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ECE 350 Lecture Notes

15. Group and Phase Velocities.

If we have two waves that are slightly different in frequency ω and phase constant β, a linear superposition of them is still a solution of the wave equation

$$E_x = E_0 \cos(\omega_1 t - \beta_1 z) + E_0 \cos(\omega_2 t - \beta_2 z). \tag{1}$$

If $\omega_1 = \omega - \Delta\omega$, $\beta_1 = \beta - \Delta\beta$, $\omega_2 = \omega + \Delta\omega$, $\beta_2 = \beta + \Delta\beta$, then

$$E_x = E_0 \cos[\omega t - \beta z - (\Delta\omega t - \Delta\beta z)] + E_0 \cos[\omega t - \beta z + (\Delta\omega t - \Delta\beta z)]. \tag{2}$$

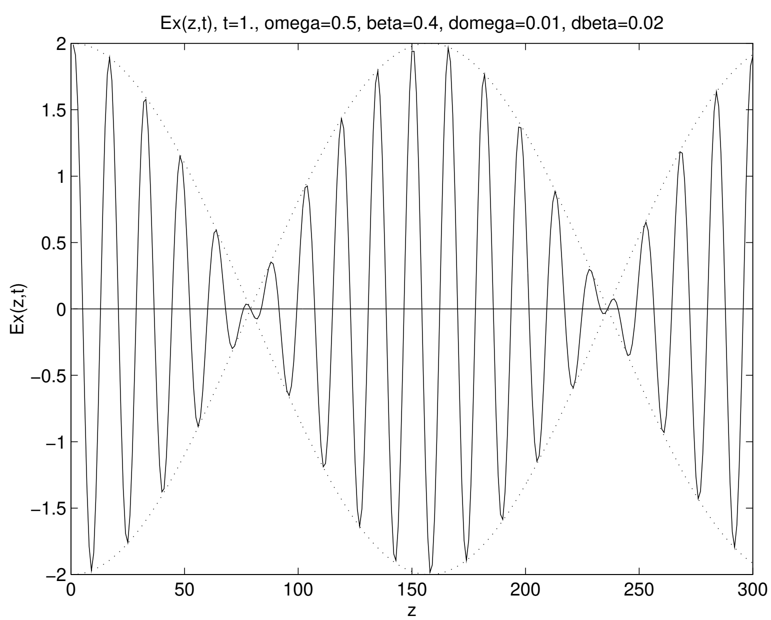
Using the fact that $\cos(A-B) + \cos(A+B) = 2\cos A \cos B$, we have

$$E_x = 2E_0 \cos(\omega t - \beta z) \cos(\Delta\omega t - \Delta\beta z), \tag{3}$$

or

$$E_x(z,t) = 2E_0 \cos\left[\beta\left(\frac{\omega}{\beta}t - z\right)\right] \cos\left[\Delta\beta\left(\frac{\Delta\omega}{\Delta\beta}t - z\right)\right]. \tag{4}$$

At $t = 0$, we have $E_x = 2E_0 \cos\beta z \cos\Delta\beta z$ which is sketched below.



The first factor in (4) is rapidly varying while the second factor is slowly varying. The slowly varying term amplitude-modulates the rapidly varying term giving rise to the picture as shown.

We have learnt that a function of the form $f(vt - z)$ propagates in the positive z-direction with velocity v. From (10.5), we see that the rapidly varying term propagates with velocity $\frac{\omega}{\beta}$. Since this represents the propagation of the phases in the rapidly oscillating part in the figure, this is also known as phase velocity,

$$v_p = \frac{\omega}{\beta}. \tag{5}$$

The slowly varying part propagates with the velocity $\frac{\Delta\omega}{\Delta\beta}$, which is $\frac{d\omega}{d\beta}$ in the limit that $\Delta\omega$ and $\Delta\beta \to 0$. This represents the velocity on the envelope in the picture and hence, it is known as the group velocity,

$$v_g = \frac{d\omega}{d\beta} \text{ or } v_g^{-1} = \frac{d\beta}{d\omega}. \tag{6}$$

If $\beta = \omega\sqrt{\mu\epsilon}$, the phase velocity $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}$, the group velocity from (6) is also $\frac{1}{\sqrt{\mu\epsilon}}$. Hence, the group and the phase velocities are the same is β is a linear function of ω.

If β is not a linear function of ω, then, the phase velocity and the group velocities are functions of frequencies, and the medium is known to be ***dispersive***. In a dispersive medium, a pulse propagates with subsequent distortions because the different harmonics in the pulse propagate with different phase velocity. Example of a dispersive medium is a conductive medium where $\beta = \frac{1}{\delta} = \sqrt{\frac{\omega\mu\sigma}{2}}$, is not a linear function of ω.

In a distortionless line, the phase velocity is made to be frequency independent so that a pulse propagates without distortions.

Furthermore, a phase velocity can be larger than the velocity of light while the group velocity is always less than the speed of light. This is because energy propagates with the group velocity so that ***special relativity*** is not violated.