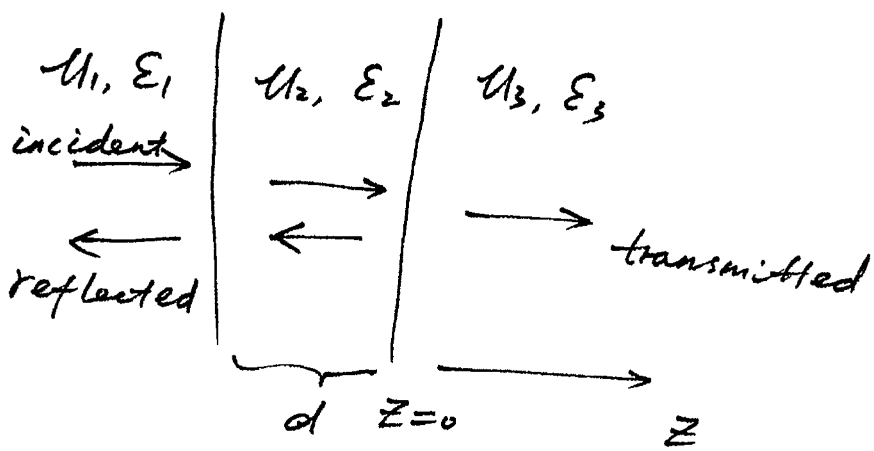
- Reflection and Transmission of Multiple Interfaces

Interested Readers:

- Textbook 6.8, 6.14; Balanis 5.5

Reflection and Transmission of Multiple Interfaces



Electric field:

$$\vec{E}_1 = \hat{x} E_0 e^{-i\beta_1 z} + \hat{x} R E_0 e^{i\beta_1 z}$$

$$\vec{E}_2 = \hat{x} A e^{-i\beta_2 z} + \hat{x} B e^{i\beta_2 z}$$

$$\vec{E}_3 = \hat{x} T E_0 e^{-i\beta_3 z}$$

Magnetic field:

$$\vec{H}_1 = \frac{E_0}{\eta_1}\left(e^{-i\beta_1 z} - R e^{i\beta_1 z}\right)\hat{y}$$

$$\vec{H}_2 = \frac{1}{\eta_2}\left(A e^{-i\beta_2 z} - B e^{i\beta_2 z}\right)\hat{y}$$

$$\vec{H}_3 = \frac{T E_0}{\eta_3} e^{-i\beta_3 z}\,\hat{y}$$

B.C. at $z=-d$:

$$E_0 e^{i\beta_1 d} + RE_0 e^{-i\beta_1 d} = Ae^{i\beta_2 d} + Be^{-i\beta_2 d}$$

$$\frac{E_0}{\eta_1}\left(e^{i\beta_1 d} - Re^{-i\beta_1 d}\right) = \frac{1}{\eta_2}\left(Ae^{i\beta_2 d} - Be^{-i\beta_2 d}\right)$$

B.C. at $z=0$:

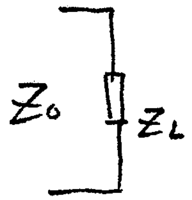
$$A + B = TE_0$$

$$\frac{1}{\eta_2}(A-B) = \frac{TE_0}{\eta_3}$$

Solution for R:

$$R = e^{2i\beta_1 d}\,\frac{\eta_2(\eta_3\cos\beta_2 d + i\eta_2\sin\beta_2 d) - \eta_1(\eta_2\cos\beta_2 d + i\eta_3\sin\beta_2 d)}{\eta_2(\eta_3\cos\beta_2 d + i\eta_2\sin\beta_2 d) + \eta_1(\eta_2\cos\beta_2 d + i\eta_3\sin\beta_2 d)}$$

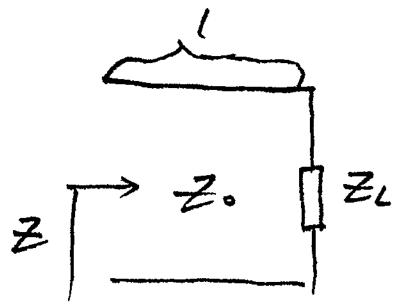
Transmission Line (lossless)



$$R = \frac{Z_L - Z_0}{Z_L + Z_0}$$

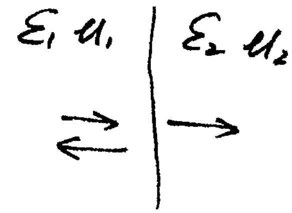
$$\frac{d^2V}{dz^2} + \beta^2 V = 0$$

$$Z = \frac{V(z)}{I(z)}, \quad Z_C = \frac{V^+}{I^+} = -\frac{V^-}{I^-}$$



$$Z = Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l}$$

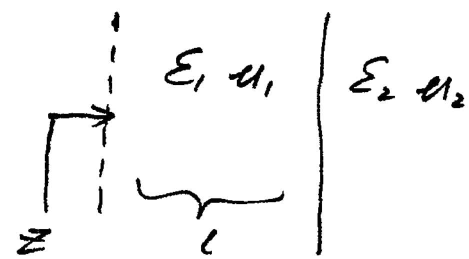
Plane wave (normal inc)



$$R = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{Z_{w2} - Z_{w1}}{Z_{w2} + Z_{w1}}$$

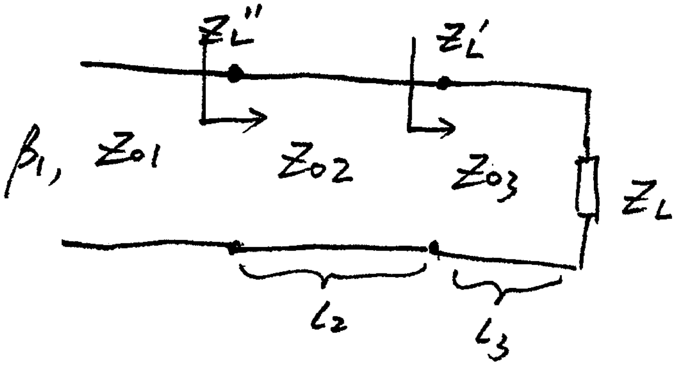
$$\frac{d^2E}{dz^2} + \beta^2 E = 0$$

$$Z = \frac{E}{H}, \quad Z_w = \frac{E^+}{H^+} = -\frac{E^-}{H^-}$$



$$Z = Z_{w1} \frac{Z_{w2} \cos\beta_1 l + jZ_{w1} \sin\beta_1 l}{Z_{w1} \cos\beta_1 l + jZ_{w2} \sin\beta_1 l}$$

Transmission line

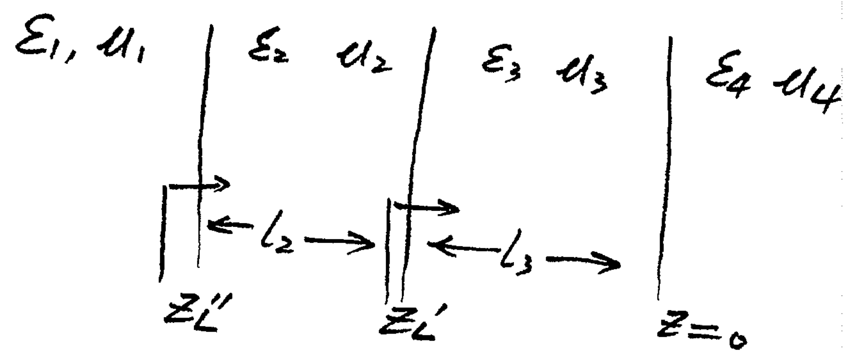


$$Z_L' = Z_{03}\frac{Z_L\cos\beta_3 l_3 + jZ_{03}\sin\beta_3 l_3}{Z_{03}\cos\beta_3 l_3 + jZ_L\sin\beta_3 l_3}$$

$$Z_L'' = Z_{02}\frac{Z_L'\cos\beta_2 l_2 + jZ_{02}\sin\beta_2 l_2}{Z_{02}\cos\beta_2 l_2 + jZ_L'\sin\beta_2 l_2}$$

$$\Gamma\Big|_{Z=Z_L''} = \frac{Z_L''-Z_{01}}{Z_L''+Z_{01}}$$

Plane Wave (normal incidence)



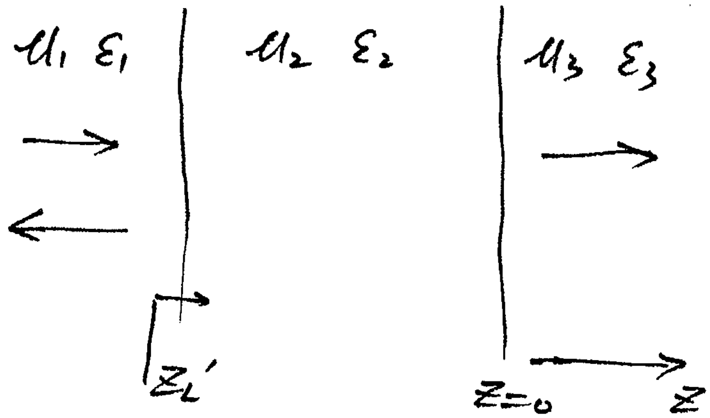
$$Z_L' = Z_{w3}\frac{Z_{w4}\cos\beta_3 l_3 + iZ_{w3}\sin\beta_3 l_3}{Z_{w3}\cos\beta_3 l_3 + iZ_{w4}\sin\beta_3 l_3}$$

$$Z_L'' = Z_{w2}\frac{Z_L'\cos\beta_2 l_2 + jZ_{w2}\sin\beta_2 l_2}{Z_{w2}\cos\beta_2 l_2 + jZ_L'\sin\beta_2 l_2}$$

$$\Gamma\Big|_{Z=Z_L''} = \frac{Z_L''-Z_{w1}}{Z_L''+Z_{w1}}$$

$$\Gamma(z) = \frac{E^r(z)}{E^i(z)} = \frac{RE_0e^{i\beta_1 z}}{E_0e^{-i\beta_1 z}}$$
$$= Re^{2i\beta_1 z}$$

Alternative Solution :



$$Z_L' = \eta_2 \frac{\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d}{\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d}$$

$j\eta_3 \sin\beta_2 d$

$$\Gamma\Big|_{z=-d} = \frac{Z_L' - \eta_1}{Z_L' + \eta_1} = \frac{\eta_2(\eta_3\cos\beta_2 d + j\eta_2 \sin\beta_2 d) - \eta_1(\eta_2\cos\beta_2 d + j\eta_3 \sin\beta_2 d)}{\eta_2(\eta_3\cos\beta_2 d + j\eta_2 \sin\beta_2 d) + \eta_1(\eta_2\cos\beta_2 d + j\eta_3 \sin\beta_2 d)}$$

$$R = \Gamma e^{-2j\beta_1 z} = \Gamma\Big|_{z=-d}\, e^{2j\beta_1 d}$$

Special cases:

1) If $\beta_2 d = n\pi \Rightarrow \frac{2\pi}{\lambda_2} d = n\pi \Rightarrow d = n\frac{\lambda_2}{2}$

then $Z_L' = \eta_3$, $\boxed{\text{If } \eta_3 = \eta_1,\ Z_L' = \eta_1, \Rightarrow \Gamma = 0}$

2) If $\beta_2 d = \frac{\pi}{2}$, $d = \frac{\pi}{2\beta_2} = \frac{\lambda_2}{4}$, then $Z_L' = \eta_2^2/\eta_3$

If $\boxed{\eta_2 = \sqrt{\eta_1 \eta_3}}$, $Z_L' = \eta_1 \Rightarrow \Gamma = 0$

For oblique incidence:

$$Z_{wz}^{+} = \frac{E_x}{H_y} = -\frac{E_y}{H_x}$$

For E-polarization:

$$\vec{E} = \hat{y} E_0 e^{-j(\beta \sin\theta\, x + \beta\cos\theta\, z)}$$

$$\vec{H} = \left(-\hat{x}\cos\theta \frac{E_0}{\eta} + \cdots\right) e^{-j(\beta\sin\theta x + \beta\cos\theta z)}$$

$$\therefore \boxed{Z_{wz}^{+} = -\frac{E_y}{H_x} = \frac{\eta}{\cos\theta}}$$

For H-polarization:

$$\vec{H} = \hat{y}\frac{E_0}{\eta} e^{-j(\beta\sin\theta x + \beta\cos\theta z)}$$

$$\vec{E} = (\hat{x}\cos\theta E_0 + \cdots) e^{-j(\beta\sin\theta x + \beta\cos\theta z)}$$

$$\therefore \boxed{Z_{wz}^{+} = \frac{E_x}{H_y} = \eta\cos\theta}$$

From <u>normal incidence</u> to <u>oblique incidence</u>

- $Z_w \longrightarrow Z_{wz}^{+}$
- $\beta \longrightarrow \beta_z = \beta\cos\theta_t$