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## CHAPTER 1

### PRELIMINARY BACKGROUND

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This chapter presents the fundamentals of electromagnetic theory necessary for reading this book. Many mathematical techniques discussed herein could be adapted for other kinds of waves. We will, however, illustrate most of the techniques with electromagnetic waves and fields. The material in this chapter is also discussed in many textbooks which are given in the reference list.

The electromagnetic field can sometimes be described by the scalar wave equation, but in most cases, it can only be described by the vector wave equation. In many instances, the mathematical techniques explained in this book can be illustrated more clearly using scalar wave equations. Since acoustic waves are always described by the scalar wave equation, the derivation of the acoustic wave equation for inhomogeneous medium is also given in Section 1.2 (on the topic of scalar wave equation).

#### §1.1 Maxwell's Equations

Maxwell's equations were established by James Clerk Maxwell in 1873. Prior to that time, the equations existed in incomplete forms as a result of the work of Faraday, Ampere, Gauss, and Poisson. Later, Maxwell added a displacement current term to the equations. Also, this was important to prove that an electromagnetic field could exist as waves. Finally, the wave nature of Maxwell's equations was verified experimentally by Heinrich Hertz in 1888. Even though the earth's surface is curved, with the aid of the ionosphere which reflects radio waves, Guglielmo Marconi was able to send a radio wave across the Atlantic Ocean in 1901. Since then, the importance of Maxwell's equations has been demonstrated in optics, microwaves, antennas, communications, radar, and many sensing applications.

##### §§1.1.1 *Differential Representations*

In vector notation and SI units, Maxwell's equations in differential representations are

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t), \quad (1.1.1)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t), \quad (1.1.2)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \quad (1.1.3)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t), \quad (1.1.4)$$

where  $\mathbf{E}$  is the electric field in volts/m,  $\mathbf{H}$  is the magnetic field in amperes/m,  $\mathbf{D}$  is the electric flux in coulombs/m<sup>2</sup>,  $\mathbf{B}$  is the magnetic flux in webers/m<sup>2</sup>,<sup>1</sup>  $\mathbf{J}(\mathbf{r}, t)$  is the current density in amperes/m<sup>2</sup>, and  $\rho(\mathbf{r}, t)$  is the charge density in coulombs/m<sup>3</sup>.

For a time-varying electromagnetic field, Equations (3) and (4) of the above Maxwell's equations can be derived from Equations (1) and (2). For example, taking the divergence of (1) gives rise to (3). Taking the divergence of (2) and using the continuity equation,

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0, \quad (1.1.5)$$

we arrive at (4).

For static problems where  $\partial/\partial t = 0$ , the electric field and the magnetic field are decoupled. In this case, Equations (3) and (4) cannot be derived from Equations (1) and (2). Then, the electric field Equations (1) and (4) are to be solved independently from the magnetic field Equations (2) and (3). However, in practice, a current is carried by a conductor. Unless the conductor is a superconductor, the current would have to be driven by an electric field or a voltage. Therefore, the magnetic field may never be completely decoupled from the electric field in statics.

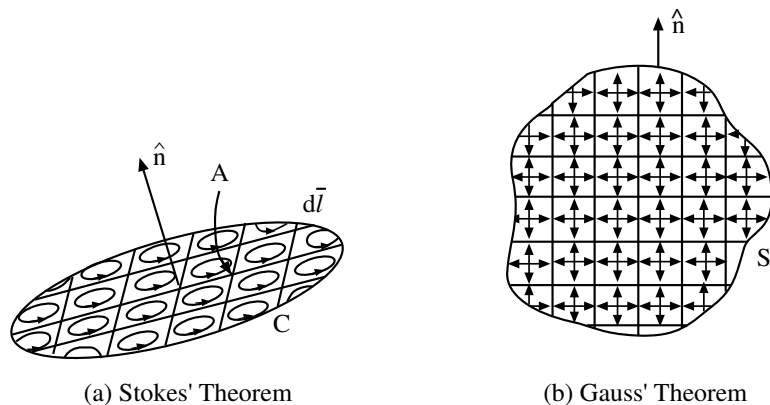
The curl operator  $\nabla \times$  is a measure of field rotation. Hence, Equation (1) indicates that a time-varying magnetic flux generates an electric field with rotation. Moreover, Equation (2) indicates that a current or a time-varying electric flux (also known as displacement current) generates a magnetic field with rotation.

The divergence operator  $\nabla \cdot$  is a measure of the total flux exuding from a point. If there is no source or sink at a point, the divergence of the flux at that point should be zero. Therefore, Equation (3) says that the divergence of the magnetic flux is always zero, since a source or a sink of magnetic flux (namely, magnetic charges) has not been found to date. Furthermore, Equation (4) states that the divergence of the electric flux at a point is proportional to the positive charge density present at the point.

Equation (1), which was discovered by Michael Faraday, is also known as Faraday's Law. Equation (2), without the  $\partial \mathbf{D}/\partial t$  term, or the displacement current term, is also known as Ampere's Law. The displacement current term, discovered by Maxwell later, is very important because it couples the magnetic field to the time-varying electric flux. Moreover, it also allows for the possible existence of electromagnetic waves which were later shown to be the same as light waves. Equations (3) and (4) are the consequences of Gauss' Law, which is a statement of the conservation of flux. More specifically,

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<sup>1</sup> 1 weber/m<sup>2</sup> = 1 Tesla = 10<sup>4</sup> Gauss. The earth's field is about 0.5 Gauss.



**Figure 1.1.1** *The parameters in the application of Stokes' theorem and Gauss' theorem.*

Equation (4) implies that the electric flux  $\mathbf{D}$  is produced by a charge density  $\rho$ .

### §§1.1.2 Integral Representations

Different insights sometimes result if we look at Maxwell's equations in their integral representations. To derive the integral forms of Equations (1) and (2), we integrate them over a cross-sectional area  $A$  and make use of Stokes' theorem,

$$\int_A d\mathbf{S} \cdot \nabla \times \mathbf{E}(\mathbf{r}, t) = \int_C d\mathbf{l} \cdot \mathbf{E}(\mathbf{r}, t). \quad (1.1.6)$$

In (6),  $C$  is a contour that forms the perimeter of the area  $A$  (Figure 1.1.1a). Moreover, (6) is a statement that the sum of all the rotations due to the field  $\mathbf{E}$  over the cross-sectional area  $A$  is equal to the “torque” produced by these rotations on the perimeter of  $A$  which is  $C$ : The left-hand side of (6) is the summation over all the rotations, while the right-hand side of (6) is the evaluation of the net “torque” on the perimeter  $C$ . The fact is that neighboring rotations within the area  $C$  cancel each other, leaving a net rotation on the perimeter.

Using Stokes' theorem, we can then convert (1) and (2) to

$$\int_C d\mathbf{l} \cdot \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \int_A d\mathbf{S} \cdot \mathbf{B}(\mathbf{r}, t), \quad (1.1.7)$$

$$\int_C d\mathbf{l} \cdot \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t} \int_A d\mathbf{S} \cdot \mathbf{D}(\mathbf{r}, t) + \int_A d\mathbf{S} \cdot \mathbf{J}(\mathbf{r}, t). \quad (1.1.8)$$

But to convert Equations (3) and (4) into integral forms, we integrate them over a volume  $V$  and make use of Gauss' theorem, which states that

$$\int_V dV \nabla \cdot \mathbf{B}(\mathbf{r}, t) = \int_S d\mathbf{S} \cdot \mathbf{B}(\mathbf{r}, t). \quad (1.1.9)$$

This is a mere statement that the sum of all divergences of a flux  $\mathbf{B}$  in a volume  $V$  is equal to the net flux which is leaving the volume  $V$  through the surface  $S$ . In other words, neighboring divergences tend to cancel each other within a volume  $V$  (Figure 1.1.1b).

Consequently, (3) and (4) become

$$\int_S d\mathbf{S} \cdot \mathbf{B}(\mathbf{r}, t) = 0, \quad (1.1.10)$$

$$\int_S d\mathbf{S} \cdot \mathbf{D}(\mathbf{r}, t) = \int_V \varrho(\mathbf{r}, t) dV = Q, \quad (1.1.11)$$

where  $Q$  is the total charge in volume  $V$ .

The left-hand side of Equation (7) is also the definition of an electromotive force. Hence, Equation (7) implies that a time-varying magnetic flux through an area  $A$  generates an electromotive force around a loop  $C$ . For instance, if  $C$  is replaced with a metallic conductor, the electromotive force will drive a current through this metallic conductor.

By the same token, Equation (8) implies that a time-varying electric flux (displacement current) or a current will generate a magnetomotive force, or simply, a magnetic field that loops around the currents.

On the other hand, Equations (10) and (11) are mere statements of the conservation of fluxes. Equation (11) implies that the net flux through a surface  $S$  equals the total charge  $Q$  inside  $S$ .

### §§1.1.3 Time Harmonic Forms

Maxwell's equations can be further simplified if we assume that the field is time harmonic. A time harmonic field can be expressed as

$$\mathbf{A}(\mathbf{r}, t) = \Re[\mathbf{A}(\mathbf{r}) e^{-i\omega t}],$$

where  $i = \sqrt{-1}$ ,  $\omega$  is frequency in radians/second, and  $\mathbf{A}(\mathbf{r})$  is a complex vector. This is also commonly referred to as the  $e^{-i\omega t}$  time convention. (The  $e^{j\omega t}$  time convention is sometimes used. Here, letting  $-i \rightarrow j$  will make the two conventions equivalent.) In this case,  $\mathbf{A}(\mathbf{r}, t)$  is a sinusoidal function of time—in other words, it is time harmonic. If this is in fact the case, it is easy to show that

$$\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) = \Re[-i\omega \mathbf{A}(\mathbf{r}) e^{-i\omega t}].$$

Subsequently, Equations (1) to (4) become

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega \mathbf{B}(\mathbf{r}), \quad (1.1.12)$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega \mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \quad (1.1.13)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0, \quad (1.1.14)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}). \quad (1.1.15)$$

In the above,  $\mathbf{E}(\mathbf{r})$ ,  $\mathbf{H}(\mathbf{r})$ ,  $\mathbf{D}(\mathbf{r})$ ,  $\mathbf{B}(\mathbf{r})$ ,  $\mathbf{J}(\mathbf{r})$ , and  $\rho(\mathbf{r})$  are complex vector or scalar functions known as *phasors*. Better still, a simple rule of thumb of obtaining (12) to (15) from (1) to (4) is to replace  $\partial/\partial t$  with  $-i\omega$ , and vice versa if we were to obtain (1) to (4) from (12) to (15). Alternatively, Equations (12) to (15) can also be obtained by Fourier transforming Equations (1) to (4) with respect to time. In this case, the phasors are actually the Fourier transforms of the fields in the time domain, and they are functions of frequency as well (see Exercise 1.1). Hence, the phasors are also known as the *frequency domain* solutions of the field. Likewise, the solutions of (1) to (4) are the *time domain* solutions. Obviously, the advantage of working with (12) to (15) is the absence of the time dependence and time derivatives.

#### §§1.1.4 Constitutive Relations

Since only two of the four Maxwell's equations are independent in electrodynamics, we need only work with the first two: Equations (12) and (13). However, there are four vector unknowns,  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{B}$ , and  $\mathbf{D}$ , with only two vector equations. Hence, in order to have a sufficient number of equations for the four unknowns, two more equations relating  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{B}$ , and  $\mathbf{D}$  are needed. This can be obtained from the constitutive relations.<sup>2</sup>

The electric and magnetic fluxes are related to the electric and magnetic fields via the constitutive relations. These general constitutive relations in the frequency domain have the form

$$\mathbf{D}(\mathbf{r}, \omega) = \bar{\epsilon}(\mathbf{r}, \omega) \cdot \mathbf{E}(\mathbf{r}, \omega) + \bar{\xi}(\mathbf{r}, \omega) \cdot \mathbf{H}(\mathbf{r}, \omega), \quad (1.1.16a)$$

$$\mathbf{B}(\mathbf{r}, \omega) = \bar{\mu}(\mathbf{r}, \omega) \cdot \mathbf{H}(\mathbf{r}, \omega) + \bar{\zeta}(\mathbf{r}, \omega) \cdot \mathbf{E}(\mathbf{r}, \omega), \quad (1.1.16b)$$

where  $\bar{\epsilon}$ ,  $\bar{\xi}$ ,  $\bar{\mu}$ , and  $\bar{\zeta}$  are  $3 \times 3$  tensors.<sup>3</sup> In addition, the constitutive relations also characterize the medium that we are describing. In fact, the above medium is also known as a *bianisotropic* medium because  $\mathbf{D}$  and  $\mathbf{B}$  are related to both  $\mathbf{E}$  and  $\mathbf{H}$ . In contrast, a medium that is *anisotropic* has constitutive relations where  $\mathbf{D}$  is only related to  $\mathbf{E}$ , and  $\mathbf{B}$  is only related to  $\mathbf{H}$ , that is,

$$\mathbf{D} = \bar{\epsilon} \cdot \mathbf{E}, \quad (1.1.17a)$$

$$\mathbf{B} = \bar{\mu} \cdot \mathbf{H}. \quad (1.1.17b)$$

<sup>2</sup> Extended discussion of this topic is given in Kong (1986).

<sup>3</sup> For a review of tensors see Appendix B.

And the word “anisotropy” implies that relationships (16) and (17) are functions of the field directions.

When  $\bar{\epsilon}$ ,  $\bar{\xi}$ ,  $\bar{\mu}$ , or  $\bar{\zeta}$  are functions of space, the medium is also known as an *inhomogeneous* medium. But when they are functions of frequency, the medium is *frequency dispersive*. In this case, the relations in the time domain correspond to convolutions. On the other hand, when the relations are convolutions over space, the medium is *spatially dispersive*. Moreover, when the tensors are functions of the fields, the medium is *nonlinear*. For *isotropic* media, however, relationship (17) is independent of field polarizations and the constitutive relations simply become

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}. \quad (1.1.18)$$

In free-space,  $\epsilon = \epsilon_0 = 8.854 \times 10^{-12}$  farad/m, while  $\mu = \mu_0 = 4\pi \times 10^{-7}$  henry/m. The constant  $c = 1/\sqrt{\mu_0\epsilon_0}$  is the velocity of light, which has been measured very accurately. The value of  $\mu_0$  is assigned, while the value of  $\epsilon_0$  is calculated from  $c$ . In fact, the value of  $\mu_0$  is chosen so that the units of voltage and current in a laboratory experiment are not inordinately large or small (see Exercise 1.2). More recently, a study (Cohen and Taylor 1986) recommends that the unit of meter be redefined so that the velocity of light is exactly 299,792,458 m/s.

### §§1.1.5 Poynting Theorem and Lossless Conditions

#### (a) Poynting Theorem

It can be easily shown that the vector  $\mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$  has a dimension of watts/m<sup>2</sup> which is that of power density. Therefore, it may be associated with the direction of power flow. If the fields are time harmonic, a time average of the vector can be defined as

$$\langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) dt. \quad (1.1.19)$$

Given the phasors of time harmonic fields  $\mathbf{E}(\mathbf{r}, t)$  and  $\mathbf{H}(\mathbf{r}, t)$ , namely,  $\mathbf{E}(\mathbf{r})$  and  $\mathbf{H}(\mathbf{r})$  respectively, we can show that (see Exercise 1.3)

$$\langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \rangle = \frac{1}{2} \Re\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})\}. \quad (1.1.20)$$

Here, the vector  $\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})$  is also known as the complex Poynting vector. Moreover, because of its aforementioned property, and its dimension of power density, we will study its conservative property. To do so, we take its divergence and use the appropriate vector identity to obtain

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^*. \quad (1.1.21)$$

Next, using Maxwell's equations for  $\nabla \times \mathbf{E}$  and  $\nabla \times \mathbf{H}^*$  and the constitutive relations for anisotropic media, we have

$$\begin{aligned}\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) &= i\omega \mathbf{H}^* \cdot \mathbf{B} - i\omega \mathbf{E} \cdot \mathbf{D}^* - \mathbf{E} \cdot \mathbf{J}^* \\ &= i\omega \mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H} - i\omega \mathbf{E} \cdot \bar{\boldsymbol{\epsilon}}^* \cdot \mathbf{E}^* - \mathbf{E} \cdot \mathbf{J}^*.\end{aligned}\quad (1.1.22)$$

The above is also known as the complex Poynting theorem. It can also be written in an integral form using Gauss' theorem, namely,

$$\int d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = i\omega \int_V dV (\mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \bar{\boldsymbol{\epsilon}}^* \cdot \mathbf{E}^*) - \int_V dV \mathbf{E} \cdot \mathbf{J}^*.\quad (1.1.23)$$

### (b) Lossless Conditions

For a region  $V$  that is lossless and source-free,  $\mathbf{J} = 0$  and

$$\Re \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = 0,$$

because there is no net time-averaged power-flow out of or into this region  $V$ . Therefore, because of energy conservation, the real part of the right-hand side of (22), without the  $\mathbf{E} \cdot \mathbf{J}^*$  term, must be zero. In other words,

$$\int_V dV (\mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \bar{\boldsymbol{\epsilon}}^* \cdot \mathbf{E}^*)\quad (1.1.24)$$

must be a real quantity.

Other than the possibility that the above is zero, the general requirement for it to be real is that  $\mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H}$  and  $\mathbf{E} \cdot \bar{\boldsymbol{\epsilon}}^* \cdot \mathbf{E}^*$  are real quantities. But since the conjugate transpose of a real number is itself, we have  $(\mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H})^\dagger = \mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H}$ . Therefore,

$$(\mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H})^\dagger = (\mathbf{H} \cdot \bar{\boldsymbol{\mu}}^* \cdot \mathbf{H}^*)^t = \mathbf{H}^* \cdot \bar{\boldsymbol{\mu}}^\dagger \cdot \mathbf{H} = \mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H}.\quad (1.1.25)$$

The last equality in the above is possible only if  $\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}^\dagger$  (where the  $\dagger$  implies conjugate transpose and  $t$  implies transpose), or that  $\bar{\boldsymbol{\mu}}$  is Hermitian. Therefore, the conditions for anisotropic media to be lossless are

$$\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}^\dagger, \quad \bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}^\dagger,\quad (1.1.26)$$

requiring the permittivity and permeability tensors to be Hermitian. Then, for an isotropic medium, the lossless conditions are simply that  $\Im m(\mu) = 0$  and  $\Im m(\epsilon) = 0$ .

If a medium is source-free, but lossy, then  $\Re \int d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) < 0$ . Consequently, from (23), this implies

$$\Im \int_V dV (\mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \bar{\boldsymbol{\epsilon}}^* \cdot \mathbf{E}^*) > 0. \quad (1.1.27)$$

But the above is the same as

$$i \int_V dV [\mathbf{H}^* \cdot (\bar{\boldsymbol{\mu}}^\dagger - \bar{\boldsymbol{\mu}}) \cdot \mathbf{H} + \mathbf{E}^* \cdot (\bar{\boldsymbol{\epsilon}}^\dagger - \bar{\boldsymbol{\epsilon}}) \cdot \mathbf{E}] > 0. \quad (1.1.28)$$

Therefore, for a medium to be lossy,  $i(\bar{\boldsymbol{\mu}}^\dagger - \bar{\boldsymbol{\mu}})$  and  $i(\bar{\boldsymbol{\epsilon}}^\dagger - \bar{\boldsymbol{\epsilon}})$  must be Hermitian, **positive definite** matrices, to ensure the inequality in (28). Similarly, for an active medium,  $i(\bar{\boldsymbol{\mu}}^\dagger - \bar{\boldsymbol{\mu}})$  and  $i(\bar{\boldsymbol{\epsilon}}^\dagger - \bar{\boldsymbol{\epsilon}})$  must be Hermitian, **negative definite** matrices (see Exercise 1.6).

For a lossy medium which is conductive, we may define  $\mathbf{J} = \bar{\boldsymbol{\sigma}} \cdot \mathbf{E}$  where  $\bar{\boldsymbol{\sigma}}$  is a conductivity tensor. In this case, Equation (23), after combining the last two terms, may be written as

$$\begin{aligned} \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) &= i\omega \int_V dV \left[ \mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \left( \bar{\boldsymbol{\epsilon}}^* - \frac{i\bar{\boldsymbol{\sigma}}^*}{\omega} \right) \cdot \mathbf{E}^* \right] \\ &= i\omega \int_V dV [\mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \tilde{\boldsymbol{\epsilon}}^* \cdot \mathbf{E}^*], \end{aligned} \quad (1.1.29)$$

where  $\tilde{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}} + \frac{i\bar{\boldsymbol{\sigma}}}{\omega}$  is known as the **complex permittivity tensor**. In this manner, (29) has the same structure as the source-free Poynting theorem.

The quantity  $\mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H}$  for lossless media is associated with the time-averaged energy density stored in the magnetic field, while the quantity  $\mathbf{E} \cdot \bar{\boldsymbol{\epsilon}}^* \cdot \mathbf{E}^*$  for lossless media is associated with the time-averaged energy density stored in the electric field. Then, for lossless, source-free media, (23) implies that

$$\Im \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = \omega \int_V dV (\mathbf{H}^* \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \bar{\boldsymbol{\epsilon}}^* \cdot \mathbf{E}^*), \quad (1.1.30)$$

or that

$$\Im \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*)$$

is proportional to the time rate of change of the difference of the time-averaged energy stored in the magnetic field and the electric field. Since this power is nondissipative, it is also known as the **reactive power** (see Exercise 1.5). Hence, the imaginary part of  $\mathbf{E} \times \mathbf{H}^*$  may be associated with the reactive power density.



### §§1.1.6 Duality Principle

Maxwell's equations exhibit a certain symmetry between  $\mathbf{E}$  and  $\mathbf{H}$ , and  $\mathbf{D}$  and  $\mathbf{B}$ . However, the absence of magnetic charges destroys the symmetry for the sources. Nevertheless, from a mathematical viewpoint, Maxwell's equations can be made symmetrical by introducing a magnetic current density  $\mathbf{M}$  and a magnetic charge density  $\varrho_m$ . In this case, (1) to (4) become

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) - \mathbf{M}(\mathbf{r}, t), \quad (1.1.31)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t), \quad (1.1.32)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = \varrho_m(\mathbf{r}, t), \quad (1.1.33)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \varrho(\mathbf{r}, t). \quad (1.1.34)$$

The symmetry exhibited by the above equations implies that given a solution to Maxwell's equations, with  $\mathbf{E}$ ,  $\mathbf{D}$ ,  $\mathbf{H}$ ,  $\mathbf{B}$ ,  $\mathbf{M}$ ,  $\mathbf{J}$ ,  $\varrho_m$ , and  $\varrho$ , another solution can be obtained by the following replacements:

$$\mathbf{E} \rightarrow \mathbf{H}, \quad \mathbf{H} \rightarrow \mathbf{E}, \quad \mathbf{B} \rightarrow -\mathbf{D}, \quad \mathbf{D} \rightarrow -\mathbf{B},$$

$$\mathbf{M} \rightarrow -\mathbf{J}, \quad \mathbf{J} \rightarrow -\mathbf{M}, \quad \varrho_m \rightarrow -\varrho, \quad \varrho \rightarrow -\varrho_m. \quad (1.1.35)$$

Notice that the above replacements are nonunique and any other replacements that make Equations (31) to (34) invariant will also suffice (see Exercise 1.7).

If the constitutive relations appear explicitly in (31) to (34), the rule for replacements can be altered accordingly. For example, for anisotropic media,  $\mathbf{B} = \bar{\boldsymbol{\mu}} \cdot \mathbf{H}$  and  $\mathbf{D} = \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}$ , a possible set of replacement rules is

$$\mathbf{E} \rightarrow \mathbf{H}, \quad \mathbf{H} \rightarrow \mathbf{E}, \quad \bar{\boldsymbol{\mu}} \rightarrow -\bar{\boldsymbol{\epsilon}}, \quad \bar{\boldsymbol{\epsilon}} \rightarrow -\bar{\boldsymbol{\mu}},$$

$$\mathbf{M} \rightarrow -\mathbf{J}, \quad \mathbf{J} \rightarrow -\mathbf{M}, \quad \varrho_m \rightarrow -\varrho, \quad \varrho \rightarrow -\varrho_m. \quad (1.1.36)$$

For source-free Maxwell's equations, this becomes simply

$$\mathbf{E} \rightarrow \mathbf{H}, \quad \mathbf{H} \rightarrow \mathbf{E}, \quad \bar{\boldsymbol{\mu}} \rightarrow -\bar{\boldsymbol{\epsilon}}, \quad \bar{\boldsymbol{\epsilon}} \rightarrow -\bar{\boldsymbol{\mu}}. \quad (1.1.37)$$

Even though there is no true magnetic current, one can still speak of equivalent magnetic current. For instance, a current loop carrying an electric current generates a field that resembles that of a magnetic dipole. Consequently, in the limit when the electric current loop is very small, it is equivalent to a magnetic dipole. Then, a series of current loops such as a solenoid or a toroid generates a field similar to that generated by a magnetic current.

## §1.2 Scalar Wave Equations

Certain physical phenomena can be described using only the scalar wave equation, for example, acoustic waves and Schrödinger waves. In fact, in

certain situations, electromagnetic waves can also be described by the scalar wave equation. Hence, we shall study the scalar wave equation and first illustrate the derivation of the acoustic wave equation.

### §§1.2.1 Acoustic Wave Equation

The acoustic wave equation can be derived based on the conservation of mass and conservation of momentum. Similar to the conservation of charge expressed by Equation (1.1.5), the conservation of mass for a fluid can be written as

$$\nabla \cdot (\mathbf{v}\rho) + \frac{\partial \rho}{\partial t} = 0, \quad (1.2.1)$$

where  $\rho$  is the mass density and  $\mathbf{v}$  is the velocity of the fluid particles. If no external force is acting on a fluid mass, the conservation law for the  $i$ -th component of the momentum can be written in a manner similar to (1), giving

$$\nabla \cdot (\mathbf{v}\rho v_i) + \frac{\partial \rho v_i}{\partial t} = 0, \quad (1.2.2)$$

where  $v_i$  is either  $v_x$ ,  $v_y$ , or  $v_z$ . Next, by writing the conservation law for three components of the momentum simultaneously, we have

$$\nabla \cdot \rho \mathbf{v}\mathbf{v} + \frac{\partial \rho \mathbf{v}}{\partial t} = 0. \quad (1.2.3)$$

Now, if an external force density  $\mathbf{F}$  is applied to the fluid mass, then (3) becomes

$$\nabla \cdot \rho \mathbf{v}\mathbf{v} + \frac{\partial \rho \mathbf{v}}{\partial t} = \mathbf{F}. \quad (1.2.4)$$

Furthermore, by using the conservation of mass equation given by (1), (4) can be rewritten more simply as (see Exercise 1.8)

$$\rho \left[ \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \right] = \mathbf{F}. \quad (1.2.5)$$

Force in a fluid sets up a disturbance, giving rise to particle velocity  $\mathbf{v}$ , and changes in mass density  $\rho$  and pressure  $p$ . Before proceeding with a perturbation analysis, we denote the equilibrium quantities by subscript 0, and the perturbed quantities by subscript 1 as follows:

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_1(\mathbf{r}, t), \quad (1.2.6a)$$

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t), \quad (1.2.6b)$$

$$p(\mathbf{r}, t) = p_0(\mathbf{r}) + p_1(\mathbf{r}, t), \quad (1.2.6c)$$

where we assume  $\mathbf{v}_0 = 0$ , i.e., the fluid particles are at rest before a wave is established. Next, assuming  $\rho_1 \ll \rho_0$ ,  $p_1 \ll p_0$ , and  $\mathbf{v}_1$  to be a small quantity, on substituting (6) into (1) yields

$$\nabla \cdot (\mathbf{v}_1 \rho_0) + \nabla \cdot (\mathbf{v}_1 \rho_1) + \frac{\partial \rho_1}{\partial t} = 0. \quad (1.2.7)$$

Then, keeping only the first order terms (assuming  $\mathbf{v}_1 \varrho_1$  to be much smaller than the other terms), we have

$$\nabla \cdot \mathbf{v}_1 \varrho_0 + \frac{\partial \varrho_1}{\partial t} = 0. \quad (1.2.8)$$

The restoring force in a fluid is provided by the pressure differential set up in it. Therefore, the force density in (5) can be shown to be (see Exercise 1.8)

$$\mathbf{F} = -\nabla p = -\nabla p_0 - \nabla p_1. \quad (1.2.9)$$

Then, using (6) and (9) in (5), we have

$$[\varrho_0 + \varrho_1] \left[ \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + \frac{\partial \mathbf{v}_1}{\partial t} \right] = -\nabla p_0 - \nabla p_1. \quad (1.2.10)$$

Now, by equating the leading-order term in (10), we have

$$\nabla p_0 = 0, \quad (1.2.11)$$

while by keeping the first order term, we have

$$\varrho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1. \quad (1.2.12)$$

Equation (11) implies that the pressure has to be uniform in the equilibrium state. Note that this quiescent pressure need not be a constant in the presence of a gravitational force. But over a short length-scale, the pressure gradient induced by gravity can be ignored.

In a compressible fluid, if  $\mathbf{v}_1$ ,  $p_1$ , and  $\varrho_1$  are small, and constant entropy is assumed for adiabatic compression and expansion, they can be further linearly related as (see Exercise 1.9, see also Pierce 1981)

$$\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p_0 = c^2 \left( \frac{\partial \varrho_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \varrho_0 \right). \quad (1.2.13)$$

Next, using (13) in (8), and making use of (11) we have

$$\varrho_0 \nabla \cdot \mathbf{v}_1 + \frac{1}{c^2} \frac{\partial p_1}{\partial t} = 0. \quad (1.2.14)$$

Then, after differentiating the above once with respect to  $t$ , we obtain

$$\varrho_0 \nabla \cdot \left( \frac{\partial \mathbf{v}_1}{\partial t} \right) + \frac{1}{c^2} \frac{\partial^2 p_1}{\partial t^2} = 0. \quad (1.2.15)$$

Finally, using (12) in (15) yields

$$\varrho_0 \nabla \cdot \varrho_0^{-1} \nabla p_1(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 p_1}{\partial t^2}(\mathbf{r}, t) = 0. \quad (1.2.16)$$

Equation (16) is a scalar wave equation for acoustic waves in inhomogeneous media. In addition,  $\rho_0(\mathbf{r})$  and  $c(\mathbf{r})$  are both functions of position. Furthermore, in the case of a homogeneous medium where  $\rho_0$  and  $c$  are constants, (16) becomes

$$\nabla^2 p_1(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p_1(\mathbf{r}, t) = 0, \quad (1.2.17)$$

where  $c$  is the velocity of the wave. For a time harmonic field, however, (16) becomes

$$\rho_0 \nabla \cdot \rho_0^{-1} \nabla p_1(\mathbf{r}, \omega) + k^2 p_1(\mathbf{r}, \omega) = 0, \quad (1.2.18)$$

where  $k = \omega/c$ . The above is the Helmholtz wave equation for inhomogeneous acoustic media.

### §§1.2.2 *Scalar Wave Equation from Electromagnetics*

Certain electromagnetic problems can even be described by the scalar wave equation. For instance, in three dimensions, the vector wave equations reduce to the scalar wave equations in a homogeneous, isotropic medium.

In a homogeneous, isotropic, and source-free medium,  $\mathbf{B} = \mu \mathbf{H}$  and  $\mathbf{D} = \epsilon \mathbf{E}$ . Next, after taking the curl of Equation (1.1.12) and substituting it for  $\nabla \times \mathbf{H}$  from Equation (1.1.13) without the current source, we have

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \mu \epsilon \mathbf{E}(\mathbf{r}) = 0, \quad (1.2.19)$$

which is the vector wave equation for a source-free homogeneous medium. Moreover, by using the vector identity that  $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$  and that  $\nabla \cdot \mathbf{E} = 0$  for a homogeneous, source-free medium, Equation (19) becomes

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = 0, \quad (1.2.20)$$

where  $k^2 = \omega^2 \mu \epsilon$ . In Cartesian coordinates,  $\mathbf{E}(\mathbf{r}) = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$ , where  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$  are unit vectors independent of position. Hence, (20) consists of three scalar wave equations,

$$(\nabla^2 + k^2) \psi(\mathbf{r}) = 0, \quad (1.2.21)$$

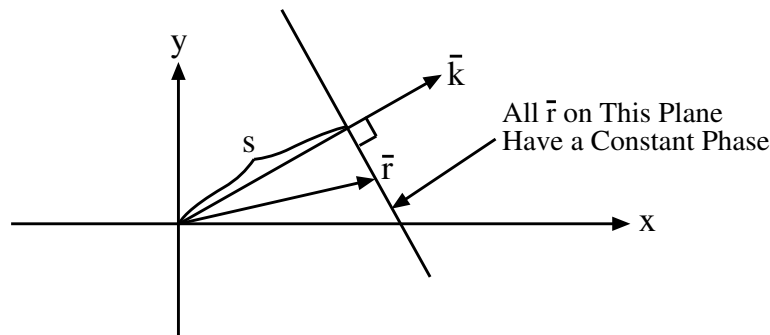
where  $\psi(\mathbf{r})$  can be either  $E_x$ ,  $E_y$ , or  $E_z$ . [Note that this statement is not true in cylindrical or spherical coordinates (see Exercise 1.10).] However, Equation (20) must be solved with  $\nabla \cdot \mathbf{E} = 0$  condition before the solution is also admissible for (19). Hence, only two out of the three equations in (20) are independent.

The above establishes the wave nature of Maxwell's equations, which is a consequence of the displacement current term discovered by Maxwell. Therefore, it is worthwhile to study more extensively the solutions of the scalar wave equation.

### §§1.2.3 *Cartesian Coordinates*

In Cartesian coordinates, the Laplacian operator in (21) becomes

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \psi(\mathbf{r}) = 0. \quad (1.2.22)$$



**Figure 1.2.1** Constant phase front of a plane wave which is perpendicular to  $\mathbf{k}$ .

Then,  $\psi(\mathbf{r})$  has the general solution

$$\psi(\mathbf{r}) = Ae^{i(k_x x + k_y y + k_z z)} = Ae^{i\mathbf{k} \cdot \mathbf{r}}, \quad (1.2.23)$$

where the vector  $\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$  is known as the propagation vector, while the vector  $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$  is known as the position vector. Next, after substituting (23) into (22), we have

$$(-k_x^2 - k_y^2 - k_z^2 + k^2) \psi(\mathbf{r}) = 0. \quad (1.2.24)$$

For nontrivial  $\psi(\mathbf{r})$ , we require that  $k^2 = k_x^2 + k_y^2 + k_z^2$ , which is also known as the **dispersion relation**. It implies that the propagation vector  $\mathbf{k}$  is of a fixed length, i.e.,  $|\mathbf{k}| = k$ , no matter what direction it is pointing.

Equation (23) denotes mathematically a plane wave propagating in the  $\mathbf{k}$  direction. For example, for a plane wave propagating in the  $x$  direction,  $\psi(\mathbf{r}) = Ae^{ikx}$  and  $\mathbf{k} = \hat{x}k$ , a vector pointing in the  $x$  direction. More specifically, the function  $e^{i\mathbf{k} \cdot \mathbf{r}}$  has a constant phase  $e^{iks}$  for all  $\mathbf{r}$  such that  $\mathbf{k} \cdot \mathbf{r} = ks$ . The locus of the tips of all such  $\mathbf{r}$ 's is a plane perpendicular to  $\mathbf{k}$  (see Figure 1.2.1).

In addition, assuming that  $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r}}$ , we can show easily that  $\nabla \rightarrow i\mathbf{k}$ . Then, by using this fact in (19), we have

$$-\mathbf{k} \times \mathbf{k} \times \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = 0. \quad (1.2.25)$$

Dotting the above with  $\mathbf{k}$  implies that  $\mathbf{k} \cdot \mathbf{E} = 0$  for all plane waves. Furthermore, for a homogeneous, isotropic and source-free medium,  $\nabla \times \mathbf{E} = i\omega\mu \mathbf{H}$ , implying that  $\mathbf{k} \times \mathbf{E} = \omega\mu \mathbf{H}$  and  $\mathbf{k} \cdot \mathbf{H} = 0$  for plane waves. Therefore,  $\mathbf{E}$ ,  $\mathbf{H}$ , and  $\mathbf{k}$  form a right-handed system: they are mutually orthogonal.

### §§1.2.4 Cylindrical Coordinates

The scalar wave equation in cylindrical coordinates is

$$\left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \psi(\mathbf{r}) = 0. \quad (1.2.26)$$

The above partial differential equation can be solved by the separation of variables. On the other hand, the simple  $\partial^2/\partial\phi^2$  and  $\partial^2/\partial z^2$  derivatives imply that the solutions are of the form<sup>4</sup>

$$\psi(\mathbf{r}) = F_n(\rho) e^{in\phi + ik_z z}, \quad (1.2.27)$$

where  $n$  is an integer since the field has to be  $2\pi$  periodic in  $\phi$ . Then, substituting (27) into (26) gives rise to

$$\left( \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} - \frac{n^2}{\rho^2} + k_\rho^2 \right) F_n(\rho) = 0 \quad (1.2.28)$$

where  $k_\rho^2 = k^2 - k_z^2$ . Notice that the above is just the Bessel equation with two linearly independent solutions. Its general solution is a linear superposition of any two of the following four special functions<sup>5</sup>:

- (i) the Bessel function  $J_n(k_\rho \rho)$ ;
- (ii) the Neumann function  $N_n(k_\rho \rho)$ ;
- (iii) the Hankel function of the first kind  $H_n^{(1)}(k_\rho \rho)$ ; and
- (iv) the Hankel function of the second kind  $H_n^{(2)}(k_\rho \rho)$ .

Since only two of these four special functions are independent, they are linearly related to each other, i.e.,

$$J_n(k_\rho \rho) = \frac{1}{2} [H_n^{(1)}(k_\rho \rho) + H_n^{(2)}(k_\rho \rho)], \quad (1.2.29a)$$

$$N_n(k_\rho \rho) = \frac{1}{2i} [H_n^{(1)}(k_\rho \rho) - H_n^{(2)}(k_\rho \rho)], \quad (1.2.29b)$$

or

$$H_n^{(1)}(k_\rho \rho) = J_n(k_\rho \rho) + iN_n(k_\rho \rho), \quad (1.2.29c)$$

$$H_n^{(2)}(k_\rho \rho) = J_n(k_\rho \rho) - iN_n(k_\rho \rho). \quad (1.2.29d)$$

These special functions behave differently around the origin when the argument  $k_\rho \rho \rightarrow 0$ . For instance, for  $n = 0$ , when  $k_\rho \rho \rightarrow 0$ ,

$$J_0(k_\rho \rho) \sim 1, \quad N_0(k_\rho \rho) \sim \frac{2}{\pi} \ln(k_\rho \rho), \quad (1.2.30a)$$

$$H_0(k_\rho \rho) \sim \frac{2i}{\pi} \ln(k_\rho \rho), \quad H_0^{(2)}(k_\rho \rho) \sim -\frac{2i}{\pi} \ln(k_\rho \rho). \quad (1.2.30b)$$

<sup>4</sup> This form is also obtainable by separation of variables.

<sup>5</sup> See Abramowitz and Stegun (1965).

But for  $n > 0$ , when  $k_\rho \rho \rightarrow 0$ ,

$$J_n(k_\rho \rho) \sim \frac{(k_\rho \rho/2)^n}{n!}, \quad N_n(k_\rho \rho) \sim -\frac{(n-1)!}{\pi} \left(\frac{2}{k_\rho \rho}\right)^n, \quad (1.2.31a)$$

$$H_n^{(1)}(k_\rho \rho) \sim -\frac{i(n-1)!}{\pi} \left(\frac{2}{k_\rho \rho}\right)^n, \quad H_n^{(2)}(k_\rho \rho) \sim \frac{i(n-1)!}{\pi} \left(\frac{2}{k_\rho \rho}\right)^n. \quad (1.2.31b)$$

Therefore, only the Bessel function is regular at the origin, while the other functions are singular.

On the other hand, when  $k_\rho \rho \rightarrow \infty$ ,

$$J_n(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} \cos\left(k_\rho \rho - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad (1.2.32a)$$

$$N_n(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} \sin\left(k_\rho \rho - \frac{n\pi}{2} - \frac{\pi}{4}\right), \quad (1.2.32b)$$

$$H_n^{(1)}(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} e^{i(k_\rho \rho - \frac{n\pi}{2} - \frac{\pi}{4})}, \quad (1.2.32c)$$

$$H_n^{(2)}(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} e^{-i(k_\rho \rho - \frac{n\pi}{2} - \frac{\pi}{4})}. \quad (1.2.32d)$$

Therefore, the Bessel function and the Neumann function are standing waves. In contrast,  $H_n^{(1)}(k_\rho \rho)$  is an outgoing wave, whereas  $H_n^{(2)}(k_\rho \rho)$  is an incoming wave (assuming  $e^{-i\omega t}$  dependence), when  $k_\rho \rho \rightarrow \infty$ . When  $k_\rho \rho$  is real,  $J_n(k_\rho \rho)$  and  $N_n(k_\rho \rho)$  are real functions, whereas  $H_n^{(1)}(k_\rho \rho)$  and  $H_n^{(2)}(k_\rho \rho)$  are complex functions. Furthermore,  $H_n^{(1)}(k_\rho \rho) = [H_n^{(2)}(k_\rho \rho)]^*$  in this case.

Equation (27) in general represents a cylindrical wave or a conical wave, since for large  $\rho$ , the wavefront has a cone shape (see Exercise 1.11). When  $F_n(\rho)$  in (27) is a  $H_n^{(1)}(k_\rho \rho)$ , then

$$\psi(\mathbf{r}) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} e^{-i\frac{n\pi}{2} - i\frac{\pi}{4}} e^{i(\frac{n}{\rho}\rho\phi + ik_z z + ik_\rho \rho)}. \quad (1.2.33)$$

In the above,  $\rho\phi$  is the arc length in the  $\phi$  direction, and  $n/\rho$  can be thought of as the  $\phi$  component of the  $\mathbf{k}$  vector if we compare (33) with (23). Consequently, (33) looks like a plane wave propagating mainly in the direction  $\mathbf{k} = \hat{z}k_z + \hat{\rho}k_\rho$ , when  $\rho \rightarrow \infty$ .

An important recurrence formula for solutions of the Bessel equation is

$$\begin{aligned} B_n'(k_\rho \rho) &= B_{n-1}(k_\rho \rho) - \frac{n}{k_\rho \rho} B_n(k_\rho \rho) \\ &= -B_{n+1}(k_\rho \rho) + \frac{n}{k_\rho \rho} B_n(k_\rho \rho), \end{aligned} \quad (1.2.34)$$

where  $B_n(k_\rho\rho)$  is either  $J_n(k_\rho\rho)$ ,  $N_n(k_\rho\rho)$ ,  $H_n^{(1)}(k_\rho\rho)$ ,  $H_n^{(2)}(k_\rho\rho)$ , or a linear combination thereof.

### §§1.2.5 Spherical Coordinates

In spherical coordinates, the scalar wave equation is

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + k^2 \right] \psi(\mathbf{r}) = 0. \quad (1.2.35)$$

Noting the  $\partial^2/\partial\phi^2$  derivative, we assume that  $\psi(\mathbf{r})$  is of the form

$$\psi(\mathbf{r}) = F(r, \theta) e^{im\phi}. \quad (1.2.36)$$

Then, (35) becomes

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{m^2}{r^2 \sin^2 \theta} + k^2 \right] F(r, \theta) = 0. \quad (1.2.37)$$

The above can be further simplified by the separation of variables by letting

$$F(r, \theta) = b_n(kr) P_n^m(\cos \theta), \quad (1.2.38)$$

where  $P_n^m(\cos \theta)$  is the associate Legendre polynomial satisfying the equation

$$\left\{ \frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta \frac{d}{d\theta} + \left[ n(n+1) - \frac{m^2}{\sin^2 \theta} \right] \right\} P_n^m(\cos \theta) = 0. \quad (1.2.39)$$

Therefore,  $b_n(kr)$  satisfies the equation

$$\left[ \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} + k^2 - \frac{n(n+1)}{r^2} \right] b_n(kr) = 0. \quad (1.2.40)$$

The above is just the spherical Bessel equation, and  $b_n(kr)$  is either the spherical Bessel function  $j_n(kr)$ , spherical Neumann function  $n_n(kr)$ , or the spherical Hankel functions  $h_n^{(1)}(kr)$  and  $h_n^{(2)}(kr)$ .

The spherical functions are related to the cylindrical functions via (see Exercise 1.12)

$$b_n(kr) = \sqrt{\frac{\pi}{2kr}} B_{n+\frac{1}{2}}(kr), \quad (1.2.40a)$$

where  $b_n(kr)$  is either  $j_n(kr)$ ,  $n_n(kr)$ ,  $h_n^{(1)}(kr)$ , or  $h_n^{(2)}(kr)$ ; while  $B_{n+\frac{1}{2}}(kr)$  is either  $J_{n+\frac{1}{2}}(kr)$ ,  $N_{n+\frac{1}{2}}(kr)$ ,  $H_{n+\frac{1}{2}}^{(1)}(kr)$ , or  $H_{n+\frac{1}{2}}^{(2)}(kr)$ . More specifically,

$$h_0^{(1)}(kr) = \frac{e^{ikr}}{ikr}, \quad h_1^{(1)}(kr) = - \left( 1 + \frac{i}{kr} \right) \frac{e^{ikr}}{kr}, \quad (1.2.41a)$$

$$h_0^{(2)}(kr) = - \frac{e^{-ikr}}{ikr}, \quad h_1^{(2)}(kr) = - \left( 1 - \frac{i}{kr} \right) \frac{e^{-ikr}}{kr}, \quad (1.2.41b)$$

$$j_0(kr) = \frac{\sin kr}{kr}, \quad j_1(kr) = - \frac{\cos kr}{kr} + \frac{\sin kr}{(kr)^2}, \quad (1.2.41c)$$

$$n_0(kr) = - \frac{\cos kr}{kr}, \quad n_1(kr) = - \frac{\sin kr}{kr} - \frac{\cos kr}{(kr)^2}. \quad (1.2.41d)$$



Hence, the spherical functions represent spherical waves, which resemble plane waves when  $r \rightarrow \infty$ . Moreover, recurrence relations similar to (34) can be derived for spherical Bessel functions (see Exercise 1.13, also see Abramowitz and Stegun 1965)

### §1.3 Vector Wave Equations

For an inhomogeneous, anisotropic medium, Maxwell's equations with a fictitious magnetic current density  $\mathbf{M}$  could be written as

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \bar{\boldsymbol{\mu}} \cdot \mathbf{H}(\mathbf{r}, t) - \mathbf{M}(\mathbf{r}, t), \quad (1.3.1)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial}{\partial t} \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}, t) + \mathbf{J}(\mathbf{r}, t). \quad (1.3.2)$$

Furthermore, if the fields are time harmonic, the above equations become

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega \bar{\boldsymbol{\mu}} \cdot \mathbf{H}(\mathbf{r}) - \mathbf{M}(\mathbf{r}), \quad (1.3.3)$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}). \quad (1.3.4)$$

Since electromagnetic fields are vector fields, the general wave equation is a vector wave equation. Hence, we will derive the general, time harmonic form of the vector wave equation first.

To do so, we take the curl of  $\bar{\boldsymbol{\mu}}^{-1} \cdot (3)$ , and use (4), to obtain

$$\nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}) - \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}(\mathbf{r}). \quad (1.3.5a)$$

Similarly,

$$\nabla \times \bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}(\mathbf{r}) - \omega^2 \bar{\boldsymbol{\mu}} \cdot \mathbf{H}(\mathbf{r}) = i\omega \mathbf{M}(\mathbf{r}) + \nabla \times \bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}(\mathbf{r}). \quad (1.3.5b)$$

The above also follows directly from the duality principle.

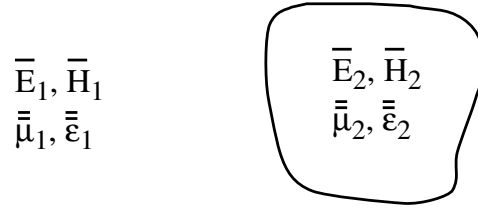
Equations (5a) and (5b) are two vector wave equations governing the solutions of an electromagnetic field in an inhomogeneous, anisotropic medium. Here,  $\bar{\boldsymbol{\mu}}$  and  $\bar{\boldsymbol{\epsilon}}$  are assumed to be functions of positions; hence, they do not commute with the  $\nabla$  operator. Moreover, for time-varying fields,  $\mathbf{E}$  and  $\mathbf{H}$  can be derived from each other; hence, only one of the two Equations (5a) and (5b) is needed to fully describe electromagnetic fields.

For an inhomogeneous isotropic medium, however, the above equations reduce to

$$\nabla \times \mu^{-1} \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \epsilon \mathbf{E}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}) - \nabla \times \mu^{-1} \mathbf{M}(\mathbf{r}), \quad (1.3.6a)$$

$$\nabla \times \epsilon^{-1} \nabla \times \mathbf{H}(\mathbf{r}) - \omega^2 \mu \mathbf{H}(\mathbf{r}) = i\omega \mathbf{M}(\mathbf{r}) + \nabla \times \epsilon^{-1} \mathbf{J}(\mathbf{r}). \quad (1.3.6b)$$

As mentioned in the preceding paragraph, either one of the above equations is self-contained. Consequently, all phenomena of electrodynamic fields in



**Figure 1.3.1** *The solution to a piecewise-constant inhomogeneity can be obtained by first obtaining the solution in each region, and then patching the solutions together via boundary conditions.*

inhomogeneous media are obtained by studying just one of them. In fact, the two equations are derivable from each other. Furthermore, since  $\nabla \cdot \bar{\epsilon} \cdot \mathbf{E} = \rho$  and  $\nabla \cdot \bar{\mu} \cdot \mathbf{H} = \rho_m$ , the three components of  $\mathbf{E}$  or  $\mathbf{H}$  are not linearly independent of each other. Hence, many electromagnetic problems can be formulated in terms of only two of the six components in  $\mathbf{E}$  and  $\mathbf{H}$ .

### §§1.3.1 *Boundary Conditions*

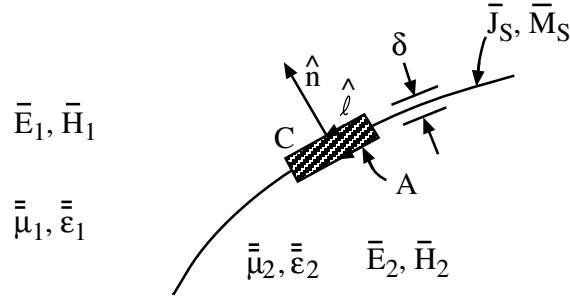
Equation (5a) or (5b) describes all the phenomena of electrodynamic wave interaction with inhomogeneity. Therefore, either one of them can be considered as the basic equation rather than Maxwell's equations for electromagnetic phenomena. Moreover, what is derivable from Maxwell's equations is also derivable from the above equations. For instance, when solving problems involving piecewise-constant inhomogeneities, a common practice is to obtain first solutions to Maxwell's equations in each region, and later, match boundary conditions at the interfaces to obtain the solution valid everywhere (see Figure 1.3.1). We shall show that these boundary conditions can be derived from either one of the two vector wave equations.

To do so, we integrate (5a) about a small area between the interface of the two inhomogeneities (see Figure 1.3.2). Then, on invoking Stokes' theorem for the surface integral of a curl, (5a) becomes

$$\oint_C d\mathbf{l} \cdot (\bar{\mu}^{-1} \cdot \nabla \times \mathbf{E}) - \omega^2 \int_A d\mathbf{S} \cdot \bar{\epsilon} \cdot \mathbf{E} = i\omega \int_A d\mathbf{S} \cdot \mathbf{J} - \oint_C d\mathbf{l} \cdot \bar{\mu}^{-1} \cdot \mathbf{M}. \quad (1.3.7)$$

But if  $\mathbf{M}$  is a current sheet, the last term on the right-hand side of (7) vanishes because  $\bar{\mu}^{-1} \cdot \mathbf{M} = 0$  on  $C$ .

Consequently, when  $\delta \rightarrow 0$  (see Figure 1.3.2), the area integral on the left-hand side of the above equation vanishes, because  $\bar{\epsilon} \cdot \mathbf{E}$  is nonsingular at the interface. Moreover, if a current sheet  $\mathbf{J}_s$  resides at the interface, then  $\mathbf{J}$



**Figure 1.3.2** Derivation of the boundary conditions.

is singular at the interface, and

$$\int_A d\mathbf{S} \cdot \mathbf{J} = \int_a^b dl (\hat{\mathbf{n}} \times \hat{\mathbf{l}} \cdot \mathbf{J}_s). \quad (1.3.8)$$

Similarly,

$$\oint_C d\mathbf{l} \cdot (\bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}) = \int_a^b d\mathbf{l} \cdot (\bar{\boldsymbol{\mu}}_1^{-1} \cdot \nabla \times \mathbf{E}_1) - \int_a^b d\mathbf{l} \cdot (\bar{\boldsymbol{\mu}}_2^{-1} \cdot \nabla \times \mathbf{E}_2), \quad (1.3.9)$$

where  $\mathbf{E}_1$  and  $\mathbf{E}_2$  are the fields and  $\bar{\boldsymbol{\mu}}_1$  and  $\bar{\boldsymbol{\mu}}_2$  are the permeability tensors in the two different regions. This further implies that

$$\hat{\mathbf{l}} \cdot (\bar{\boldsymbol{\mu}}_1^{-1} \cdot \nabla \times \mathbf{E}_1) - \hat{\mathbf{l}} \cdot (\bar{\boldsymbol{\mu}}_2^{-1} \cdot \nabla \times \mathbf{E}_2) = i\omega \hat{\mathbf{n}} \times \hat{\mathbf{l}} \cdot \mathbf{J}_s. \quad (1.3.10)$$

On noticing that  $\hat{\mathbf{l}} = (\hat{\mathbf{n}} \times \hat{\mathbf{l}}) \times \hat{\mathbf{n}}$  on the left-hand side, and using the appropriate vector identity, we have

$$\hat{\mathbf{n}} \times (\bar{\boldsymbol{\mu}}_1^{-1} \cdot \nabla \times \mathbf{E}_1) - \hat{\mathbf{n}} \times (\bar{\boldsymbol{\mu}}_2^{-1} \cdot \nabla \times \mathbf{E}_2) = i\omega \mathbf{J}_s. \quad (1.3.11)$$

Since  $\nabla \times \mathbf{E} = i\omega \bar{\boldsymbol{\mu}} \cdot \mathbf{H}$ , the above is also the same as

$$\hat{\mathbf{n}} \times \mathbf{H}_1 - \hat{\mathbf{n}} \times \mathbf{H}_2 = \mathbf{J}_s, \quad (1.3.12)$$

which states that the discontinuity in the tangential component of the magnetic field is proportional to the electric current sheet  $\mathbf{J}_s$ .

To derive another boundary condition, we rewrite Equation (5a) as

$$\nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot [\nabla \times \mathbf{E}(\mathbf{r}) + \mathbf{M}(\mathbf{r})] - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}). \quad (1.3.13)$$

Now, on the right-hand side, if  $\mathbf{J}(\mathbf{r})$  is a current sheet  $\mathbf{J}_s$ , it will give rise to a discontinuity in  $\bar{\boldsymbol{\mu}}^{-1} \cdot [\nabla \times \mathbf{E}(\mathbf{r}) + \mathbf{M}]$ . However,  $\bar{\boldsymbol{\mu}}^{-1} \cdot [\nabla \times \mathbf{E}(\mathbf{r}) + \mathbf{M}]$  must be regular or nonsingular, for if it is singular, its curl will make it doubly singular, which cannot be cancelled by any other terms in (13). But if  $\bar{\boldsymbol{\mu}}^{-1} \cdot [\nabla \times \mathbf{E}(\mathbf{r}) + \mathbf{M}(\mathbf{r})]$  is regular, so must  $\nabla \times \mathbf{E}(\mathbf{r}) + \mathbf{M}$  since  $\bar{\boldsymbol{\mu}}^{-1}$  is nonsingular. Therefore, after integrating  $\nabla \times \mathbf{E}(\mathbf{r}) + \mathbf{M}$  over  $A$  as in (7) and letting  $\delta \rightarrow 0$ , we conclude that

$$\hat{n} \times \mathbf{E}_1 - \hat{n} \times \mathbf{E}_2 = -\mathbf{M}_s, \quad (1.3.14)$$

where  $\mathbf{M}_s$  is a magnetic current sheet at the interface. Thus, the discontinuity in the tangential component of the electric field is proportional to the magnetic current sheet  $\mathbf{M}_s$ .

The boundary conditions (12) and (14) can also be derived more directly from Maxwell's equations. Though the derivation here is less direct, it illustrates that these boundary conditions are inherently buried in (5a). Similarly, they can also be extracted from (5b). In general, boundary conditions are buried in the partial differential equation that governs the field (see Exercise 1.14). This further reinforces the point that either (5a) or (5b) alone is sufficient to describe electrodynamic phenomena in an inhomogeneous, anisotropic medium.

### §§1.3.2 Reciprocity Theorem

The reciprocity theorem relates in a simple manner the mutual interactions between two groups of sources, under certain conditions on the medium. Such a medium is then known as a reciprocal medium. We shall show how such a reciprocity relation can be derived from the vector wave equations.

If there are two groups of sources  $\mathbf{J}_1, \mathbf{M}_1$ ; and  $\mathbf{J}_2, \mathbf{M}_2$  radiating in an anisotropic, inhomogeneous medium, where  $\mathbf{J}_1, \mathbf{M}_1$  produces the field  $\mathbf{E}_1$ , and  $\mathbf{J}_2, \mathbf{M}_2$  produces the field  $\mathbf{E}_2$ , the vector wave equations that  $\mathbf{E}_1$  and  $\mathbf{E}_2$  satisfy are then

$$\nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_1 - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_1 = i\omega \mathbf{J}_1 - \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}_1, \quad (1.3.15a)$$

and

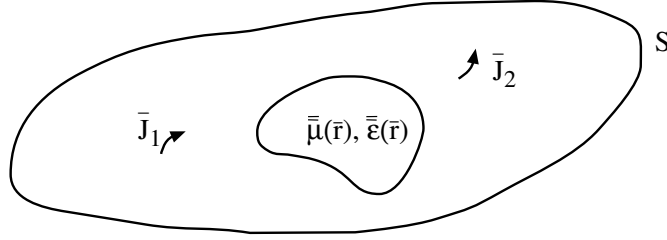
$$\nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_2 = i\omega \mathbf{J}_2 - \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}_2. \quad (1.3.15b)$$

Next, by dotting (15a) by  $\mathbf{E}_2$  and integrating over volume, we obtain

$$\begin{aligned} i\omega \langle \mathbf{E}_2, \mathbf{J}_1 \rangle - \langle \mathbf{E}_2, \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}_1 \rangle \\ = \langle \mathbf{E}_2, \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_1 \rangle - \omega^2 \langle \mathbf{E}_2, \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_1 \rangle, \end{aligned} \quad (1.3.16a)$$

where the reaction or the inner product  $\langle \mathbf{A}, \mathbf{B} \rangle = \int d\mathbf{r} \mathbf{A} \cdot \mathbf{B}$  (Rumsey 1954). By the same token, from (15b),

$$\begin{aligned} i\omega \langle \mathbf{E}_1, \mathbf{J}_2 \rangle - \langle \mathbf{E}_1, \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}_2 \rangle \\ = \langle \mathbf{E}_1, \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2 \rangle - \omega^2 \langle \mathbf{E}_1, \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_2 \rangle. \end{aligned} \quad (1.3.16b)$$



**Figure 1.3.3** *Derivation of the reciprocity relation.*

The second terms on the right-hand side of (16a) and (16b) are equal if  $\bar{\epsilon} = \bar{\epsilon}^t$ . It is unclear if the first terms on the right-hand side of (16a) and (16b) are the same, but they are of the form

$$\langle \mathbf{E}_i, \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_j \rangle = \int_V d\mathbf{r} \mathbf{E}_i \cdot \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_j. \quad (1.3.17)$$

The above can be rewritten using the vector identity  $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$  and Gauss' theorem<sup>6</sup>:

$$\begin{aligned} \langle \mathbf{E}_i, \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_j \rangle &= \int_V d\mathbf{r} (\nabla \times \mathbf{E}_i) \cdot \bar{\boldsymbol{\mu}}^{-1} \cdot (\nabla \times \mathbf{E}_j) \\ &\quad + \int_S dS \hat{n} \cdot (\bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_j) \times \mathbf{E}_i, \end{aligned} \quad (1.3.18)$$

where  $V$  and  $S$  are volume and surface respectively, tending to infinity (see Figure 1.3.3). Note that now, the first term on the right-hand side of (18) is symmetric about  $\mathbf{E}_i$  and  $\mathbf{E}_j$  if  $\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}^t$ .

To show the symmetry of the second term, however, more manipulation is needed as follows: When  $S \rightarrow \infty$ , it is reasonable to assume that  $\bar{\boldsymbol{\mu}}$  is isotropic and homogeneous. Furthermore, the fields, which are produced by sources of finite extent, become plane waves in the far field. Hence,  $\nabla \rightarrow i\mathbf{k}$ , which is the case for plane waves. Consequently,

$$(\bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_j) \times \mathbf{E}_i = i\mu_0^{-1}(\mathbf{k} \times \mathbf{E}_j) \times \mathbf{E}_i = -i\mu_0^{-1}\mathbf{k}(\mathbf{E}_i \cdot \mathbf{E}_j). \quad (1.3.19)$$

In arriving at the above, we have used  $\mathbf{k} \cdot \mathbf{E}_i = 0$  because of the plane-wave assumption. In this manner, the surface integral in (18) is symmetric about  $\mathbf{E}_i$  and  $\mathbf{E}_j$ .

<sup>6</sup> This manipulation is also referred to as integration by parts. It is the generalization of integration by parts in one dimension to higher dimensions and vector fields.

From the above, the right-hand sides of (16a) and (16b) are equal only if

$$\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}^t, \quad \bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}^t, \quad (1.3.20)$$

i.e., when  $\bar{\boldsymbol{\mu}}$  and  $\bar{\boldsymbol{\epsilon}}$  are symmetric tensors. Consequently, Equation (20) implies that

$$i\omega \langle \mathbf{E}_2, \mathbf{J}_1 \rangle - \langle \mathbf{E}_2, \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}_1 \rangle = i\omega \langle \mathbf{E}_1, \mathbf{J}_2 \rangle - \langle \mathbf{E}_1, \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}_2 \rangle. \quad (1.3.21)$$

Moreover, using the vector identity used for Equation (18), we can show that

$$\langle \mathbf{E}_2, \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}_1 \rangle = \int_V d\mathbf{r} (\nabla \times \mathbf{E}_2) \cdot \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}_1 = i\omega \int_V d\mathbf{r} \mathbf{H}_2 \cdot \mathbf{M}_1. \quad (1.3.22)$$

The last equality follows because  $\bar{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}^t$ . Hence, (21) is identical to

$$\langle \mathbf{E}_2, \mathbf{J}_1 \rangle - \langle \mathbf{H}_2, \mathbf{M}_1 \rangle = \langle \mathbf{E}_1, \mathbf{J}_2 \rangle - \langle \mathbf{H}_1, \mathbf{M}_2 \rangle, \quad (1.3.23)$$

which is the reciprocal theorem. Note that the above describes a mutually reciprocal relationship.

One side of Equation (23) describes the mutual interaction between the field of one group of sources with another group of sources. This mutual interaction is only reciprocal if the medium satisfies the conditions of Equation (20). A medium for which conditions given by Equation (20) hold, implying the reciprocal relationship (23), is known as a *reciprocal medium*. We shall see later that the reciprocal nature of (23) is due to the symmetric nature of the vector wave Equation (15). Scalar wave equations with similar symmetry also have an analogous reciprocal relation (see Exercise 1.14).

The reaction  $\langle \mathbf{E}_i, \mathbf{J}_j \rangle$  and  $\langle \mathbf{H}_i, \mathbf{M}_j \rangle$  can be thought of as generalized measurements. Physically, Equation (23) states that the field resulting from  $\mathbf{J}_1, \mathbf{M}_1$  measured by  $\mathbf{J}_2, \mathbf{M}_2$  is the same as the field resulting from  $\mathbf{J}_2, \mathbf{M}_2$  measured by  $\mathbf{J}_1, \mathbf{M}_1$ . Examples of reciprocal media are free-space and lossy media—a medium can be lossy and still be reciprocal! Examples of nonreciprocal media are plasma and ferrite media biased by a magnetic field.

### §§1.3.3 Plane Wave in Homogeneous, Anisotropic Media

A plane wave is the simplest of the wave solutions. All wave types can be expanded in terms of plane waves as shall be shown later in Chapter 2. Therefore, we shall look for a plane-wave solution in a homogeneous, anisotropic and source-free medium. In such a medium, the vector wave equation from (5a) is

$$\nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E} - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E} = 0. \quad (1.3.24)$$

To look for a plane-wave solution to (24), we assume  $\mathbf{E}$  to be of the form

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (1.3.25)$$

where  $\mathbf{k}$  is the  $\mathbf{k}$  vector denoting the direction of propagation of the plane wave. Then, substituting (25) into (24) yields

$$\mathbf{k} \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{k} \times \mathbf{E}_0 + \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_0 = 0. \quad (1.3.26)$$

Next,  $\mathbf{k} \times \mathbf{E}_0$  can be written as  $k \bar{\mathbf{K}} \cdot \mathbf{E}_0$  where  $\bar{\mathbf{K}}$  is a tensor:

$$k \bar{\mathbf{K}} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}, \quad (1.3.27)$$

which is an antisymmetric matrix in Cartesian coordinates (see Appendix B for a review of tensors). Moreover, we can write the Cartesian components of  $\mathbf{k}$  in terms of direction cosines to obtain

$$k \bar{\mathbf{K}}(\theta, \phi) = k \begin{bmatrix} 0 & -\cos \theta & \sin \theta \sin \phi \\ \cos \theta & 0 & -\sin \theta \cos \phi \\ -\sin \theta \sin \phi & \sin \theta \cos \phi & 0 \end{bmatrix}, \quad (1.3.28)$$

where  $k$ , which is yet to be found, is the length of the  $\mathbf{k}$  vector. Alternatively, Equation (26) can be written as

$$[k^2 \bar{\mathbf{F}}(\theta, \phi) + \omega^2 \bar{\mathbf{I}}] \cdot \mathbf{D}_0 = 0, \quad (1.3.29)$$

where

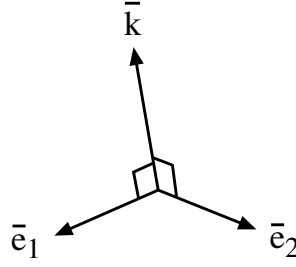
$$\bar{\mathbf{F}}(\theta, \phi) = \bar{\mathbf{K}}(\theta, \phi) \cdot \bar{\boldsymbol{\mu}}^{-1} \cdot \bar{\mathbf{K}}(\theta, \phi) \cdot \bar{\boldsymbol{\epsilon}}^{-1}, \quad \mathbf{D}_0 = \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_0. \quad (1.3.29a)$$

$\bar{\mathbf{F}}(\theta, \phi)$  is only a function of angles and tensors  $\bar{\boldsymbol{\mu}}$  and  $\bar{\boldsymbol{\epsilon}}$ . Notice that for a plane-wave solution propagating in a fixed direction,  $\bar{\mathbf{F}}(\theta, \phi)$  is a constant matrix.

Equation (29) corresponds to an eigenvalue problem where  $\omega^2/k^2$  is the eigenvalue and  $\mathbf{D}_0$  is the eigenvector. Since  $\bar{\mathbf{F}}$  is a  $3 \times 3$  matrix, we expect the above equations to have three eigenvalues and three eigenvectors. However, from the  $\nabla \cdot \mathbf{D} = 0$  condition,  $\mathbf{k} \cdot \mathbf{D} = 0$ . This can also be seen by dotting Equation (26) with  $\mathbf{k}$  and noting that  $\mathbf{k} \cdot \mathbf{k} \times \mathbf{A} = 0$ . Therefore, only two out of three components of the electric flux  $\mathbf{D}$  are independent, implying that only two equations in (29) are independent. Consequently, it will only yield two eigenvalues and two eigenvectors. This can be shown easily by expressing the field and the tensors in a coordinate system where the  $z$  axis corresponds to the  $\mathbf{k}$  direction of the  $\mathbf{k}$  vector (see Exercise 1.15). Hence, the general wave solution to (24) is of the form

$$\mathbf{E} = a_1 \mathbf{e}_1 e^{i\mathbf{k}_1 \cdot \mathbf{r}} + a_2 \mathbf{e}_2 e^{i\mathbf{k}_2 \cdot \mathbf{r}} = \sum_{j=1}^2 a_j \mathbf{e}_j e^{i\mathbf{k}_j \cdot \mathbf{r}}, \quad (1.3.30)$$

where  $\mathbf{k}_j = k_j \hat{\mathbf{k}}, k_j$  is derived from the  $j$ -th eigenvalue, and  $\mathbf{e}_j$  is derived from the  $j$ -th eigenvector of (29). Because  $k$  is different for the two waves, they



**Figure 1.3.4** *The electric field vectors and the  $\mathbf{k}$  vector in an isotropic medium.*

have  $\mathbf{k}$  vectors of different lengths. Moreover, since the phase velocity for a plane wave is defined by  $v_p = \omega/k$ , the phase velocities for the two waves will be different. These two waves are generally known as type I and type II waves. Furthermore, their corresponding magnetic field can easily be derived to be

$$\mathbf{H} = \sum_{j=1}^2 a_j (\omega \bar{\boldsymbol{\mu}})^{-1} \cdot \mathbf{k}_j \times \mathbf{e}_j e^{i\mathbf{k}_j \cdot \mathbf{r}} = \sum_{j=1}^2 a_j \mathbf{h}_j e^{i\mathbf{k}_j \cdot \mathbf{r}}. \quad (1.3.31)$$

Note that since  $\bar{\mathbf{F}}$  is a function of angles, the eigenvalues  $k_j$  or the lengths of the  $\mathbf{k}_j$  vectors change as a function of angle, i.e.,  $k_j(\theta, \phi)$ . Therefore, the phase velocity in an anisotropic medium is also a function of angles for each of the two types of waves. With  $\mathbf{k}_j$  changing as a function of angles,  $\mathbf{e}_j$  also changes as a function of angles. Furthermore, since  $\nabla \cdot \mathbf{E} \neq 0$  in general,  $\mathbf{k} \cdot \mathbf{E} \neq 0$  for anisotropic media, unlike the case for isotropic media.

For a homogeneous, isotropic medium, the eigenvalues are degenerate where  $k_j = \omega \sqrt{\bar{\mu}\bar{\epsilon}}$  and  $\mathbf{k}_j \cdot \mathbf{e}_j = 0$  for  $j = 1, 2$ . The eigenvectors of (29) then are any two linearly independent vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  that are orthogonal to  $\mathbf{k}$  (Figure 1.3.4). Moreover,  $\mathbf{e}_1$  and  $\mathbf{e}_2$  can also be made orthogonal to each other without loss of generality. The general solution then becomes

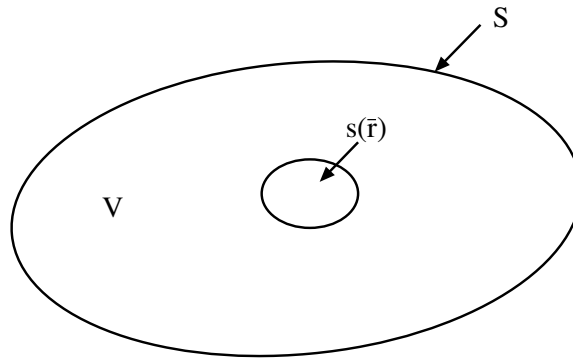
$$\mathbf{E} = a_1 (\mathbf{k} \times \mathbf{c}) e^{i\mathbf{k} \cdot \mathbf{r}} + a_2 (\mathbf{k} \times \mathbf{k} \times \mathbf{c}) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (1.3.32)$$

where  $\mathbf{c}$  is an arbitrary constant vector. For instance, if  $\mathbf{c} = \hat{z}$  which points upward, then the above two waves correspond to a horizontal polarization and a vertical polarization. The first corresponds to a transverse electric (TE) to  $z$  wave while the second corresponds to a transverse magnetic (TM) to  $z$  wave.

### §§1.3.4 Green's Function

The Green's function of a wave equation is the solution of the wave equation for a point source. And when the solution to the wave equation due to a





**Figure 1.3.5** *The radiation of a source  $s(\mathbf{r})$  in a volume  $V$ .*

point source is known, the solution due to a general source can be obtained by the principle of linear superposition (see Figure 1.3.5). This is merely a result of the linearity of the wave equation, and that a general source is just a linear superposition of point sources.

For example, to obtain the solution to the scalar wave equation in  $V$  in Figure 1.3.5,

$$(\nabla^2 + k^2) \psi(\mathbf{r}) = s(\mathbf{r}), \quad (1.3.33)$$

we first seek the Green's function in the same  $V$ , which is the solution to the following equation:

$$(\nabla^2 + k^2) g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \quad (1.3.34)$$

Given  $g(\mathbf{r}, \mathbf{r}')$ ,  $\psi(\mathbf{r})$  can be found easily from the principle of linear superposition, since  $g(\mathbf{r}, \mathbf{r}')$  is the solution to (33) with a point source on the right-hand side. To see this more clearly, note that an arbitrary source  $s(\mathbf{r})$  is just

$$s(\mathbf{r}) = \int d\mathbf{r}' s(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}'), \quad (1.3.35)$$

which is actually a linear superposition of point sources in mathematical terms. Consequently, the solution to (33) is just

$$\psi(\mathbf{r}) = - \int_V d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') s(\mathbf{r}'), \quad (1.3.36)$$

which is an integral linear superposition of the solution of (34). Moreover, it can be seen that  $g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}', \mathbf{r})$  from reciprocity irrespective of the shape of  $V$  (see Exercises 1.14, 1.17).

To find the solution of Equation (34) for an unbounded, homogeneous medium, one solves it in spherical coordinates with the origin at  $\mathbf{r}'$ . By so doing, (34) becomes

$$(\nabla^2 + k^2) g(\mathbf{r}) = -\delta(\mathbf{r}) = -\delta(x) \delta(y) \delta(z). \quad (1.3.37)$$

But due to the spherical symmetry of a point source,  $g(\mathbf{r})$  must also be spherically symmetric. Then, for  $r \neq 0$ , the homogeneous, spherically symmetric solution to (37) is given by

$$g(\mathbf{r}) = C \frac{e^{ikr}}{r} + D \frac{e^{-ikr}}{r}. \quad (1.3.38)$$

Since sources are absent at infinity, physical grounds then imply that only an outgoing solution can exist; hence,

$$g(\mathbf{r}) = C \frac{e^{ikr}}{r}. \quad (1.3.39)$$

The constant  $C$  is found by matching the singularities at the origin on both sides of (37). To do this, we substitute (39) into (37) and integrate Equation (37) over a small volume about the origin to yield

$$\int_{\Delta V} dV \nabla \cdot \nabla \frac{C e^{ikr}}{r} + \int_{\Delta V} dV k^2 \frac{C e^{ikr}}{r} = -1. \quad (1.3.40)$$

Note that the second integral vanishes when  $\Delta V \rightarrow 0$ , because  $dV = 4\pi r^2 dr$ . Moreover, the first integral in (40) can be converted into a surface integral using Gauss' theorem to obtain

$$\lim_{r \rightarrow 0} 4\pi r^2 \frac{d}{dr} C \frac{e^{ikr}}{r} = -1, \quad (1.3.41)$$

or  $C = 1/4\pi$ .

The solution to (34) must depend only on  $|\mathbf{r} - \mathbf{r}'|$ . Therefore, in general,

$$g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r} - \mathbf{r}') = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}, \quad (1.3.42)$$

implying that  $g(\mathbf{r}, \mathbf{r}')$  is translationally invariant for unbounded, homogeneous media. Consequently, the solution to (33), from Equation (36), is then

$$\psi(\mathbf{r}) = - \int_V d\mathbf{r}' \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} s(\mathbf{r}'). \quad (1.3.43)$$

The Green's function for the scalar wave equation could be used to find the dyadic Green's function for the vector wave equation in a homogeneous,

isotropic medium. First, notice that the vector wave equation in a homogeneous, isotropic medium is

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2 \mathbf{E}(\mathbf{r}) = i\omega\mu \mathbf{J}(\mathbf{r}). \quad (1.3.44)$$

Then, by using the fact that  $\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E} + \nabla \nabla \cdot \mathbf{E}$  and that  $\nabla \cdot \mathbf{E} = \rho/\epsilon = \nabla \cdot \mathbf{J}/i\omega\epsilon$ , which follows from the continuity equation, we can rewrite (44) as

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = -i\omega\mu \left[ \bar{\mathbf{I}} + \frac{\nabla \nabla}{k^2} \right] \cdot \mathbf{J}(\mathbf{r}), \quad (1.3.45)$$

where  $\bar{\mathbf{I}}$  is an identity operator. In Cartesian coordinates, there are actually three scalar wave equations embedded in the above vector equation, each of which can be solved easily in the manner of Equation (36). Consequently,

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \int_V d\mathbf{r}' g(\mathbf{r}' - \mathbf{r}) \left[ \bar{\mathbf{I}} + \frac{\nabla' \nabla'}{k^2} \right] \cdot \mathbf{J}(\mathbf{r}') \quad (1.3.46)$$

where  $g(\mathbf{r}' - \mathbf{r})$  is the unbounded medium scalar Green's function. Moreover, by using the vector identities<sup>7</sup>  $\nabla g f = f \nabla g + g \nabla f$  and  $\nabla \cdot g \mathbf{F} = g \nabla \cdot \mathbf{F} + (\nabla g) \cdot \mathbf{F}$ , it can be shown that

$$\int_V d\mathbf{r}' g(\mathbf{r}' - \mathbf{r}) \nabla' f(\mathbf{r}') = - \int_V d\mathbf{r}' [\nabla' g(\mathbf{r}' - \mathbf{r})] f(\mathbf{r}'), \quad (1.3.47)$$

and

$$\int_V d\mathbf{r}' [\nabla' g(\mathbf{r}' - \mathbf{r})] \nabla' \cdot \mathbf{J}(\mathbf{r}') = - \int_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') \cdot \nabla' \nabla' g(\mathbf{r}' - \mathbf{r}). \quad (1.3.48)$$

Hence, Equation (46) can be rewritten as

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \int_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') \cdot \left[ \bar{\mathbf{I}} + \frac{\nabla' \nabla'}{k^2} \right] g(\mathbf{r}' - \mathbf{r}). \quad (1.3.49)$$

It can also be derived using scalar and vector potentials (see Exercise 1.16 and Chapter 7).

Alternatively, Equation (49) can be written as

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \int_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') \cdot \bar{\mathbf{G}}_e(\mathbf{r}', \mathbf{r}), \quad (1.3.50)$$

---

<sup>7</sup> The first identity is the same as the second identity if we think of  $\mathbf{F}(\mathbf{r}) = \mathbf{a}f(\mathbf{r})$  where  $\mathbf{a}$  is an arbitrary constant vector. Since  $\mathbf{a}$  is an arbitrary constant vector, we can cancel it from both sides of the equation to obtain the first identity.

where

$$\overline{\mathbf{G}}_e(\mathbf{r}', \mathbf{r}) = \left[ \overline{\mathbf{I}} + \frac{\nabla' \nabla'}{k^2} \right] g(\mathbf{r}' - \mathbf{r}) \quad (1.3.51)$$

is a dyad known as the dyadic Green's function for the electric field in an unbounded, homogeneous medium. (A *dyad* is a  $3 \times 3$  matrix that transforms a vector to a vector. It is also a second rank tensor. See Appendix B for details.) Even though (50) is established for an unbounded, homogeneous medium, such a general relationship also exists in a bounded, homogeneous medium. It could easily be shown from reciprocity that

$$\begin{aligned} \langle \mathbf{J}_1(\mathbf{r}), \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'), \mathbf{J}_2(\mathbf{r}') \rangle &= \langle \mathbf{J}_2(\mathbf{r}'), \overline{\mathbf{G}}_e(\mathbf{r}', \mathbf{r}), \mathbf{J}_1(\mathbf{r}) \rangle \\ &= \langle \mathbf{J}_1(\mathbf{r}), \overline{\mathbf{G}}_e^t(\mathbf{r}', \mathbf{r}), \mathbf{J}_2(\mathbf{r}') \rangle, \end{aligned} \quad (1.3.52)$$

where

$$\langle \mathbf{J}_i(\mathbf{r}'), \overline{\mathbf{G}}_e(\mathbf{r}', \mathbf{r}), \mathbf{J}_j(\mathbf{r}) \rangle = \int_V \int_V d\mathbf{r}' d\mathbf{r} \mathbf{J}_i(\mathbf{r}') \cdot \overline{\mathbf{G}}_e(\mathbf{r}', \mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}), \quad (1.3.52a)$$

is the reaction between  $\mathbf{J}_i$  and the electric field produced by  $\mathbf{J}_j$ . Notice that the above implies that<sup>8</sup>

$$\overline{\mathbf{G}}_e^t(\mathbf{r}', \mathbf{r}) = \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'). \quad (1.3.52b)$$

Then, by taking its transpose, (50) becomes

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \int_V d\mathbf{r}' \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}'). \quad (1.3.53)$$

Alternatively, the dyadic Green's function for an unbounded, homogeneous medium can also be written as

$$\overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = \frac{1}{k^2} [\nabla \times \nabla \times \overline{\mathbf{I}} g(\mathbf{r} - \mathbf{r}') - \overline{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}')]. \quad (1.3.54)$$

By substituting (53) back into (44) and writing

$$\mathbf{J}(\mathbf{r}) = \int d\mathbf{r}' \overline{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}'), \quad (1.3.55)$$

we can show quite easily that

$$\nabla \times \nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') - k^2 \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}'). \quad (1.3.56)$$

Equation (50) or (53), due to the  $\nabla \nabla$  operator inside the integration operating on  $g(\mathbf{r} - \mathbf{r}')$ , has a singularity of  $1/|\mathbf{r} - \mathbf{r}'|^3$  when  $\mathbf{r} \rightarrow \mathbf{r}'$ . Consequently, it has to be redefined in this case for it does not converge uniformly,

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<sup>8</sup> Similar relations also hold for scalar wave equation (see Exercise 1.17).

specifically, when  $\mathbf{r}$  is also in the source region occupied by  $\mathbf{J}(\mathbf{r})$ . Hence, at this point, the evaluation of Equation (53) in a source region is undefined. This singular nature of the dyadic Green's function will be addressed later in Chapter 7.

### §1.4 Huygens' Principle

Huygens' principle shows how a wave field on a surface  $S$  determines the wave field off the surface  $S$ . This concept can be expressed mathematically for both scalar and vector waves. We shall first discuss the scalar wave case first, followed by the electromagnetic wave case.

#### §§1.4.1 Scalar Waves

For a  $\psi(\mathbf{r})$  that satisfies the scalar wave equation

$$(\nabla^2 + k^2) \psi(\mathbf{r}) = 0, \quad (1.4.1)$$

the corresponding scalar Green's function  $g(\mathbf{r}, \mathbf{r}')$  satisfies

$$(\nabla^2 + k^2) g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \quad (1.4.2)$$

Next, on multiplying (1) by  $g(\mathbf{r}, \mathbf{r}')$  and (2) by  $\psi(\mathbf{r})$ , subtracting the resultant equations and integrating over a volume containing  $\mathbf{r}'$ , we have

$$\int_V d\mathbf{r} [g(\mathbf{r}, \mathbf{r}') \nabla^2 \psi(\mathbf{r}) - \psi(\mathbf{r}) \nabla^2 g(\mathbf{r}, \mathbf{r}')] = \psi(\mathbf{r}'). \quad (1.4.3)$$

Since  $g \nabla^2 \psi - \psi \nabla^2 g = \nabla \cdot (g \nabla \psi - \psi \nabla g)$ , the left-hand side of (3) can be rewritten using Gauss' divergence theorem, giving<sup>9</sup>

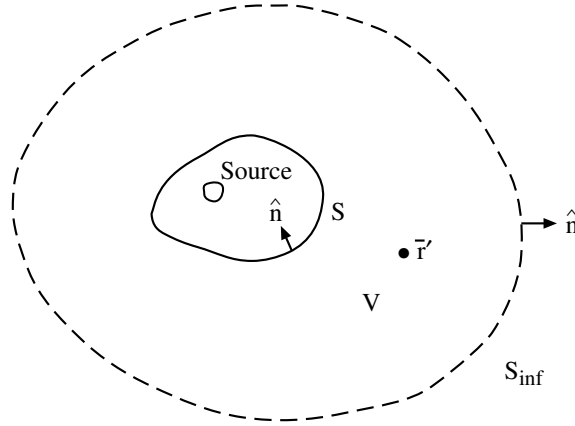
$$\psi(\mathbf{r}') = \oint_S dS \hat{n} \cdot [g(\mathbf{r}, \mathbf{r}') \nabla \psi(\mathbf{r}) - \psi(\mathbf{r}) \nabla g(\mathbf{r}, \mathbf{r}')], \quad (1.4.4)$$

where  $S$  is the surface bounding  $V$ . The above is the mathematical expression that once  $\psi(\mathbf{r})$  and  $\hat{n} \cdot \nabla \psi(\mathbf{r})$  are known on  $S$ , then  $\psi(\mathbf{r}')$  away from  $S$  could be found.

If the volume  $V$  is bounded by  $S$  and  $S_{inf}$  as shown in Figure 1.4.1, then the surface integral in (4) should include an integral over  $S_{inf}$ . But when  $S_{inf} \rightarrow \infty$ , all fields look like plane wave, and  $\nabla \rightarrow \hat{r}ik$  on  $S_{inf}$ . Furthermore,  $g(\mathbf{r} - \mathbf{r}') \sim O(1/r)$ ,<sup>10</sup> when  $r \rightarrow \infty$ , and  $\psi(\mathbf{r}) \sim O(1/r)$ , when  $r \rightarrow \infty$ , if  $\psi(\mathbf{r})$  is due to a source of finite extent. Then, the integral over  $S_{inf}$  in (4) vanishes,

<sup>9</sup> The equivalence of the volume integral in (3) to the surface integral in (4) is also known as Green's theorem.

<sup>10</sup> The symbol "O" means "of the order."



**Figure 1.4.1** *The geometry for the derivation of Huygens' principle.*

and (4) is valid for the case shown in Figure 1.4.1 as well. Here, the field outside  $S$  at  $\mathbf{r}'$  is expressible in terms of the field on  $S$ .

Notice that in deriving (4),  $g(\mathbf{r}, \mathbf{r}')$  has only to satisfy (2) for both  $\mathbf{r}$  and  $\mathbf{r}'$  in  $V$  but no boundary condition has yet been imposed on  $g(\mathbf{r}, \mathbf{r}')$ . Therefore, if we further require that  $g(\mathbf{r}, \mathbf{r}') = 0$  for  $\mathbf{r} \in S$ , then (4) becomes

$$\psi(\mathbf{r}') = - \oint_S dS \psi(\mathbf{r}) \hat{n} \cdot \nabla g(\mathbf{r}, \mathbf{r}'). \quad (1.4.5)$$

On the other hand, if we require additionally that  $g(\mathbf{r}, \mathbf{r}')$  satisfies (2) with the boundary condition  $\hat{n} \cdot \nabla g(\mathbf{r}, \mathbf{r}') = 0$  for  $\mathbf{r} \in S$ , then (4) becomes

$$\psi(\mathbf{r}') = \oint_S dS g(\mathbf{r}, \mathbf{r}') \hat{n} \cdot \nabla \psi(\mathbf{r}). \quad (1.4.6)$$

Equations (4), (5), and (6) are various forms of Huygens' principle depending on the definition of  $g(\mathbf{r}, \mathbf{r}')$ . Equations (5) and (6) stipulate that only  $\psi(\mathbf{r})$  or  $\hat{n} \cdot \nabla \psi(\mathbf{r})$  need be known on the surface  $S$  in order to determine  $\psi(\mathbf{r}')$ . (Note that in the above derivation,  $k^2$  could be a function of position as well.)

### §§1.4.2 *Electromagnetic Waves*

In a source-free region, an electromagnetic wave satisfies the vector wave equation

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2 \mathbf{E}(\mathbf{r}) = 0. \quad (1.4.7)$$

Moreover, the dyadic Green's function satisfies the equation

$$\nabla \times \nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') - k^2 \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}'). \quad (1.4.8)$$

Then, after post-multiplying (7) by  $\overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')$ , pre-multiplying (8) by  $\mathbf{E}(\mathbf{r})$ , subtracting the resultant equations and integrating the difference over volume  $V$ , we have

$$\mathbf{E}(\mathbf{r}') = \int_V dV [\mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') + \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')]. \quad (1.4.9)$$

Next, using the vector identity that<sup>11</sup>

$$\begin{aligned} -\nabla \cdot [\mathbf{E}(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') + \nabla \times \mathbf{E}(\mathbf{r}) \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')] \\ = \mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') - \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'), \end{aligned} \quad (1.4.10)$$

Equation (9), with the help of Gauss' divergence theorem, can be written as

$$\begin{aligned} \mathbf{E}(\mathbf{r}') &= - \oint_S dS \hat{n} \cdot [\mathbf{E}(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') + \nabla \times \mathbf{E}(\mathbf{r}) \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')] \\ &= - \oint_S dS [\hat{n} \times \mathbf{E}(\mathbf{r}) \cdot \nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') + i\omega\mu \hat{n} \times \mathbf{H}(\mathbf{r}) \cdot \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')]. \end{aligned} \quad (1.4.11)$$

The above is just the vector analogue of (4). Again, notice that (11) is derived via the use of (8), but no boundary condition has yet been imposed on  $\overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')$  on  $S$ . Now, if we require that  $\hat{n} \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = 0$  for  $\mathbf{r} \in S$ , then (11) becomes

$$\mathbf{E}(\mathbf{r}') = - \oint_S dS \hat{n} \times \mathbf{E}(\mathbf{r}) \cdot \nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'), \quad (1.4.12)$$

for it could be shown that  $\hat{n} \times \mathbf{H} \cdot \overline{\mathbf{G}}_e = \mathbf{H} \cdot \hat{n} \times \overline{\mathbf{G}}_e$  implying that the second term in (11) is zero. On the other hand, if we require that  $\hat{n} \times \nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = 0$  for  $\mathbf{r} \in S$ , then (11) becomes

$$\mathbf{E}(\mathbf{r}') = -i\omega\mu \oint_S dS \hat{n} \times \mathbf{H}(\mathbf{r}) \cdot \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}'). \quad (1.4.13)$$

Equations (12) and (13) state that  $\mathbf{E}(\mathbf{r}')$  is determined if either  $\hat{n} \times \mathbf{E}(\mathbf{r})$  or  $\hat{n} \times \mathbf{H}(\mathbf{r})$  is specified on  $S$ .

It can be shown from reciprocity that (Exercise 1.18)

$$[\overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')]^t = \overline{\mathbf{G}}_e(\mathbf{r}', \mathbf{r}), \quad (1.4.14a)$$

---

<sup>11</sup> This identity can be established by using the identity  $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$ . The equality of the volume integral in (9) to the surface integral in (11) is also known as vector Green's theorem.

$$[\nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')]^t = \nabla' \times \overline{\mathbf{G}}_m(\mathbf{r}', \mathbf{r}), \quad (1.4.14b)$$

where  $\overline{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}')$  is the dyadic Green's function for magnetic field, and  $\overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')$  is the dyadic Green's function for electric field. Then, by taking its transpose, Equation (11) becomes

$$\mathbf{E}(\mathbf{r}') = -\nabla' \times \oint_S dS \overline{\mathbf{G}}_m(\mathbf{r}', \mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r}) - i\omega\mu \oint_S dS \overline{\mathbf{G}}_e(\mathbf{r}', \mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r}). \quad (1.4.15)$$

Moreover, Equation (12) then becomes

$$\mathbf{E}(\mathbf{r}') = -\nabla' \times \oint_S dS \overline{\mathbf{G}}_m(\mathbf{r}', \mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r}), \quad (1.4.16)$$

while Equation (13) becomes

$$\mathbf{E}(\mathbf{r}') = -i\omega\mu \oint_S dS \overline{\mathbf{G}}_e(\mathbf{r}', \mathbf{r}) \cdot \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r}). \quad (1.4.17)$$

The dyadic Green's functions in (12), (13), (16), and (17) are for a closed cavity since boundary conditions are imposed on  $S$  for them. But the dyadic Green's function for an unbounded, homogeneous medium can be written as

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \frac{1}{k^2} [\nabla \times \nabla \times \overline{\mathbf{I}}g(\mathbf{r} - \mathbf{r}') - \overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}')], \quad (1.4.18)$$

$$\nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \nabla \times \overline{\mathbf{I}}g(\mathbf{r} - \mathbf{r}'). \quad (1.4.19)$$

Also, for unbounded homogeneous medium,  $\overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}')$ . Then, (15) becomes

$$\mathbf{E}(\mathbf{r}') = -\nabla' \times \oint_S dS g(\mathbf{r} - \mathbf{r}') \hat{\mathbf{n}} \times \mathbf{E}(\mathbf{r}) + \frac{1}{i\omega\epsilon} \nabla' \times \nabla' \times \oint_S dS g(\mathbf{r} - \mathbf{r}') \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r}). \quad (1.4.20)$$

The above can be applied to the geometry in Figure 1.4.1 where  $\mathbf{r}'$  is enclosed in  $S$  and  $S_{inf}$ . However, the integral over  $S_{inf}$  vanishes by virtue of the radiation condition as for (4). Then, (20) relates the field outside  $S$  at  $\mathbf{r}'$  in terms of only the field on  $S$ .

### §1.5 Uniqueness Theorem

The uniqueness theorem provides conditions under which the solution to the wave equation is unique. This is especially important because the solutions to a problem should not be indeterminate. These conditions under which a solution to a wave equation is unique are the boundary conditions and the radiation condition. Uniqueness also allows one to construct solutions by inspections; if a candidate solution satisfies the conditions of uniqueness,



it is the unique solution. Because of its simplicity, the scalar wave equation shall be examined first for greater insight into this problem.

### §§1.5.1 *Scalar Wave Equation*

Given a scalar wave equation with a source term on the right-hand side, we shall derive the conditions under which a solution is unique. First, assume that there are two different solutions to the scalar wave equation, namely,

$$[\nabla^2 + k^2(\mathbf{r})] \phi_1(\mathbf{r}) = s(\mathbf{r}), \quad (1.5.1a)$$

$$[\nabla^2 + k^2(\mathbf{r})] \phi_2(\mathbf{r}) = s(\mathbf{r}), \quad (1.5.1b)$$

where  $k^2(\mathbf{r})$  includes inhomogeneities of finite extent. Then, on subtracting the two equations, we have

$$[\nabla^2 + k^2(\mathbf{r})] \delta\phi(\mathbf{r}) = 0, \quad (1.5.2)$$

where  $\delta\phi(\mathbf{r}) = \phi_1(\mathbf{r}) - \phi_2(\mathbf{r})$ . Note that the solution is unique if and only if  $\delta\phi = 0$  for all  $\mathbf{r}$ .

Then, after multiplying (2) by  $\delta\phi^*$ , integrating over volume, and using the vector identity  $\nabla \cdot \psi \mathbf{A} = \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{A}$ , we have

$$\int_S \hat{n} \cdot (\delta\phi^* \nabla \delta\phi) dS - \int_V |\nabla \delta\phi|^2 dV + \int_V k^2 |\delta\phi|^2 dV = 0, \quad (1.5.3)$$

where  $\hat{n}$  is a unit normal to the surface  $S$ . Then, the imaginary part of the above equation is

$$\Im \int_S \hat{n} \cdot (\delta\phi^* \nabla \delta\phi) dS + \int_V \Im(k^2) |\delta\phi|^2 dV = 0. \quad (1.5.4)$$

Hence, if  $\Im[k^2(\mathbf{r})] \neq 0$  in  $V$ , and

- (i)  $\delta\phi = 0$  or  $\hat{n} \cdot \nabla \delta\phi = 0$  on  $S$ , or
- (ii)  $\delta\phi = 0$  on part of  $S$  and  $\hat{n} \cdot \nabla \delta\phi = 0$  on the rest of  $S$ ,

then

$$\int_V \Im[k^2(\mathbf{r})] |\delta\phi|^2 dV = 0. \quad (1.5.5)$$

Since  $|\delta\phi|^2$  is positive definite for  $\delta\phi \neq 0$ , and  $\Im(k^2) \neq 0$  in  $V$ ,<sup>12</sup> the above is only possible if  $\delta\phi = 0$  everywhere inside  $V$ .

Therefore, in order to guarantee uniqueness, so that  $\phi_1 = \phi_2$  in  $V$ , either

- (i)  $\phi_1 = \phi_2$  on  $S$  or  $\hat{n} \cdot \nabla \phi_1 = \hat{n} \cdot \nabla \phi_2$  on  $S$ , or

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<sup>12</sup> More specifically,  $\Im[k^2(\mathbf{r})] > 0, \forall \mathbf{r} \in V$ , or  $\Im[k^2(\mathbf{r})] < 0, \forall \mathbf{r} \in V$ .

(ii)  $\phi_1 = \phi_2$  on one part of  $S$ , and  $\hat{n} \cdot \nabla \phi_1 = \hat{n} \cdot \nabla \phi_2$  on the rest of  $S$ .

The specification of  $\phi$  on  $S$  is also known as the *Dirichlet* boundary condition, while the specification of  $\hat{n} \cdot \nabla \phi$ , namely, the normal derivative, is also known as the *Neumann* boundary condition. In words, the uniqueness theorem says that if two solutions satisfy the same Dirichlet or Neumann boundary condition or a mixture thereof on  $S$ , the two solutions must be identical.

When  $\Im m(k^2) = 0$ , i.e., when  $k^2$  is real, the condition  $\delta\phi = 0$  or  $\hat{n} \cdot \nabla \delta\phi = 0$  on  $S$  in (3) does not necessarily lead to  $\delta\phi = 0$  in  $V$ , or uniqueness. The reason is that solutions for  $\delta\phi = \phi_1 - \phi_2$  where

$$\int_V |\nabla \delta\phi|^2 dV = \int_V k^2 |\delta\phi|^2 dV \quad (1.5.6)$$

can exist. These are the resonance solutions in the volume  $V$  (see Exercise 1.19). These resonance solutions are the homogeneous solutions<sup>13</sup> to the wave Equation (1) at the real resonance frequencies of the volume  $V$ . Because the medium is lossless, they are time harmonic solutions which satisfies the boundary conditions, and hence, can be added to the particular solution of (1). In fact, the particular solution usually becomes infinite at these resonance frequencies if  $S(\mathbf{r}) \neq 0$ .

Equation (6) implies the balance of two energies. In the case of acoustic waves, for example, it represents the balance of the kinetic energy and the potential energy in a volume  $V$ .

When  $\Im m(k^2) \neq 0$ , however, the resonance solutions of the volume  $V$  are exponentially decaying with time for a lossy medium [ $\Im m(k^2) > 0$ ], and they are exponentially growing with time for an active medium [ $\Im m(k^2) < 0$ ]. But if only time harmonic solutions  $\phi_1$  and  $\phi_2$  are permitted in (1), these resonance solutions are automatically eliminated from the class of permissible solutions. Hence, for a lossy medium [ $\Im m(k^2) > 0$ ] or an active medium [ $\Im m(k^2) < 0$ ], the uniqueness of the solution is guaranteed if we consider only time harmonic solutions, namely, two solutions will be identical if they have the same boundary conditions for  $\phi$  and  $\hat{n} \cdot \nabla \phi$  on  $S$ .<sup>14</sup>

When  $S \rightarrow \infty$  or  $V \rightarrow \infty$ , the number of resonance frequencies of  $V$  becomes denser. In fact, when  $S \rightarrow \infty$ , the resonance frequencies of  $V$  become a continuum implying that any real frequency could be the resonant frequency of  $V$ . Hence, if the medium is lossless, the uniqueness of the solution is not

<sup>13</sup> “Homogeneous solutions” is a mathematical parlance for solutions to (1) without the source term.

<sup>14</sup> The nonuniqueness associated with the resonance solution for a lossless medium can be eliminated if we consider time domain solutions. In the time domain, we can set up an initial value problem in time, e.g., by requiring all fields be zero for  $t < 0$ ; thus, the nonuniqueness problem can be removed via the causality requirement. The resonance solution, being time harmonic, is noncausal.

guaranteed at any frequency, even appropriate boundary conditions on  $S$  at infinity, as a result of the presence of the continuum of resonance frequencies. One remedy then is to introduce a small loss. With this small loss [ $\Im m(k) > 0$ ], the solution is either exponentially small when  $r \rightarrow \infty$  (if a solution corresponds to an outgoing wave,  $e^{ikr}$ ), or exponentially large when  $r \rightarrow \infty$  (if a solution corresponds to an incoming wave,  $e^{-ikr}$ ). Now, if the solution is exponentially small, namely, keeping only the outgoing wave solutions, it is clear that the surface integral term in (4) vanishes when  $S \rightarrow \infty$ , and the uniqueness of the solution is guaranteed. This manner of imposing the outgoing wave condition at infinity is also known as the **Sommerfeld radiation condition** (Sommerfeld 1949, p. 188). This radiation condition can be used in the limit of a vanishing loss for an unbounded medium to guarantee uniqueness.

### §§1.5.2 Vector Wave Equation

Similar to the uniqueness conditions for the scalar wave equation, analogous conditions for the vector wave equation can also be derived. First, assume that there are two different solutions to a vector wave Equation (1.3.5), i.e.,

$$\nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_1(\mathbf{r}) - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_1(\mathbf{r}) = \mathbf{S}(\mathbf{r}), \quad (1.5.7a)$$

$$\nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_2(\mathbf{r}) - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}_2(\mathbf{r}) = \mathbf{S}(\mathbf{r}), \quad (1.5.7b)$$

where  $\mathbf{S}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}) - \nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}(\mathbf{r})$  corresponds to a source of finite extent. Similarly,  $\bar{\boldsymbol{\mu}}$  and  $\bar{\boldsymbol{\epsilon}}$  correspond to an inhomogeneity of finite extent. Subtracting (7a) from (7b) then yields

$$\nabla \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E} - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \delta \mathbf{E} = 0, \quad (1.5.8)$$

where  $\delta \mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2$ . The solution is unique if and only if  $\delta \mathbf{E} = 0$ . Next, on multiplying the above by  $\delta \mathbf{E}^*$ , integrating over volume  $V$ , and using the vector identity  $\mathbf{A} \cdot \nabla \times \mathbf{B} = -\nabla \cdot (\mathbf{A} \times \mathbf{B}) + \mathbf{B} \cdot \nabla \times \mathbf{A}$ , we have

$$\begin{aligned} - \int_S \hat{\mathbf{n}} \cdot (\delta \mathbf{E}^* \times \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E}) dS + \int_V \nabla \times \delta \mathbf{E}^* \cdot \bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \delta \mathbf{E} dV \\ - \omega^2 \int_V \delta \mathbf{E}^* \cdot \bar{\boldsymbol{\epsilon}} \cdot \delta \mathbf{E} dV = 0. \end{aligned} \quad (1.5.9)$$

Since  $\nabla \times \delta \mathbf{E} = i\omega \bar{\boldsymbol{\mu}} \cdot \delta \mathbf{H}$ , the above can be rewritten as

$$i\omega \int_S \hat{\mathbf{n}} \cdot (\delta \mathbf{E}^* \times \delta \mathbf{H}) dS + \omega^2 \int_V (\delta \mathbf{H}^* \cdot \bar{\boldsymbol{\mu}}^\dagger \cdot \delta \mathbf{H} - \delta \mathbf{E}^* \cdot \bar{\boldsymbol{\epsilon}} \cdot \delta \mathbf{E}) dV = 0. \quad (1.5.10)$$

Then, taking the imaginary part of (10) yields

$$\begin{aligned} \Im m \left\{ i\omega \int_S \hat{n} \cdot (\delta \mathbf{E}^* \times \delta \mathbf{H}) dS \right\} \\ - \frac{i\omega^2}{2} \int_V [\delta \mathbf{H}^* \cdot (\bar{\boldsymbol{\mu}}^\dagger - \bar{\boldsymbol{\mu}}) \cdot \delta \mathbf{H} + \delta \mathbf{E}^* \cdot (\bar{\boldsymbol{\epsilon}}^\dagger - \bar{\boldsymbol{\epsilon}}) \cdot \delta \mathbf{E}] dV = 0. \end{aligned} \quad (1.5.11)$$

But if the medium is not lossless (either lossy or active), then  $\bar{\boldsymbol{\mu}}^\dagger \neq \bar{\boldsymbol{\mu}}$  and  $\bar{\boldsymbol{\epsilon}}^\dagger \neq \bar{\boldsymbol{\epsilon}}$  [see (1.1.2b)], and the second integral in (11) may not be zero. Moreover, if

- (i)  $\hat{n} \times \delta \mathbf{E} = 0$  or  $\hat{n} \times \delta \mathbf{H} = 0$  on  $S$ , or
- (ii)  $\hat{n} \times \delta \mathbf{E} = 0$  on one part of  $S$  and  $\hat{n} \times \delta \mathbf{H} = 0$  on the rest of  $S$ ,

the first integral in (11) vanishes. In this case,

$$\frac{\omega^2}{2} \int_V [\delta \mathbf{H}^* \cdot i(\bar{\boldsymbol{\mu}}^\dagger - \bar{\boldsymbol{\mu}}) \cdot \delta \mathbf{H} + \delta \mathbf{E}^* \cdot i(\bar{\boldsymbol{\epsilon}}^\dagger - \bar{\boldsymbol{\epsilon}}) \cdot \delta \mathbf{E}] dV = 0. \quad (1.5.12)$$

In the above,  $i(\bar{\boldsymbol{\mu}}^\dagger - \bar{\boldsymbol{\mu}})$  and  $i(\bar{\boldsymbol{\epsilon}}^\dagger - \bar{\boldsymbol{\epsilon}})$  are Hermitian matrices. Moreover, the integrand will be positive definite if both  $\bar{\boldsymbol{\mu}}$  and  $\bar{\boldsymbol{\epsilon}}$  are lossy, and the integrand will be negative definite if both  $\bar{\boldsymbol{\mu}}$  and  $\bar{\boldsymbol{\epsilon}}$  are active (see Subsection 1.1.5). Hence, the only way for (12) to be satisfied is for  $\delta \mathbf{E} = 0$  and  $\delta \mathbf{H} = 0$ , or that  $\mathbf{E}_1 = \mathbf{E}_2$  and  $\mathbf{H}_1 = \mathbf{H}_2$ , implying uniqueness.

Consequently, in order for uniqueness to be guaranteed, either

- (i)  $\hat{n} \times \mathbf{E}_1 = \hat{n} \times \mathbf{E}_2$  on  $S$  or  $\hat{n} \times \mathbf{H}_1 = \hat{n} \times \mathbf{H}_2$  on  $S$ , or
- (ii)  $\hat{n} \times \mathbf{E}_1 = \hat{n} \times \mathbf{E}_2$  on a part of  $S$  while  $\hat{n} \times \mathbf{H}_1 = \hat{n} \times \mathbf{H}_2$  on the rest of  $S$ .

In other words, if two solutions satisfy the same boundary conditions for tangential  $\mathbf{E}$  or tangential  $\mathbf{H}$ , or a mixture thereof on  $S$ , the two solutions must be identical.

Again, the requirement for a nonlossless condition is to eliminate the real resonance solutions which could otherwise be time harmonic, homogeneous solutions to (7) satisfying the boundary conditions. For example, if the appropriate boundary conditions for  $\delta \mathbf{E}$  and  $\delta \mathbf{H}$  are imposed so that the first term of (10) is zero, then

$$\int_V (\delta \mathbf{H}^* \cdot \bar{\boldsymbol{\mu}}^\dagger \cdot \delta \mathbf{H} - \delta \mathbf{E}^* \cdot \bar{\boldsymbol{\epsilon}} \cdot \delta \mathbf{E}) dV = 0. \quad (1.5.13)$$

The above does not imply that  $\delta \mathbf{E}$  or  $\delta \mathbf{H}$  equals zero, because at resonances, a perfect balance between the energy stored in the electric field and the energy

stored in the magnetic field is maintained. As a result, the left-hand side of the above could vanish without having  $\delta\mathbf{E}$  and  $\delta\mathbf{H}$  be zero, which is necessary for uniqueness. But away from the resonances of the volume  $V$ , the energy stored in the electric field is not equal to that stored in the magnetic field. Hence, in order for (13) to be satisfied,  $\delta\mathbf{E}$  and  $\delta\mathbf{H}$  have to be zero since each term in (13) is positive definite for lossless media due to the Hermitian nature of  $\bar{\boldsymbol{\mu}}$  and  $\bar{\boldsymbol{\epsilon}}$ .

When  $V \rightarrow \infty$ , as in the scalar wave equation case, some loss has to be imposed to guarantee uniqueness. This is the same as requiring the wave to be outgoing at infinity, namely, the radiation condition. Again, the radiation condition can be imposed for an unbounded medium with vanishing loss to guarantee uniqueness.

### Exercises for Chapter 1

**1.1** Show that Equations (1.1.12) to (1.1.15) can also be obtained from Equations (1.1.1) to (1.1.4) by Fourier transforms. In this case, we define a

$$\text{field } \mathbf{A}(\mathbf{r}, t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \mathbf{A}(\mathbf{r}, \omega).$$

**1.2** (a) The fundamental units in electromagnetics can be considered to be meter, kilogram, second, and coulomb. Show that 1 volt, which is 1 watt/amp, has the dimension of (kilogram meter<sup>2</sup>)/(coulomb sec<sup>2</sup>).

(b) From Maxwell's equations, show that  $\mu_0$  has the dimension of (second volt)/(meter amp), and hence, its dimension is (kilogram meter)/(coulomb<sup>2</sup>) in the more fundamental units.

(c) If we assign the value of  $\mu_0$  to be  $4\pi$  instead of  $4\pi \times 10^{-7}$ , what would be the unit of coulomb in this new assignment compared to the old unit? What would be the present value of 1 volt and 1 amp in this new assignment?

**1.3** Show that for two time harmonic functions,

$$\langle A(\mathbf{r}, t), B(\mathbf{r}, t) \rangle = \frac{1}{2} \Re e[A(\mathbf{r})B^*(\mathbf{r})],$$

where  $A(\mathbf{r})$  and  $B(\mathbf{r})$  are the phasors of  $A(\mathbf{r}, t)$  and  $B(\mathbf{r}, t)$ .

**1.4** By putting an electrostatic field next to a magnetostatic field, show that  $\mathbf{E} \times \mathbf{H}$  is not zero, but the quantity cannot possibly correspond to power flow.

**1.5** Assume that a voltage is time harmonic, i.e.,  $V(t) = V_0 \cos \omega t$ , and that a current  $I(t) = I_I \cos \omega t + I_Q \sin \omega t$ , i.e., it consists of an in-phase and a quadrature component.

(a) Find the instantaneous power due to this voltage and current, namely,  $V(t)I(t)$ .

- (b) Find the phasor representations of the voltage and current and the complex power due to this voltage and current.
- (c) Establish a relationship between the real part and reactive part of the complex power to the instantaneous power.
- (d) Show that the reactive power is due to the quadrature component of the current, which is related to a time-varying part of the instantaneous power with zero-time average.
- 1.6** Show that the matrices  $i(\bar{\boldsymbol{\mu}}^\dagger - \bar{\boldsymbol{\mu}})$  and  $i(\bar{\boldsymbol{\epsilon}}^\dagger - \bar{\boldsymbol{\epsilon}})$  are either zero, positive, or negative definite. Explain the physical interpretation of each case.
- 1.7** Find another set of replacement rules different from Equation (1.1.36) that will leave Maxwell's equations invariant.
- 1.8** (a) Derive Equation (1.2.5).

- (b) In one dimension, a pressure gradient  $p(x)$  (force/unit area) is established. Show that the force on an elemental sheet between  $x$  and  $x+\Delta x$  is  $[p(x)-p(x+\Delta x)]A$  where  $A$  is the area of the elemental sheet. Hence, show that the force per unit volume is  $[p(x) - p(x + \Delta x)]/\Delta x$ , implying that the force density  $F_x = -\partial p/\partial x$ .
- (c) Apply the same derivation to a cube and show that  $\mathbf{F} = -\nabla p$ .

- 1.9** (a) In hydrodynamic problems, it is easier to formulate a concept if one moves with the particles in a fluid. For example, if the density is described by  $\rho(\mathbf{r}, t)$ , in the coordinate system which moves with a fluid particle, then  $\mathbf{r}(t)$  is a function of time as well. Consequently, the total change in density in the neighborhood of the particle that one observes is affected by  $\mathbf{r}$  being a function of  $t$  as well. This total change of  $\rho$  with respect to  $t$  is usually denoted  $\frac{D}{Dt}\rho(\mathbf{r}, t)$ . Show that

$$\frac{D}{Dt}\rho(\mathbf{r}, t) = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho.$$

- (b) The pressure in a fluid is a function of both the density  $\rho$  and entropy  $S$ , i.e.,  $p(\rho, S)$ . If one follows a fluid particle's motion, the entropy in the vicinity of the fluid particle is constant. This can be denoted by  $\frac{DS}{Dt} = 0$ . From this, deduce that

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial \rho} \frac{D\rho}{Dt} + \frac{\partial p}{\partial S} \frac{DS}{Dt}$$

and then

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = \frac{\partial p}{\partial \rho} \left[ \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \right].$$

Hence, derive Equation (1.2.13).

- 1.10** Explain why Equation (1.2.20) is not equivalent to three scalar wave equations if  $\mathbf{E}$  is decomposed into three components not in the Cartesian coordinates.

- 1.11** Sketch the wavefront of Equation (1.2.33) in three dimensions and explain why it is called a conical wave.
- 1.12** Using Equations (1.2.28) and (1.2.40), establish the relationship in Equation (1.2.40a).
- 1.13** Show that a recurrence relationship similar to (1.2.34) for the solutions of (1.2.40) is

$$b'_n(kr) = b_{n-1}(kr) - \frac{n+1}{kr}b_n(kr) = -b_{n+1}(kr) + \frac{n}{kr}b_n(kr).$$

- 1.14** For a scalar wave equation,  $\nabla \cdot p^{-1}(\mathbf{r})\nabla\phi(\mathbf{r}) + k^2\phi(\mathbf{r}) = s(\mathbf{r})$ :
- (a) What is the boundary condition at an interface where  $p$  is discontinuous?
- (b) Show that a reciprocal relationship  $\langle\phi_1(\mathbf{r}), s_2(\mathbf{r})\rangle = \langle\phi_2(\mathbf{r}), s_1(\mathbf{r})\rangle$  exists.
- 1.15** By considering the case where  $\mathbf{k}$  is pointing in the  $z$  direction, prove that Equation (1.3.29) has only two nontrivial eigenvalues, and hence, only two nontrivial eigenvectors. Find these eigenvalues and eigenvectors.
- 1.16** By letting  $\mathbf{B} = \nabla \times \mathbf{A}$  and  $\mathbf{E} = -\nabla\phi + i\omega\mathbf{A}$ , and starting from Maxwell's equations, derive an expression similar to (1.3.49).
- 1.17** (a) Define a Green's function to be a solution of  $\nabla \cdot p^{-1}(\mathbf{r})\nabla g(\mathbf{r}, \mathbf{r}') + k^2g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$  and show that the solution to the equation  $\nabla \cdot p^{-1}(\mathbf{r})\nabla\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = s(\mathbf{r})$  can be written as

$$\psi(\mathbf{r}) = - \int d\mathbf{r}' g(\mathbf{r}, \mathbf{r}') s(\mathbf{r}').$$

- (b) Using the result of Exercise 1.14, show that  $g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r}', \mathbf{r})$ .
- 1.18** (a) In the manner of Equation (1.3.52), show that  $[\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')]^t = \overline{\mathbf{G}}(\mathbf{r}', \mathbf{r})$  for a dyadic Green's function defined over a bounded region.
- (b) Define a magnetic field dyadic Green's function such that

$$\mathbf{H}(\mathbf{r}) = i\omega\epsilon \int_V d\mathbf{r}' \overline{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}(\mathbf{r}').$$

From the reciprocity requirement that  $\langle\mathbf{M}_2, \mathbf{H}_1\rangle = -\langle\mathbf{J}_1, \mathbf{E}_2\rangle$ , show that  $[\nabla \times \overline{\mathbf{G}}_e(\mathbf{r}, \mathbf{r}')]^t = \nabla' \times \overline{\mathbf{G}}_m(\mathbf{r}', \mathbf{r})$ .

- 1.19** For the lossless scalar wave equation in a homogeneous medium like (1.5.1):
- (a) Find the resonance solutions to a box of dimension  $a \times b \times d$ , with homogeneous Neumann boundary condition ( $\hat{n} \cdot \nabla\phi = 0$ ) on the sides of the box.

- (b) Show that the resonance solutions satisfy (1.5.6).
- (c) At resonance, show that the nontrivial difference between two solutions is still a solution satisfying the boundary condition.
- (d) Describe what happens to the resonance solutions when  $a$ ,  $b$ , and  $d \rightarrow \infty$ .



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