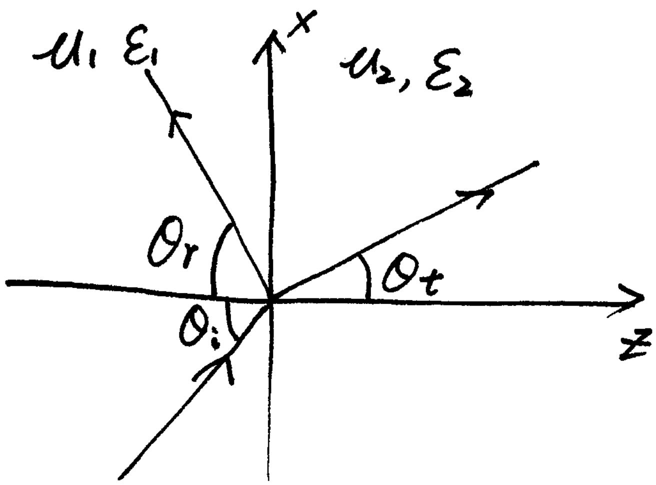
- Reflection and Transmission
  - Oblique Incidence
    - Perpendicular Pol, Parallel Pol
    - Brewster Angle, Critical Angle

Interested Readers:

- Textbook 6.9, Balanis 5.3.1, 5.3.2
- Textbook 6.12-6.13, Balanis 5.3.3, 5.3.4

2. Oblique Incidence

$\boxed{A}$ Perpendicular polarization or E polarization



$$\vec{E}^i = \hat{y} E_0 e^{-j\vec{\beta}^i\cdot\vec{r}} = \hat{y} E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}^r = \hat{y} R_\perp E_0 e^{-j\vec{\beta}^r\cdot\vec{r}} = \hat{y} R_\perp E_0 e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{E}^t = \hat{y} T_\perp E_0 e^{-j\vec{\beta}^t\cdot\vec{r}} = \hat{y} T_\perp E_0 e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\underline{\nabla\times\vec{E} = -j\omega\mu\vec{H} \Rightarrow}$$

$$\vec{H}^i = (-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)\frac{E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}^r = (\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)\frac{R_\perp E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}^t = (-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\frac{T_\perp E_0}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$\hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0$ or $E_{1tan} = E_{2tan} \Rightarrow$

$$E_0 e^{-i\beta_1 \sin\theta_i x} + R_\perp E_0 e^{-i\beta_1 \sin\theta_r x}$$

$$= T_\perp E_0 e^{-i\beta_2 \sin\theta_t x}$$

$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = 0$, or $H_{1tan} = H_{2tan} \Rightarrow$

$$-\cos\theta_i \frac{E_0}{\eta_1} e^{-i\beta_1 \sin\theta_i x} + \cos\theta_r \frac{R_\perp E_0}{\eta_1} e^{-i\beta_1 \sin\theta_r x}$$

$$= -\cos\theta_t \frac{T_\perp E_0}{\eta_2} e^{-i\beta_2 \sin\theta_t x}$$

Phase matching:

$$\boxed{\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t}$$

$\sin\theta_i = \sin\theta_r \Rightarrow \boxed{\theta_i = \theta_r}$ Snell's Law of reflection

$\beta_1 \sin\theta_i = \beta_2 \sin\theta_t \Rightarrow \boxed{\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2 \varepsilon_2}{\mu_1 \varepsilon_1}}}$

$$1 + R_\perp = T_\perp$$

$$-\cos\theta_i \frac{1}{\eta_1} + \cos\theta_r \frac{R_\perp}{\eta_1} = -\cos\theta_t \frac{T_\perp}{\eta_2}$$

Solution:

$$\boxed{R_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \qquad T_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}}$$

$$R_\perp = \frac{\dfrac{\eta_2}{\cos\theta_t} - \dfrac{\eta_1}{\cos\theta_i}}{\dfrac{\eta_2}{\cos\theta_t} + \dfrac{\eta_1}{\cos\theta_i}} = \frac{Z_{z2} - Z_{z1}}{Z_{z2} + Z_{z1}}$$

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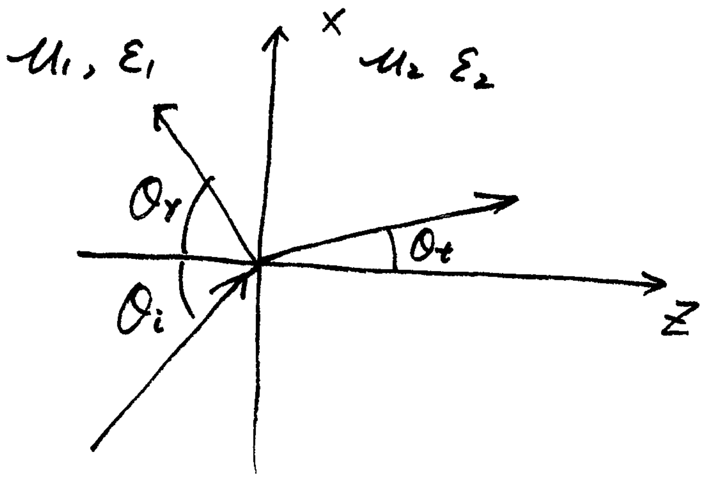
$\boxed{B}$ Parallel polarization or H polarization



$$\vec{H}^i = \hat{y} e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}^r = -\hat{y} R_{\parallel} H_0 e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}^t = \hat{y} T_{\parallel} H_0 e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

$$\frac{Z_{z1}}{Z_{z2}} = \frac{\eta_1 \cos\theta_i}{\eta_2 \cos\theta_t}$$

$\boxed{\text{Result}}$

$$\theta_r = \theta_i$$

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}$$

$$R_{\parallel} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$T_{\parallel} = \frac{2\eta_2\cos\theta_t}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$R_{\parallel} = \frac{Z_{z2} - Z_{z1}}{Z_{z2} + Z_{z1}} \qquad Z_z = \eta\cos\theta$$

Brewster Angle (No reflection)

For (E polarization) $R_\perp = \dfrac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$

If $\eta_2 \cos\theta_{iB} = \eta_1 \cos\theta_t \Rightarrow R_\perp = 0$

$$\sqrt{\frac{\mu_2}{\varepsilon_2}} \cos\theta_{iB} = \sqrt{\frac{\mu_1}{\varepsilon_1}} \cos\theta_t = \sqrt{\frac{\mu_1}{\varepsilon_1}} \sqrt{1-\sin^2\theta_t}$$

From Snell's law:

$$\frac{\sin\theta_{iB}}{\sin\theta_t} = \sqrt{\frac{\mu_2 \varepsilon_2}{\mu_1 \varepsilon_1}}$$

$$\sin^2\theta_t = \frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2} \sin^2\theta_{iB}$$

Solution: $\sin^2\theta_{iB} = \dfrac{\dfrac{\varepsilon_2}{\varepsilon_1} - \dfrac{\mu_2}{\mu_1}}{\dfrac{\mu_1}{\mu_2} - \dfrac{\mu_2}{\mu_1}}$

$$\sin\theta_{iB} = \pm\sqrt{\frac{\dfrac{\varepsilon_2}{\varepsilon_1} - \dfrac{\mu_2}{\mu_1}}{\dfrac{\mu_1}{\mu_2} - \dfrac{\mu_2}{\mu_1}}}$$

$\boxed{\text{For } \mu_1 = \mu_2, \text{ no total transmission unless } \varepsilon_1 = \varepsilon_2}$

For H ~~pto~~ polarization: $R_{\parallel} = \dfrac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$

If $\eta_2 \cos\theta_t = \eta_1 \cos\theta_{iB}$, $R_{\parallel} = 0$

$$\sin\theta_{iB} = \pm\sqrt{\frac{\varepsilon_2/\varepsilon_1 - \mu_2/\mu_1}{\varepsilon_2/\varepsilon_1 - \varepsilon_1/\varepsilon_2}}$$

If $\mu_1 = \mu_2$, $\sin\theta_{iB} = \pm\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}$

Critical Angle (Total reflection)

Snell's law of refraction:

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2 \varepsilon_2}{\varepsilon_1 \mu_1}} \qquad \sin\theta_t = \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} \sin\theta_i$$

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2} \sin^2\theta_i}$$

When $\sin\theta_i = \sqrt{\dfrac{\mu_2 \varepsilon_2}{\mu_1 \varepsilon_1}}$, $\sin\theta_t = 1 \Rightarrow \theta_t = \dfrac{\pi}{2}$

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\mu_2 \varepsilon_2}{\mu_1 \varepsilon_1}}} \Leftarrow \text{critical angle}$$

when $\theta_i = \theta_c$:

$$R_\perp = 1, \quad T_\perp = 2$$

$$\vec{E}^t = \hat{y}\, T_\perp E_0 e^{-i\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$= \hat{y}\, 2E_0 e^{-i\beta_2 x}$$

$$\vec{H}^t = (-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\frac{T_\perp E_0}{\eta_2} e^{-i\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$= \hat{z}\,\frac{2E_0}{\eta_2} e^{-i\beta_2 x}$$

$$\vec{S}_{av}^t \Big|_{\theta_i=\theta_c} = \frac{1}{2}\mathrm{Re}(\vec{E}^t \times \vec{H}^{t*}) = \hat{x}\,\frac{2|E_0|^2}{\eta_2}$$

$$\left|\vec{S}_{av}^i\right|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1}$$

$$\left|\vec{S}_{av}^r\right|_{\theta_i=\theta_c} = \frac{|E_0|^2}{2\eta_1} \Leftarrow \text{All power is reflected back!}$$

Point of discussion: Where does the power in medium 2 come from?

$\boxed{\text{What happens when } \theta_i > \theta_c\text{?}}$

$$\vec{E}^t = \hat{y} T_\perp E_0 e^{-j\beta_2 (x \sin\theta_t + z\cos\theta_t)}$$

When $\theta_i > \theta_c$, $\sin\theta_t > 1$

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \pm j\sqrt{\sin^2\theta_t - 1}$$

$$\vec{E}^t = \hat{y} T_\perp E_0 e^{-j\beta_2 x \sin\theta_t} e^{-\beta_2\sqrt{\sin^2\theta_t - 1}\, z}$$

Attenuation constant:

$$\boxed{\alpha_e = \beta_2\sqrt{\sin^2\theta_t - 1}}$$

Phase velocity: $V_{pe} = \dfrac{\omega}{\beta_2 \sin\theta_t} < V_{p2}$

Application: Optics fiber