- General Uniform Plane Wave
  (Solution of 3D wave equation)
- Reflection and Transmission
  - Normal Incidence

Interested Readers:

- Balanis 4.2.2
- Textbook 6.5-6.7, Balanis 5.2

General Uniform Plane Wave

Wave equation:

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0 \qquad \text{(classnote p57)}$$

Scalar form

$$\begin{cases} \nabla^2 E_x - \gamma^2 E_x = 0 \\ \nabla^2 E_y - \gamma^2 E_y = 0 \\ \nabla^2 E_z - \gamma^2 E_z = 0 \end{cases}$$

Consider

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0$$

Let $E_x = X(x) Y(y) Z(z)$ Separation of variables

$$Y(y) Z(z) \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} - \gamma^2 XYZ = 0$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} - \gamma^2 = 0$$

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$$\begin{cases} \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \gamma_x^2 \\ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \gamma_y^2 \\ \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \gamma_z^2 \\ \gamma_x^2 + \gamma_y^2 + \gamma_z^2 = \gamma^2 \end{cases}$$

$$\frac{\partial^2 X}{\partial x^2} - \gamma_x^2 X = 0 \Longrightarrow A_x e^{\pm \gamma_x x}$$

$$\frac{\partial^2 Y}{\partial y^2} - \gamma_y^2 Y = 0 \Longrightarrow A_y e^{\pm \gamma_y y}$$

$$\frac{\partial^2 Z}{\partial z^2} - \gamma_z^2 Z = 0 \Longrightarrow A_z e^{\pm \gamma_z z}$$

$$E_x = XYZ = A_x e^{\pm \gamma_x x} \cdot A_y e^{\pm \gamma_y y} \cdot A_z e^{\pm \gamma_z z}$$

$$= A_x A_y A_z \, e^{\pm(\gamma_x x + \gamma_y y + \gamma_z z)}$$

$$= A e^{\pm \vec{\gamma} \cdot \vec{r}}$$

$$\vec{\gamma} = \gamma_x \hat{x} + \gamma_y \hat{y} + \gamma_z \hat{z}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Similarly,

$$E_y = B e^{\pm \vec{\gamma} \cdot \vec{r}}$$

$$E_z = C e^{\pm \vec{\gamma} \cdot \vec{r}}$$

$$\therefore \vec{E} = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z$$

$$= \vec{E}_0 e^{\pm \vec{\gamma} \cdot \vec{r}}$$

$\gamma = \alpha + j\beta \quad \vec{E} = \vec{E}_0 e^{-(\vec{\alpha} + j\vec{\beta}) \cdot \vec{r}}$

$$\vec{E}(r, t) = Re[\vec{E}_0 e^{-(\vec{\alpha} + j\vec{\beta}) \cdot \vec{r}} e^{j\omega t}]$$

$$= |\vec{E}_0| e^{-\vec{\alpha} \cdot \vec{r}} \cos(\omega t - \vec{\beta} \cdot \vec{r} + \angle \vec{E}_0)$$

Equi-amplitude: $\vec{\alpha} \cdot \vec{r} = \text{constant}$

Equi-phase: $\vec{\beta} \cdot \vec{r} = \text{constant}$

Plane wave propagating along the $\vec{\beta}$ direction

$$\text{phase} = \omega t - \vec{\beta}\cdot\vec{r} + \angle\vec{E}_0$$

Equi-phase plane: $\omega t - \vec{\beta}\cdot\vec{r} + \angle\vec{E}_0 = C$

<u>Consider</u> $\vec{r} = r\hat{\beta}$

$$\vec{V}_p = \frac{d\vec{r}}{dt} = \hat{\beta}\frac{\omega}{\beta} \Leftarrow \text{phase velocity}$$

<u>Consider</u> the phase velocity along any $\vec{r}$:

$$\omega t - \hat{\beta}\beta\cdot\hat{r}r + \angle\vec{E}_0 = C$$

$$\vec{V}_{ap} = \hat{r}\frac{dr}{dt} = \hat{r}\frac{\omega}{\beta}\frac{1}{\hat{\beta}\cdot\hat{r}} \Leftarrow \text{Apparent phase velocity}$$

$$\lambda_{ap} = \frac{V_{ap}}{f} = \frac{\lambda}{\hat{r}\cdot\hat{\beta}} \Leftarrow \text{Apparent wavelength}$$

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For $e^{\alpha x}$, $\frac{d}{dx} e^{\alpha x} = \alpha e^{\alpha x}$

For $e^{\vec{\gamma}\cdot\vec{r}}$, $\nabla e^{\vec{\gamma}\cdot\vec{r}} = \vec{\gamma} e^{\vec{\gamma}\cdot\vec{r}}$

$$\nabla\cdot\left(\hat{e} e^{\vec{\gamma}\cdot\vec{r}}\right) = \vec{\gamma}\cdot\hat{e}\, e^{\vec{\gamma}\cdot\vec{r}}$$

$$\nabla\times\left(\hat{e} e^{\vec{\gamma}\cdot\vec{r}}\right) = \vec{\gamma}\times\hat{e}\, e^{\vec{\gamma}\cdot\vec{r}}$$

Consider a uniform plane wave

$$\vec{E} = \vec{E}_0 e^{-j\vec{\beta}\cdot\vec{r}} \qquad \vec{H} = \vec{H}_0 e^{-j\vec{\beta}\cdot\vec{r}}$$

$$\nabla\times\vec{E} = -j\omega\mu\vec{H} \qquad \nabla\times\vec{H} = j\omega\varepsilon\vec{E}$$

$$-j\vec{\beta}\times\vec{E}_0 e^{-j\vec{\beta}\cdot\vec{r}} = -j\omega\mu\vec{H} \qquad -j\vec{\beta}\times\vec{H}_0 e^{-j\vec{\beta}\cdot\vec{r}} = j\omega\varepsilon\vec{E}$$

$$\Downarrow \qquad \Downarrow$$

$$\vec{\beta}\times\vec{E} = \omega\mu\vec{H} \qquad \vec{\beta}\times\vec{H} = -\omega\varepsilon\vec{E}$$

$*$ $\boxed{\vec{\beta}\cdot\vec{E} = 0, \quad \vec{\beta}\cdot\vec{H} = 0}$

$\Rightarrow$ For lossless plane waves, $\vec{E}$ and $\vec{H}$ are always perpendicular to $\vec{\beta}$

$*$ $\vec{E}\times(\vec{\beta}\times\vec{E}) = \omega\mu\vec{E}\times\vec{H}$

$\because$ $\boxed{\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}}$

$$\vec{E}\times(\vec{\beta}\times\vec{E}) = (\vec{E}\cdot\vec{E})\vec{\beta} - (\vec{E}\cdot\vec{\beta})\vec{E} = E^2\vec{\beta}$$

$$\boxed{\omega\mu\vec{E}\times\vec{H} = E^2\vec{\beta}} \Rightarrow \vec{E},\ \vec{H},\ \vec{\beta} \text{ form a triad}$$

$$* \quad \vec{\beta}\times(\vec{\beta}\times\vec{E}) = \omega\mu\,\vec{\beta}\times\vec{H} = -\omega^2\mu\varepsilon\vec{E} = -k^2\vec{E}$$

$$\because \quad \vec{\beta}\times(\vec{\beta}\times\vec{E}) = -\vec{\beta}\cdot\vec{\beta}\vec{E}$$

$$\therefore \quad \vec{\beta}\cdot\vec{\beta}\vec{E} = k^2\vec{E} \Rightarrow \boxed{\vec{\beta}\cdot\vec{\beta} = k^2}$$ Dispersion relation

Summary:

$$\beta = \omega\sqrt{\mu\varepsilon}$$

$$Z_W = \frac{E}{H} = \frac{H\beta/\omega\varepsilon}{H} = \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

$$\vec{V}_P = \hat{\beta}\frac{\omega}{\beta} = \hat{\beta}\frac{1}{\sqrt{\mu\varepsilon}}$$

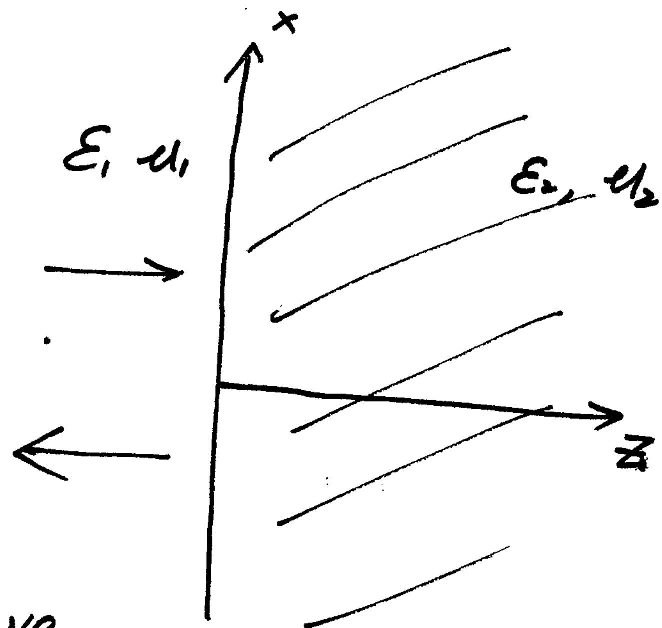
$$\vec{W}_e = \frac{1}{4}\varepsilon|E_0|^2$$

$$\vec{W}_m = \frac{1}{4}\mu|H_0|^2 = \frac{1}{4}\varepsilon|E_0|^2 = \vec{W}_e$$

$$\boxed{\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{|E_0|^2}{2\omega\mu}\vec{\beta} = \frac{|E_0|^2}{2\eta}\hat{\beta}}$$

Reflection and Transmission

1. Normal incidence



Consider a plane wave:

$$\vec{E}^i = \hat{x} E_0 e^{-j\beta_1 z} \qquad \beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$$

$$\vec{E}^r = \hat{x} R E_0 e^{j\beta_1 z}$$

$$\vec{E}^t = \hat{x} T E_0 e^{-j\beta_2 z} \qquad \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

$$\nabla\times\vec{E} = -j\omega\mu\vec{H} \Longrightarrow$$

$$\vec{H}^i = \hat{y}\frac{E_0}{\eta_1} e^{-j\beta_1 z} \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\vec{H}^r = -\hat{y}\frac{RE_0}{\eta_1} e^{j\beta_1 z}$$

$$\vec{H}^t = \hat{y}\frac{TE_0}{\eta_2} e^{-j\beta_2 z} \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

BC: $\hat{n}\times(\vec{E}_1-\vec{E}_2)=0$, $\hat{n}\times(\vec{H}_1-\vec{H}_2)=0$

$$\vec{E}_1=\vec{E}^i+\vec{E}^r \qquad \vec{E}_2=\vec{E}^t$$

$$\vec{H}_1=\vec{H}^i+\vec{H}^r \qquad \vec{H}_2=\vec{H}^t$$

$$E_0+RE_0=TE_0 \Rightarrow 1+R=T$$

$$\frac{E_0}{\eta_1}-\frac{RE_0}{\eta_1}=\frac{TE_0}{\eta_2} \Rightarrow \frac{1}{\eta_1}(1-R)=\frac{1}{\eta_2}T$$

Solution: $\boxed{R=\frac{\eta_2-\eta_1}{\eta_2+\eta_1}, \quad T=\frac{2\eta_2}{\eta_1+\eta_2}}$

Special cases:

1. If $\eta_2=\eta_1$, then $R=0$, $T=1$
2. If $\eta_2=0$, then $R=-1$, $T=0$

$$\vec{E}_1=\hat{x}E_0(e^{-j\beta_1 z}-e^{j\beta_1 z})$$

$$=-\hat{x}E_0\, 2j\sin\beta_1 z \Rightarrow \text{Pure standing wave}$$

3. $$SWR=\frac{1+|R|}{1-|R|}=\frac{1+\left|\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\right|}{1-\left|\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\right|}$$