Chapter VI

Fundamental Theorems and Principles of Electromagnetics

- Uniqueness Theorem
- Image Theory

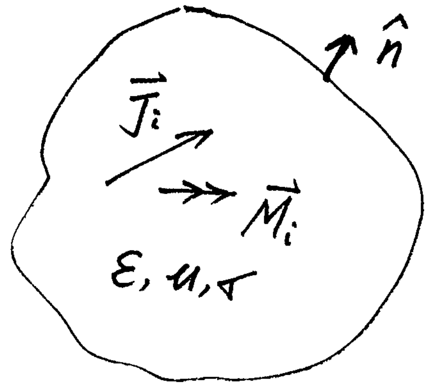
Interested Readers:

- Balanis 7.3
- Balanis Chapter 7

Uniqueness Theorem

Assume two solutions:

$$\vec{E}^a,\ \vec{H}^a;\ \vec{E}^b,\ \vec{H}^b$$



Both satisfy Maxwell's equations and B.C.

$$\begin{cases} \nabla\times\vec{E}^a = -j\omega\mu\vec{H}^a - \vec{M}_i \\ \nabla\times\vec{H}^a = j\omega\varepsilon\vec{E}^a + \sigma\vec{E}^a + \vec{J}_i \end{cases}$$

$$\begin{cases} \nabla\times\vec{E}^b = -j\omega\mu\vec{H}^b - \vec{M}_i \\ \nabla\times\vec{H}^b = j\omega\varepsilon\vec{E}^b + \sigma\vec{E}^b + \vec{J}_i \end{cases}$$

$$\begin{cases} \nabla\times(\vec{E}^a-\vec{E}^b) = -j\omega\mu(\vec{H}^a-\vec{H}^b) \\ \nabla\times(\vec{H}^a-\vec{H}^b) = (j\omega\varepsilon+\sigma)(\vec{E}^a-\vec{E}^b) \end{cases}$$

$$\begin{cases} \nabla\times\delta\vec{E} = -j\omega\mu\,\delta\vec{H} \\ \nabla\times\delta\vec{H} = (j\omega\varepsilon+\sigma)\,\delta\vec{E} \end{cases}$$

$$\delta\vec{H}^* \cdot \nabla\times\delta\vec{E} = -j\omega\mu\,\delta\vec{H}^*\cdot\delta\vec{H} = -j\omega\mu|\delta\vec{H}|^2$$

$$\delta\vec{E}\cdot\nabla\times\delta\vec{H}^* = (-j\omega\varepsilon^* + \sigma)\,\delta\vec{E}\cdot\delta\vec{E}^* = (-j\omega\varepsilon^*+\sigma)|\delta\vec{E}|^2$$

$$\delta\vec{H}^*\cdot\nabla\times\delta\vec{E} - \delta\vec{E}\cdot\nabla\times\delta\vec{H}^* = \nabla\cdot(\delta\vec{E}\times\delta\vec{H}^*)$$

$$= -j\omega\mu|\delta\vec{H}|^2 - (-j\omega\varepsilon^*+\sigma)|\delta\vec{E}|^2$$

$$\iiint_V \nabla\cdot(\delta\vec{E}\times\delta\vec{H}^*)\,dV = \oiint_S (\delta\vec{E}\times\delta\vec{H}^*)\cdot d\vec{s}$$

$$= \iiint_V \left[-j\omega\mu|\delta\vec{H}|^2 + (j\omega\varepsilon^*-\sigma)|\delta\vec{E}|^2\right]dV$$

1. If $\hat{n}\times\vec{E}$ is specified over S, then $\hat{n}\times\delta\vec{E}=0$

$$\oiint_S (\delta\vec{E}\times\delta\vec{H}^*)\cdot\hat{n}\,ds = \oiint_S (\hat{n}\times\delta\vec{E})\cdot\delta\vec{H}^*\,ds = 0$$

2. If $\hat{n}\times\vec{H}$ is specified over S, then $\hat{n}\times\delta\vec{H}=0$

$$\oiint_S (\delta\vec{E}\times\delta\vec{H}^*)\cdot\hat{n}\,ds = -\oiint_S (\hat{n}\times\delta\vec{H}^*)\cdot\delta\vec{E} = 0$$

3. If $\hat{n}\times\vec{E}$ is specified over S_1,

$\hat{n}\times\vec{H}$ is specified over S_2 $(S_1+S_2=S)$

then $\hat{n}\times\delta\vec{E}=0$ on S_1

$\hat{n}\times\delta\vec{H}=0$ on S_2

$$\oiint_S (\delta\vec{E}\times\delta\vec{H}^*)\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2}$$

$$= \iint_{S_1} (\hat{n}\times\delta\vec{E})\cdot\delta\vec{H}^*\,ds - \iint_{S_2} (\hat{n}\times\delta\vec{H}^*)\cdot\delta\vec{E}\,ds$$

$$= 0$$

Thus,

$$\iiint_V \left[-j\omega\mu|\delta\vec{H}|^2 + (j\omega\varepsilon^* - \sigma)|\delta\vec{E}|\right]dV = 0$$

For lossy material,

$$\mu = \mu' - j\mu'' \quad \mu''>0$$

$$\varepsilon = \varepsilon' - j\varepsilon'' \quad \varepsilon''>0$$

$$\iiint_V \left[(-j\omega\mu' - \omega\mu'')|\delta\vec{H}|^2 + (j\omega\varepsilon' - \omega\varepsilon'' - \sigma)|\delta\vec{E}|^2\right]dV = 0$$

Real part:

$$\iiint_V [(\omega\varepsilon'' + \sigma)|\delta\vec{E}|^2 + \omega\mu''|\delta\vec{H}|^2]\, dV = 0$$

Since $\omega\varepsilon'' + \sigma > 0$, $\omega\mu'' > 0$

$$|\delta\vec{E}|^2 = 0, \quad |\delta\vec{H}|^2 = 0$$

$$\Rightarrow \delta\vec{E} = 0, \quad \delta\vec{H} = 0$$

If $\mu'' = 0$, $|\delta\vec{E}|^2 = 0 \Rightarrow \delta\vec{E} = 0$

Imaginary part:

$$\iiint_V [\omega\varepsilon'|\delta\vec{E}|^2 - \omega\mu'|\delta\vec{H}|^2]\, dV = 0$$

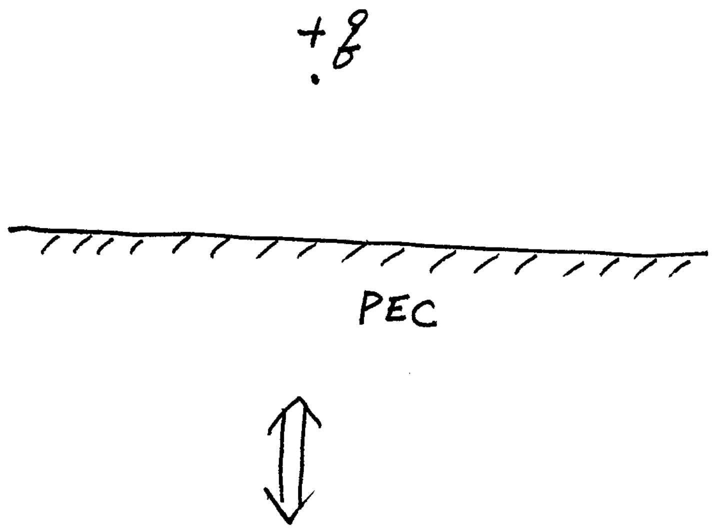
$$\iiint_V \omega\mu'|\delta\vec{H}|^2\, dV = 0$$

$$\Rightarrow |\delta\vec{H}|^2 = 0 \Rightarrow \delta\vec{H} = 0$$

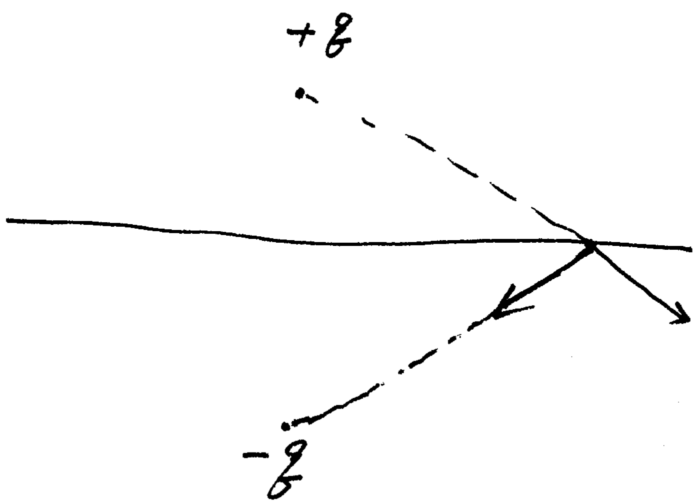
Note: Static and lossless cases are considered as special cases.

Image Theory

Static:

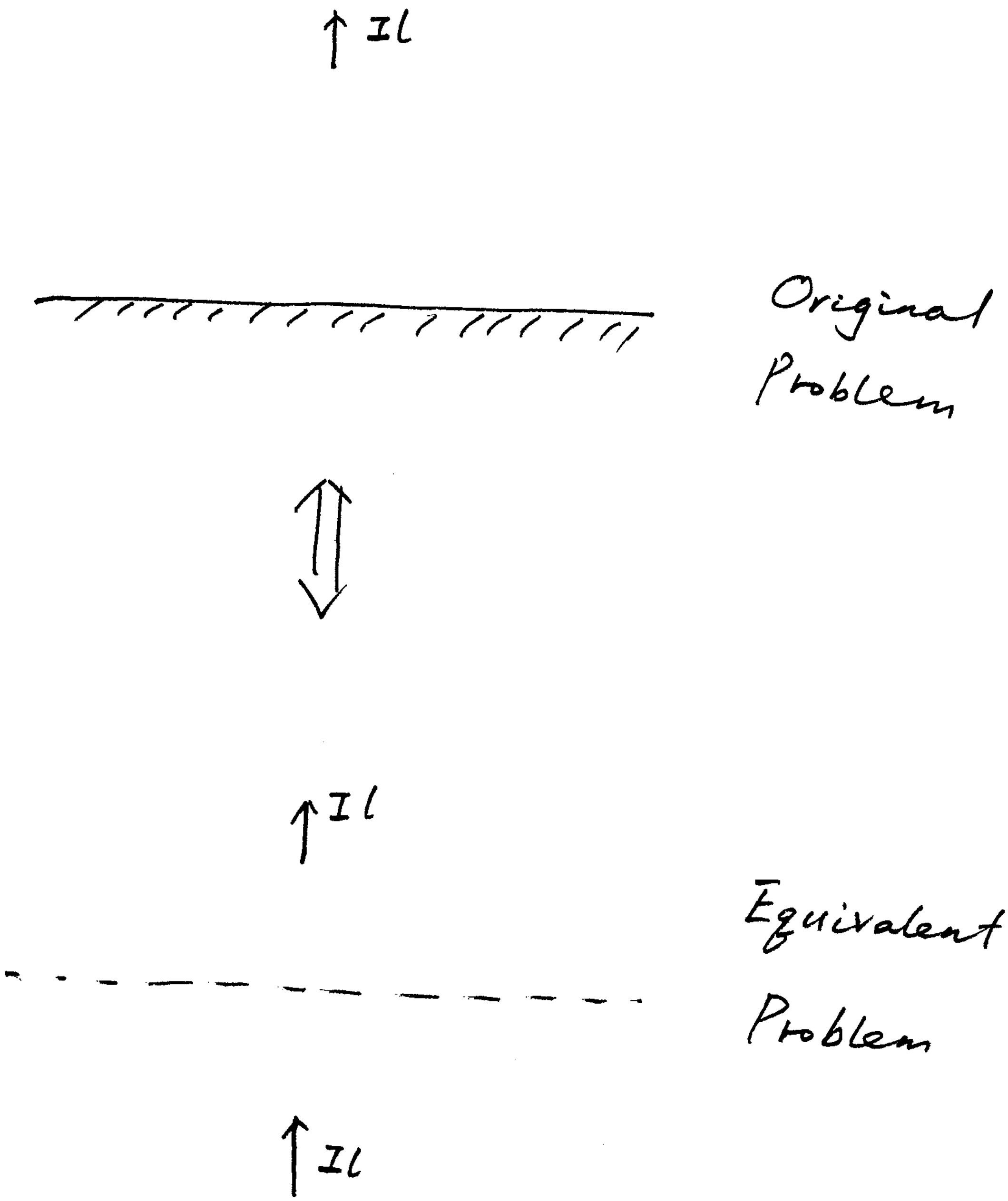


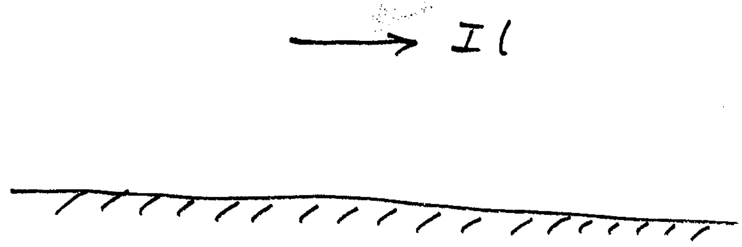
Original Problem



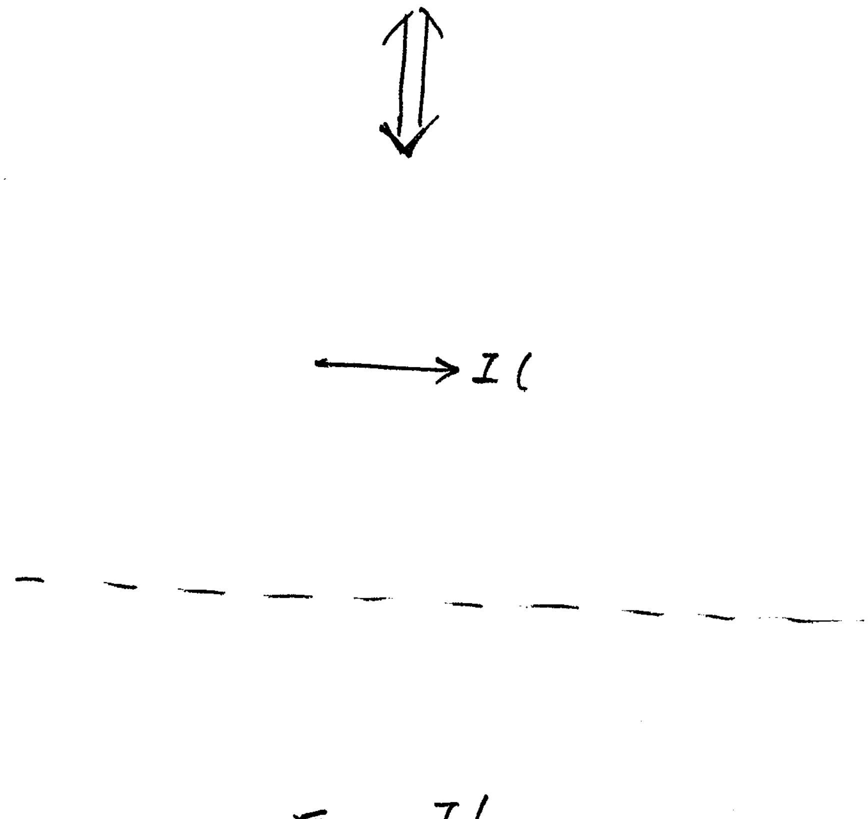
equivalent Problem

Dynamic





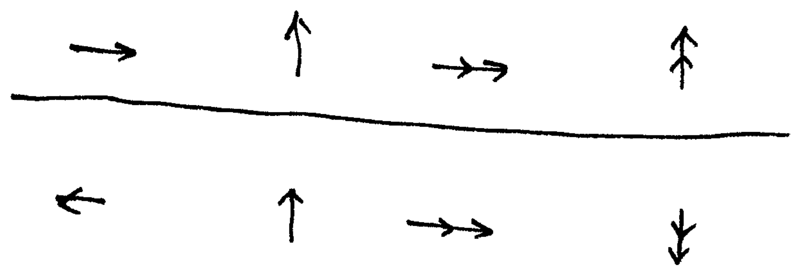
Original Problem



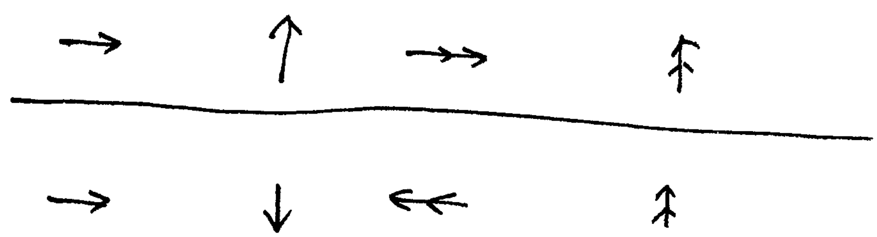
Equivalent Problem

Summary:

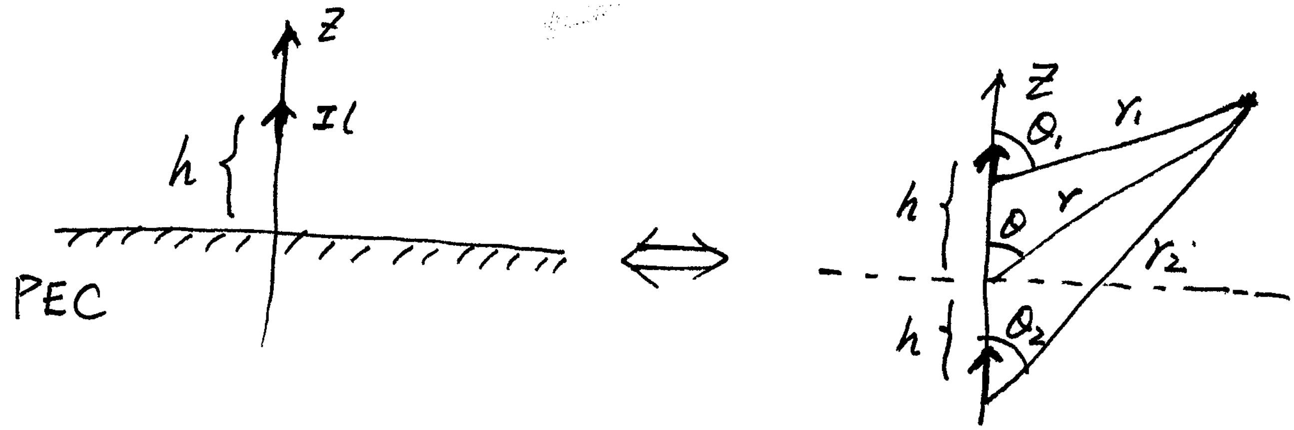
electric conductor



magnetic conductor



Application example:



$$E_\theta^1 = j\eta \frac{kIl}{4\pi r_1} e^{-jkr_1} \sin\theta_1$$

$$E_\theta^2 = j\eta \frac{kIl}{4\pi r_2} e^{-jkr_2} \sin\theta_2$$

$$r_1 = \sqrt{r^2 + h^2 - 2rh\cos\theta} \approx r - h\cos\theta$$

$$r_2 = \sqrt{r^2 + h^2 - 2rh\cos(\pi - \theta)} \approx r + h\cos\theta$$

$$\theta_1 \approx \theta, \quad \theta_2 \approx \theta$$

$$E_\theta = E_\theta^1 + E_\theta^2 = j\eta \frac{kIl}{4\pi r} \sin\theta \left[e^{-jk(r - h\cos\theta)} + e^{-jk(r + h\cos\theta)} \right]$$

$$= j\eta \frac{kIl}{4\pi r} e^{-jkr} \sin\theta \; \underline{2\cos(kh\cos\theta)}$$

$$\theta \le \frac{\pi}{2}$$

Other examples: