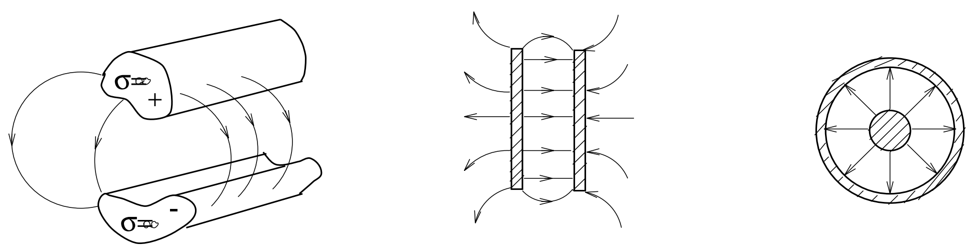
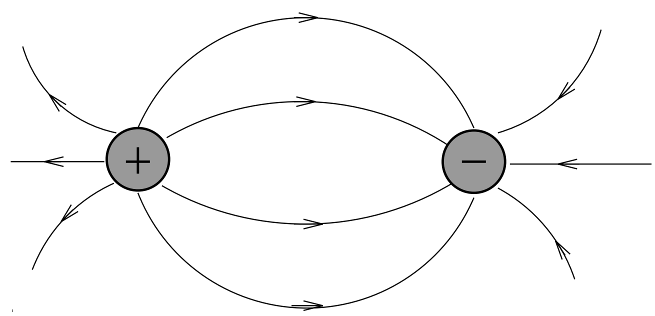
W.C.Chew
ECE 350 Lecture Notes

5. Transmission Lines

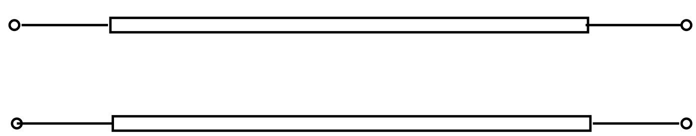


GENERAL | STRIP LINE | COAXIAL

Examples of Transmission lines

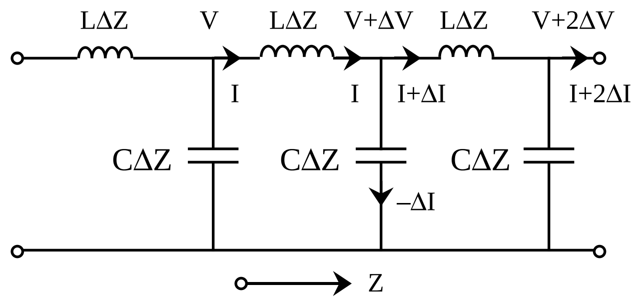


Symbol of a Transmission Line



Symbol of a Transmission Line

Another place where wave phenomenon is often encountered is on transmission lines. A transmission line consists of two parallel conductors of arbitrary cross-sections that can carry two opposite currents or two opposite charges. A transmission line has capacitances between the two conductors, and the conductors have inductances to them. We can characterize this capacitance by a line capacitance C which has the unit of farad m^{-1}, and the inductance by a line inductance L, which has the unit of henry m^{-1}. Hence a transmission line can be approximated by a lumped element equivalent as shown



We can derive the voltage equation between nodes (1) and (2) to get

$$V - (V + \Delta V) = L\Delta z \frac{\partial I}{\partial t}, \tag{1}$$

or

$$\Delta V = -L\Delta z \frac{\partial I}{\partial t}. \tag{2}$$

Similarly, the current relation at node (3) says that

$$-\Delta I = C\Delta z \frac{\partial (V + \Delta V)}{\partial t} \simeq C\Delta z \frac{\partial V}{\partial t}. \tag{3}$$

In the limit when we let our discrete or lumped element model become very small, or $\Delta z \to 0$, we have

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}, \tag{4}$$

and

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}. \tag{5}$$

The above are known as the telegrapher's equations. Wave equations can be easily derived from the above

$$\frac{\partial^2 V}{\partial z^2} - LC\frac{\partial^2 V}{\partial t^2} = 0, \tag{6}$$

and

$$\frac{\partial^2 I}{\partial z^2} - LC\frac{\partial^2 I}{\partial t^2} = 0. \tag{7}$$

Comparing with Equation (3.17), we deduce that the ***velocity*** of the current and voltage waves on a transmission line is

$$v = \frac{1}{\sqrt{LC}}. \tag{8}$$

The solution to (6) may be of the form

$$V(z,t) = f(z - vt). \tag{9}$$

Substituting into (4), we have

$$-L\frac{\partial I}{\partial t} = f'(z - vt) \tag{10}$$

or

$$I(z,t) = \frac{1}{Lv}f(z - vt). \tag{11}$$

Hence,

$$\frac{V(z,t)}{I(z,t)} = Lv = \sqrt{\frac{L}{C}} \tag{12}$$

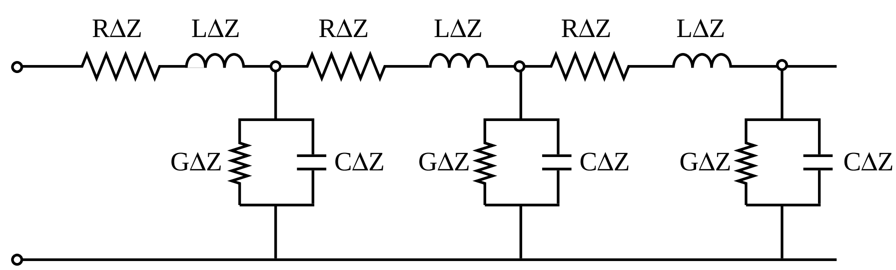
for a forward going wave. The quantity

$$Z_0 = \sqrt{\frac{L}{C}} \tag{13}$$

is the ***characteristic impedance*** of a transmission line.

Lossy Transmission Line

Often time, a transmission line has loss to it. For example, the conductor has a finite conductivity and hence is a little resistive. The insulation between the conductors may have current leakage, thus not forming an ideal capacitor. A more appropriate lumped element model is as follows.



The above circuit is more easily treated using phasor techniques. If we have applied phasor technique to (4) and (5), we would have obtained

$$\frac{d\tilde{V}}{dz} = -j\omega L\tilde{I}, \tag{14}$$

$$\frac{d\tilde{I}}{dz} = -j\omega C\tilde{V}. \tag{15}$$

Note that $j\omega L$ is the series impedance per unit length of the lossless line while $j\omega C$ is the shunt admittance per unit length of the lossless line. In the lossy line case, the series impedance per unit length becomes

$$Z = j\omega L + R \tag{16}$$

while the shunt admittance per unit length becomes

$$Y = j\omega C + G \tag{17}$$

where R and G are line resistance and line conductance respectively. The telegraphers equations become

$$\frac{d\tilde{V}}{dz} = -Z\tilde{I}, \tag{18}$$

$$\frac{d\tilde{I}}{dz} = -Y\tilde{V}, \tag{19}$$

and the corresponding Helmholtz wave equations are

$$\frac{d^2\tilde{V}}{dz^2} - ZY\tilde{V} = 0, \tag{20}$$

$$\frac{d^2\tilde{I}}{dz^2} - ZY\tilde{I} = 0. \tag{21}$$

Similarly, the characteristic impedance, is

$$Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} \quad \Rightarrow Z_0 = \sqrt{\frac{j\omega L + R}{j\omega C + G}} = \sqrt{\frac{Z}{Y}}. \tag{22}$$

Equations (20) and (21) are of the same form as (4.22) or

$$\frac{d^2\tilde{V}}{dz^2} - \gamma^2\tilde{V} = 0, \tag{23}$$

$$\frac{d^2\tilde{I}}{dz^2} - \gamma^2\tilde{I} = 0, \tag{24}$$

where

$$\gamma = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)} = \alpha + j\beta. \tag{25}$$

The general solution is of the form (4.23). For example,

$$\begin{aligned}\tilde{V}(z) &= V_+ e^{-\gamma z} + V_- e^{+\gamma z} \\ &= V_+ e^{-\alpha z - j\beta z} + V_- e^{\alpha z + j\beta z}.\end{aligned} \tag{26}$$

If $V_+ = |V_+|\, e^{j\phi_+}, \quad V_- = |V_-|\, e^{+j\phi_-}, \quad$ then the real time representation of V is

$$\begin{aligned}V(z,t) &= \Re e[\tilde{V}(z)e^{j\omega t}] \\ &= |V_+|\, e^{-\alpha z}\cos(\omega t - \beta z + \phi_1) + |V_-|\, e^{\alpha z}\cos(\omega t + \beta z + \phi_2).\end{aligned} \tag{27}$$

The first term corresponds to a decaying wave moving in the positive z-direction while the second term corresponds to a wave decaying and moving in the negative z-direction. Hence, $e^{-\gamma z}$ corresponds to a positive going wave, while $e^{+\gamma z}$ corresponds to a negative going wave.

If the transmission line is lossless, i.e., $R = G = 0$, then, the attenuation constant $\alpha = 0$, and the propagation constant γ becomes $\gamma = j\beta$. In this case, there is no attenuation, and (26) becomes

$$\tilde{V}(z) = V_+ e^{-j\beta z} + V_- e^{+j\beta z}, \tag{28}$$

and (27) becomes

$$V(z,t) = |V_+| \cos(\omega t - \beta z + \phi_1) + |V_-| \cos(\omega t + \beta z + \phi_2). \tag{29}$$

The wave propagates without attenuation or without decay in this case. The velocity of propagation is $v = \omega/\beta$.

Furthermore, we can derive the current that corresponds to the voltage in (26) using Equation (18). Hence

$$\tilde{I} = -\frac{1}{Z}\frac{d\tilde{V}}{dz} = \frac{\gamma}{Z} V_+ e^{-\gamma Z} - \frac{\gamma}{Z} V_- e^{+\gamma Z}. \tag{30}$$

But

$$\frac{\gamma}{Z} = \sqrt{\frac{Y}{Z}} = \frac{1}{Z_0} \tag{31}$$

where Z_0 is the characteristic impedance given by Equation (22). Hence,

$$\tilde{I} = \frac{V_+}{Z_0} e^{-\gamma Z} - \frac{V_-}{Z_0} e^{\gamma Z} = I_+ e^{-\gamma Z} + I_- e^{\gamma Z}, \tag{32}$$

where

$$\frac{V_+}{I_+} = Z_0, \qquad \frac{V_-}{I_-} = -Z_0. \tag{33}$$