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ECE 350 Lecture Notes

18. Wave Polarization.

We learnt that

$$\mathbf{E} = \hat{x}E_x = \hat{x}E_1 \cos(\omega t - \beta z), \tag{1}$$

is a solution to the wave equation because $\nabla \cdot \mathbf{E} = 0$. Similarly,

$$\mathbf{E} = \hat{y}E_y = \hat{y}E_2 \cos(\omega t - \beta z + \phi), \tag{2}$$

is also a solution to the wave equation. Solutions (1) and (2) are known as ***linearly polarized waves***, because the electric field or the magnetic field are polarized in only one direction. However, a linear superposition of (1) and (2) are still a solution to Maxwell's equation

$$\mathbf{E} = \hat{x}E_x(z, t) + \hat{y}E_y(z, t). \tag{3}$$

If we observe this field at $z = 0$, it is

$$\mathbf{E} = \hat{x}E_1 \cos \omega t + \hat{y}E_2 \cos(\omega t + \phi). \tag{4}$$

When $\phi = 90^\circ$,

$$E_x = E_1 \cos \omega t \quad E_y = E_2 \cos(\omega t + 90^\circ), \tag{5}$$

$$\text{When } \omega t = 0^\circ, \quad E_x = E_1, \quad E_y = 0. \tag{6}$$

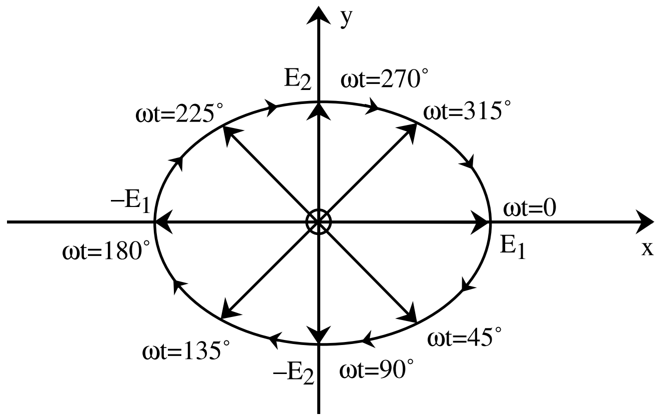
$$\text{When } \omega t = 45^\circ, \quad E_x = \frac{E_1}{\sqrt{2}}, \quad E_y = -\frac{E_2}{\sqrt{2}}. \tag{7}$$

$$\text{When } \omega t = 90^\circ, \quad E_x = 0, \quad E_y = -E_2. \tag{8}$$

$$\text{When } \omega t = 135^\circ, \quad E_x = -\frac{E_1}{\sqrt{2}}, \quad E_y = -\frac{E_2}{\sqrt{2}}. \tag{9}$$

$$\text{When } \omega t = 180^\circ, \quad E_x = -E_1, \quad E_y = 0. \tag{10}$$

If we continue further, we can sketch out the tip of the vector field $\mathbf{E}$. It traces out an ellipse as shown when $E_1 \neq E_2$. Such a wave is known as an ***elliptically polarized wave***.



When $E_1 = E_2$, the ellipse becomes a circle, and the wave is known as a ***circularly polarized wave***. When ϕ is $-90°$, the vector $\mathbf{E}$ rotates in the counter-clockwise direction.

A wave is classified as ***left hand elliptically (circularly) polarized*** when the wave is approaching the viewer. A counterclockwise rotation is classified as ***right hand elliptically (circularly) polarized***.

When $\phi \neq \pm 90°$, the tip of the vector $\mathbf{E}$ traces out a tilted ellipse. We can show this by expanding E_y in (5).

$$\begin{aligned} E_y &= E_2 \cos \omega t \cos \phi - E_2 \sin \omega t \sin \phi \\ &= \frac{E_2}{E_1} E_x \cos \phi - E_2 \left[1 - \left(\frac{E_x}{E_1} \right)^2 \right]^{\frac{1}{2}} \sin \phi. \end{aligned} \tag{11}$$

Rearranging terms, we get

$$aE_x^2 - bE_xE_y + cE_y^2 = 1, \tag{12}$$

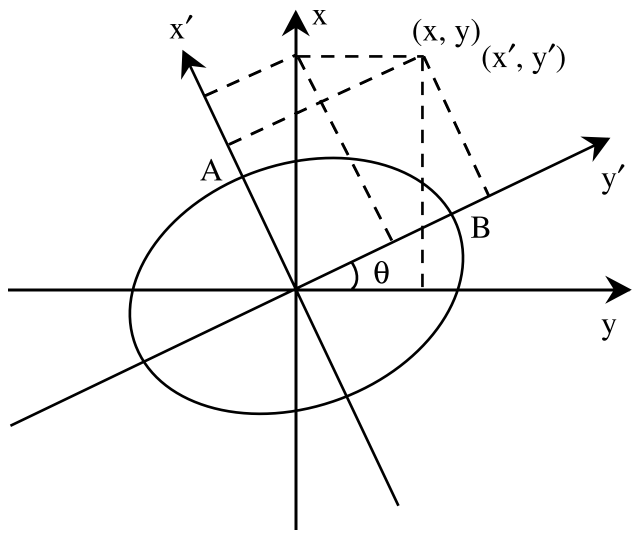
where

$$a = \frac{1}{E_1^2 \sin^2 \phi}, \quad b = \frac{2 \cos \phi}{E_1 E_2 \sin^2 \phi}, \quad c = \frac{1}{E_2^2 \sin^2 \phi}. \tag{13}$$

Equation (12) is of the form

$$ax^2 - bxy + cy^2 = 1, \tag{14}$$

which is the equation of a tilted ellipse.



The equation of an ellipse in its self coordinate is

$$\left(\frac{x'}{A}\right)^2 + \left(\frac{y'}{B}\right)^2 = 1, \tag{15}$$

where A and B are the semi-axes of the ellipse. However,

$$x' = x\cos\theta - y\sin\theta, \tag{16}$$
$$y' = x\sin\theta + y\cos\theta, \tag{17}$$

we have

$$x^2\left(\frac{\cos^2\theta}{A^2} + \frac{\sin^2\theta}{B^2}\right) - xy\sin 2\theta\left(\frac{1}{A^2} - \frac{1}{B^2}\right) + y^2\left(\frac{\sin^2\theta}{A^2} + \frac{\cos^2\theta}{B^2}\right) = 1. \tag{18}$$

Equating (14) and (18), we can deduce that

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\cos\phi E_1 E_2}{E_2^2 - E_1^2}\right), \tag{19}$$

$$AR = \left(\frac{1+\Delta}{1-\Delta}\right)^{\frac{1}{2}}, \tag{20}$$

where

$$\Delta = \left[1 - \frac{4E_1^2 E_2^2 \sin^2\phi}{E_1^2 + E_2^2}\right]^{\frac{1}{2}}. \tag{21}$$

AR is the axial ratio which is the ratio of the two axes of the ellipse. It is defined to be larger than one always.