Chapter V: Plane Wave Propagation and Reflection

- Uniform plane wave in a lossless medium
- Uniform plane wave in a lossy medium

Interested Readers:

- Textbook 6.2, 6.4, Balanis 4.1-4.3

Chapter V

Plane wave Propagation and Reflection

Uniform Plane Waves

Consider

$$\nabla \times \vec{E} = -j\omega\mu \vec{H} - \vec{M_i}$$

$$\nabla \times \vec{H} = j\omega\varepsilon \vec{E} + \sigma \vec{E} + \vec{J_i}$$

* For a lossless medium, ε and μ are real, and $\sigma = 0$

$$\nabla \times (\nabla \times \vec{E}) = -j\omega\mu \nabla \times \vec{H} - \nabla \times \vec{M_i}$$

$$= -j\omega\mu [j\omega\varepsilon \vec{E} + \vec{J_i}] - \nabla \times \vec{M_i}$$

$$= \omega^2 \mu\varepsilon \vec{E} - j\omega\mu \vec{J_i} - \nabla \times \vec{M_i}$$

$$\because \nabla \times \nabla \times \vec{E} = \nabla \nabla \cdot \vec{E} - \nabla^2 \vec{E} = \nabla \left(\frac{\rho_e}{\varepsilon} \right) - \nabla^2 \vec{E}$$

$$\nabla^2 \vec{E} + \omega^2 \mu\varepsilon \vec{E} = j\omega\mu \vec{J_i} + \nabla \times \vec{M_i} + \nabla \left(\frac{1}{\varepsilon} \rho_e \right)$$

Similarly

$$\nabla^2 \vec{H} + \omega^2 \mu\varepsilon \vec{H} = j\omega\varepsilon \vec{M_i} - \nabla \times \vec{J_i} + \nabla \left(\frac{\rho_m}{\mu} \right)$$

* For a source-free region, $\vec{J_i} = \vec{M_i} = \rho_e = \rho_m = 0$

Wave equation

$$\boxed{\begin{aligned}\nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} = 0 \\ \nabla^2 \vec{H} + \omega^2 \mu \varepsilon \vec{H} = 0\end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned}\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0 \\ \nabla^2 \vec{H} - \gamma^2 \vec{H} = 0\end{aligned}}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \gamma = j\beta = j\omega\sqrt{\mu\varepsilon}$$

Consider a special case:

$$\vec{E} = \hat{x} E_x(z)$$

Wave equation $\Longrightarrow \dfrac{d^2 E_x}{dz^2} - \gamma^2 E_x = 0$

Two solutions: $Ae^{-\gamma z}$, $Be^{\gamma z}$

Consider $E_x^+ = Ae^{-\gamma z}$

$\nabla \times \vec{E} = -j\omega\mu\vec{H} \Longrightarrow H_y^+ = \dfrac{\gamma}{j\omega\mu} Ae^{-\gamma z}$

$$= \sqrt{\frac{\varepsilon}{\mu}} Ae^{-\gamma z}$$

Define the intrinsic impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

then

$$H_y^+ = \frac{A}{\eta} e^{-\gamma z}$$

Wave impedance:

$$Z_W = \frac{E_x^+}{H_y^+} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

In time domain:

$$E_x^+(z,t) = Re[E_x^+ e^{j\omega t}] = Re[A e^{j(\omega t - \beta z)}]$$

$$= A\cos(\omega t - \beta z)$$

$\Longrightarrow$ Plane, uniform wave

Equi-phase plane:

$$\omega t - \beta z_p = \text{constant}$$

$$\omega - \beta \frac{dz_p}{dt} = 0$$

$$\Rightarrow \boxed{\text{phase velocity: } V_p = \frac{dz_p}{dt} = \frac{\omega}{\beta}}$$

In this case, $\beta = \omega\sqrt{\mu\varepsilon}$, $V_p = \frac{1}{\sqrt{\mu\varepsilon}}$

In vacuum, $\mu = \mu_0$, $\varepsilon = \varepsilon_0$, $V_p = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8\ m/s$

(Speed of light)

$\boxed{\text{Wavelength}}$ $\lambda = T V_p = \frac{1}{f} V_p = \frac{2\pi}{\omega} V_p = \frac{2\pi}{\beta}$

$$\Rightarrow \boxed{\beta = \frac{2\pi}{\lambda}} \quad \text{Wave number}$$

$$E_x^+(z,t) = A\cos(\omega t - \beta z)$$

$$H_y^+(z,t) = \frac{A}{\eta}\cos(\omega t - \beta z)$$

$$W_e = \frac{1}{2}\varepsilon|E|^2 = \frac{\varepsilon A^2}{2}\cos^2(\omega t - \beta z)$$

$$W_m = \frac{1}{2}\mu|H|^2 = \frac{\mu A^2}{2\eta^2}\cos^2(\omega t - \beta z) = \frac{\varepsilon A^2}{2}\cos^2(\omega t - \beta z)$$

$$\vec{S}(z,t) = \vec{E}\times\vec{H}) = \hat{z}\frac{A^2}{\eta}\cos^2(\omega t - \beta z)$$

$\boxed{\text{Energy velocity}}$

$$V_e = \frac{\text{Power flow density}}{\text{energy density}} = \frac{S}{W_e + W_m} =$$

$$= \frac{1}{\sqrt{\mu\varepsilon}}$$

Group velocity:

Consider $\vec{E} = \hat{x} E_0 \cos[(\omega+\Delta\omega)t - (\beta+\Delta\beta)z]$

$$+ \hat{x} E_0 \cos[(\omega-\Delta\omega)t - (\beta-\Delta\beta)z]$$

$$= \hat{x}\, 2E_0 \cos(\Delta\omega t - \Delta\beta z)\cos(\omega t - \beta z)$$

$\Delta\omega t - \Delta\beta z = \text{constant}$ ← wave packet (envelope)

$$\boxed{V_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta}} = \frac{1}{\frac{\Delta\beta}{\Delta\omega}} = \frac{1}{\frac{d\beta}{d\omega}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

Since $V_p = \dfrac{\omega}{\beta}$

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{V_p}\right) = \frac{1}{V_p} - \frac{\omega}{V_p^2}\frac{dV_p}{d\omega}$$

$$V_g = \frac{V_p}{1 - \frac{\omega}{V_p}\frac{dV_p}{d\omega}}$$

(1) No dispersion $\frac{dV_p}{d\omega}=0 \Rightarrow V_g=V_p$

(2) Normal dispersion $\frac{dV_p}{d\omega}<0 \Rightarrow V_g<V_p$

(3) Anomalous dispersion $\frac{dV_p}{d\omega}>0 \Rightarrow V_g>V_p$

Standing Waves

$$E_x = Ae^{-j\beta z} + Ae^{j\beta z} = 2A\cos\beta z$$

$$H_y = \frac{2A}{j\eta}\sin\beta z$$

$$E_x(z,t) = Re(E_x e^{j\omega t}) = 2A\cos\beta z\cos\omega t$$

$$H_y = \frac{2A}{\eta}\sin\beta z\sin\omega t$$

1. The phase is independent of z, or $V_p=0$

2. $W_e = \frac{1}{2}\varepsilon|E|^2 = 2\varepsilon A^2\cos^2\beta z\cos^2\omega t$

$W_m = \frac{1}{2}\mu|H|^2 = 2\varepsilon A^2\sin^2\beta z\sin^2\omega t$

$$\vec{S}(z,t) = \vec{E} \times \vec{H}$$

$$= \hat{z}\,\frac{4A^2}{\eta}\cos\beta z\cos\omega t\sin\beta z\sin\omega t$$

$$= \hat{z}\,\frac{A^2}{\eta}\sin(2\beta z)\sin(2\omega t)$$

$$\hat{\vec{S}} = \frac{1}{2}\vec{E}\times\vec{H}^* = \hat{z}\,\frac{2jA^2}{\eta}\sin\beta z\cos\beta z$$

$$= \hat{z}\,\frac{jA^2}{\eta}\sin(2\beta z)$$

Time-average power flow $= Re(\hat{\vec{S}}) = 0$

$\Longrightarrow$ no power flow on average !

$\boxed{\text{For a more general case}}$

$$E_x = Ae^{-j\beta z} + Be^{j\beta z}$$

$$= A(\cos\beta z - j\sin\beta z) + B(\cos\beta z + j\sin\beta z)$$

$$= (A+B)\cos\beta z - j(A-B)\sin\beta z$$

$$|E_x| = \sqrt{(A+B)^2\cos^2\beta z + (A-B)^2\sin^2\beta z} = \sqrt{A^2+B^2+2AB\cos(2\beta z)}$$

$$\angle E_x = \tan^{-1}\left[\frac{-(A-B)}{A+B}\tan\beta z\right]$$

$$|E_x|_{max} = |A| + |B|$$

$$|E_x|_{min} = |A| - |B|$$

Standing wave ratio (SWR)

$$SWR = \frac{|E_x|_{max}}{|E_x|_{min}} = \frac{|A| + |B|}{|A| - |B|} = \frac{1 + \frac{|B|}{|A|}}{1 - \frac{|B|}{|A|}}$$

$$= \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

For a pure traveling wave: $B = 0$, $SWR = 1$

For a pure standing wave: $B = A$, $SWR = \infty$

For a general case, $1 \le SWR < \infty$

Uniform plane wave in a lossy medium

$$\nabla \times \vec{E} = -j\omega\mu \vec{H} - \vec{M}_i$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \sigma\vec{E} + \vec{J}_i$$

$$= (j\omega\varepsilon + \sigma)\vec{E} + \vec{J}_i$$

For a source-free region:

$$\nabla^2\vec{E} - \gamma^2\vec{E} = 0 \qquad \boxed{\gamma^2 = j\omega\mu(j\omega\varepsilon + \sigma)}$$

$$Z_w = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon + \sigma}}$$

Consider a simple case: $\vec{E} = \hat{x} E_x(z)$

$$\frac{d^2 E_x}{dz^2} - \gamma^2 E_x = 0 \qquad \gamma = \alpha + j\beta$$

$$E_x^+ = A e^{-\gamma z} = A e^{-\alpha z} e^{-j\beta z}$$

$$E_x^- = B e^{\gamma z} = B e^{\alpha z} e^{j\beta z}$$

How to find α and β ?

$$\gamma = \sqrt{j\omega\mu(j\omega\varepsilon + \sigma)} = \alpha + j\beta$$

$$-\omega^2\mu\varepsilon = \alpha^2 - \beta^2, \quad \omega\mu\sigma = 2\alpha\beta$$

$$\beta = \frac{\omega\mu\sigma}{2\alpha}, \quad \alpha^2 - \frac{(\omega\mu\sigma)^2}{4\alpha^2} = -\omega^2\mu\varepsilon$$

$$\alpha^4 + \omega^2\mu\varepsilon\,\alpha^2 - \frac{1}{4}(\omega\mu\sigma)^2 = 0$$

$$\alpha = \pm\,\omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right)}$$

$$\beta = \pm\,\omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right)}$$

$$\alpha = \pm|\alpha|, \quad \beta = \pm|\beta|$$

Four solutions:

$$\gamma_1 = |\alpha| + j|\beta| \Longleftarrow \text{desired solution}$$

$$\gamma_2 = -|\alpha| + j|\beta|$$

$$\gamma_3 = |\alpha| - j|\beta|$$

$$\gamma_4 = -|\alpha| - j|\beta|$$

$$\boxed{\alpha = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}}$$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}}$$

$$\boxed{\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon+\sigma}}}$$

1. For good dielectric, $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 \ll 1$

$$\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} = 1 + \frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^4 + \cdots$$

$$\approx 1 + \frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2$$

$$\alpha \doteq \omega\sqrt{\mu\varepsilon}\,\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right) = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\beta \doteq \omega\sqrt{\mu\varepsilon}, \quad \eta \doteq \sqrt{\frac{\mu}{\varepsilon}}$$

2. For good conductor, $\frac{\sigma}{\omega\varepsilon} \gg 1$

$$\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} = \frac{\sigma}{\omega\varepsilon} + \frac{1}{2}\left(\frac{\omega\varepsilon}{\sigma}\right) - \frac{1}{8}\left(\frac{\omega\varepsilon}{\sigma}\right)^3 + \cdots$$

$$\alpha \approx \omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{2\omega\varepsilon}} = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\beta \approx \sqrt{\frac{\omega\mu\sigma}{2}}, \quad \eta \approx \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j)$$

3. For perfect conductor, $\sigma = \infty$

$$\alpha \to \infty, \quad \beta \to \infty, \quad \eta \to 0$$

4. For perfect dielectric, $\sigma = 0$

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu\varepsilon}, \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

<u>Skin depth</u>: the amplitude of the wave reduces its value to $e^{-1} \approx 36.8\%$

$$E_x^+ = Ae^{-\gamma z} = Ae^{-\alpha z}e^{-j\beta z}$$

$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\mu\varepsilon}\sqrt{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}} \quad (m)$$

For perfect conductor, $\delta \to 0$;

For perfect dielectric, $\delta \to \infty$;

For good conductor, $\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f\mu\sigma}}$

For good dielectric, $\delta = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$