$$



$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$Z_{ij} = \frac{V_{ij}}{I_j}$$

$$\langle 1, 2 \rangle = \int \vec{E}_1 \cdot \vec{I}_2 \, dl = I_2 \int \vec{E}_1 \cdot d\vec{l} = -I_2 V_{21}$$

$$\langle 2, 1 \rangle = \int \vec{E}_2 \cdot \vec{I}_1 \, dl = I_1 \int \vec{E}_2 \cdot d\vec{l} = -I_1 V_{12}$$

$$\langle j, i \rangle = -I_i V_{ij} \qquad V_{ij} = -\frac{\langle j, i \rangle}{I_i}$$

$$Z_{ij} = \frac{V_{ij}}{I_j} = -\frac{\langle j, i \rangle}{I_i I_j}$$

$$Z_{ji} = \frac{V_{ji}}{I_i} = -\frac{\langle i, j \rangle}{I_j I_i}$$

Since $\langle j, i \rangle = \langle i, j \rangle$

$$\boxed{Z_{ij} = Z_{ji}}$$