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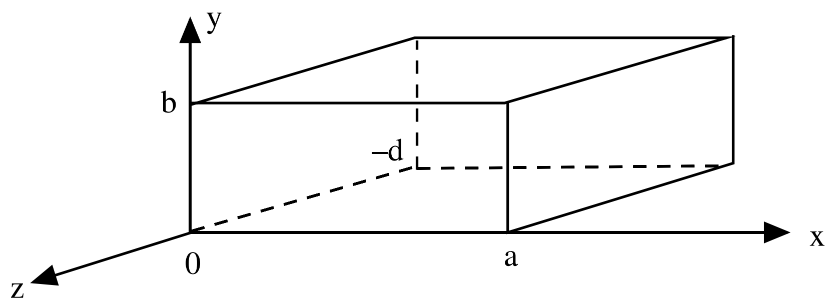
ECE 350 Lecture Notes

23. Cavity Resonator.



A cavity resonator is a useful microwave device. If we close off two ends of a rectangular waveguide with metallic walls, we have a rectangular cavity resonator. In this case, the wave propagating in the $\hat{z}$-direction will bounce off the two walls resulting in a standing wave in the $\hat{z}$-direction. For the **TM case**, we have

$$E_z = E_0 \sin(\beta_x x)\sin(\beta_y y)(e^{-j\beta_z z} + \rho e^{j\beta_z z}), \tag{1}$$

$$E_x = \frac{-j\beta_x\beta_z}{\beta_x^2+\beta_y^2}E_0\cos(\beta_x x)\sin(\beta_y y)(e^{-j\beta_z z} - \rho e^{j\beta_z z}), \tag{2}$$

$$E_y = \frac{-j\beta_y\beta_z}{\beta_x^2+\beta_y^2}E_0\sin(\beta_x x)\cos(\beta_y y)(e^{-j\beta_z z} - \rho e^{j\beta_z z}). \tag{3}$$

For the boundary conditions to be satisfied, we require that $E_x(z=0) = E_y(z=0) = 0$. Hence, $\rho = 1$, and

$$E_z = 2E_0\sin(\beta_x x)\sin(\beta_y y)\cos(\beta_z z), \tag{4}$$

$$E_x = \frac{-2\beta_x\beta_z}{\beta_x^2+\beta_y^2}E_0\cos(\beta_x x)\sin(\beta_y y)\sin(\beta_z z), \tag{5}$$

$$E_y = \frac{-2\beta_y\beta_z}{\beta_x^2+\beta_y^2}E_0\sin(\beta_x x)\cos(\beta_y y)\sin(\beta_z z). \tag{6}$$

Furthermore, $E_x(z=-d) = E_y(z=-d) = 0$, implying that

$$\beta_z = \frac{p\pi}{d}, \qquad p = 0, 1, 2, 3, \ldots. \tag{7}$$

The guidance conditions for a waveguide demand that $\beta_x = \frac{m\pi}{a}$ and $\beta_y = \frac{n\pi}{b}$, where for TM case, neither m or n can be zero. Now that β_z has to satisfy (7), the TM mode in a cavity is classified as TM_{mnp} mode. We note from (4) that p can be zero while $E_z \neq 0$. Hence, the TM$_{mn0}$ cavity mode can exist. In order for (4), (5), and (6) to be solutions to the wave equation, we require that

$$\omega^2\mu\epsilon = \beta_x^2 + \beta_y^2 + \beta_z^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2. \tag{8}$$

For a given choice of m, n, and p, only a single frequency can satisfy (8). This frequency is the **resonant frequency** of the cavity. It is only at this frequency that the cavity can sustain a free oscillation. At other frequencies, the fields interfere destructively and the free oscillation is not sustained. From (8), we gather that the resonant frequency for the TM$_{mnp}$ mode is

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right]^{\frac{1}{2}}. \tag{9}$$

For the **TE case**, similar derivation shows that

$$H_z = H_0\cos(\beta_x x)\cos(\beta_y y)\sin(\beta_z z), \tag{10}$$

$$E_x = \frac{j\omega\mu\beta_y}{\beta_x^2 + \beta_y^2}H_0\cos(\beta_x x)\sin(\beta_y y)\sin(\beta_z z), \tag{11}$$

$$E_y = -\frac{j\omega\mu\beta_x}{\beta_x^2 + \beta_y^2}H_0\sin(\beta_x x)\cos(\beta_y y)\sin(\beta_z z). \tag{12}$$

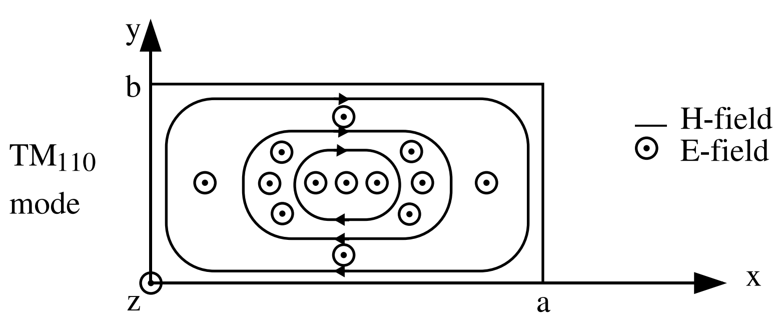
Similarly, the boundary conditions require that

$$\beta_x = \frac{m\pi}{a}, \beta_y = \frac{n\pi}{b}, \beta_z = \frac{p\pi}{d}. \tag{13}$$

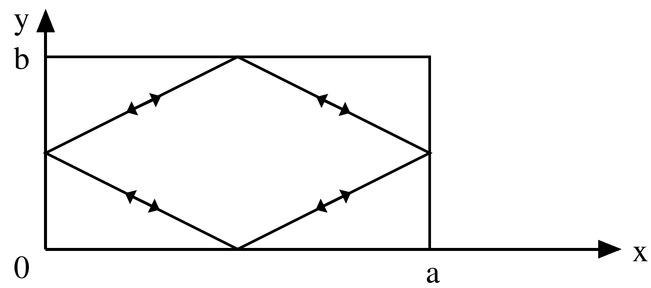
When $p = 0$, $H_z = 0$, hence TE$_{mn0}$ mode does not exist. However, TE$_{0np}$ or TE$_{m0p}$ modes can exist. The resonant frequency formula is as given in (9). If $a > b > d$, the lowest resonant frequency is the TM$_{110}$ mode. In this case,

$$\omega_{110} = \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]^{\frac{1}{2}}, \tag{14}$$

and $E_z \neq 0$, $H_x \neq 0$, $H_y \neq 0$, $E_x = E_y = 0$. A sketch of the field is as shown.



We can decompose the wave into plane waves bouncing off the four walls of the cavity.



As an example, for $a = 2$ cm, $b = 1$ cm, $d = 0.5$ cm, the resonant frequency of the TM_{110} mode is

$$2\pi f_{110} = 3 \times 10^8 \sqrt{\frac{5\pi^2}{4(10^{-2})^2}} = \frac{3 \times 10^8 \pi}{2 \times 10^{-2}} \sqrt{5}\mathrm{Hz}, \tag{15}$$

or

$$f_{110} = \frac{3}{4} \times 10^{10} \times \sqrt{5}\mathrm{Hz} = 1.68 \times 10^{10}\mathrm{Hz} = 16.8\mathrm{GHz}. \tag{16}$$

Cavity resonators are useful as filters and tuners in microwave circuits, as LC resonators are in RF circuits. Cavity resonators can also be used to measure the frequency of an electromagnetic signal.