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ECE 350 Lecture Notes

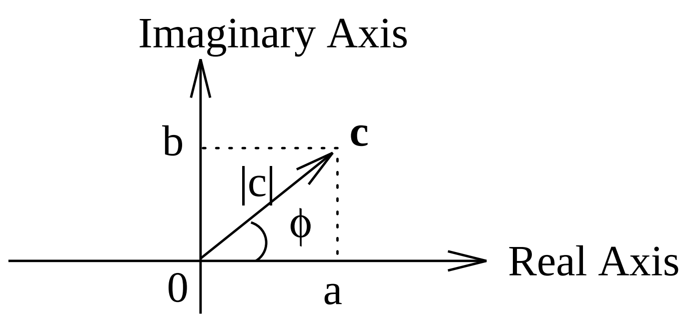
1. Elements of Complex Algebra

Complex numbers are extensions of real numbers, and they make the number fields complete in the sense that an n-th order polynomial has n-th roots in a complex field while it is not always true in the real field. Complex numbers are also very useful in time harmonic analysis of engineering and physical systems, because they considerably simplify the analysis.

A complex number can be represented in cartesian form as

$$c = a + jb \tag{1}$$

where $j = \sqrt{-1}$. a is the ***real part*** of c while b is the ***imaginary part*** of c. On the complex plane, c is represented by a point c or sometimes an arrow oc as shown.



Sometimes it is more convenient to represent c in polar form, i.e.

$$c = a + jb = |c|\, e^{j\phi} = |c| \cos\phi + j\, |c| \sin\phi \tag{2}$$

where $|c| = \sqrt{a^2 + b^2}$ is the magnitude or the absolute value of c.
From (2), it is seen that

$$\tan\phi = \frac{b}{a} \quad \Rightarrow \quad \phi = \tan^{-1}\frac{b}{a} \tag{3}$$

where ϕ is the phase of c.

Addition and Subtraction

Addition and subtraction of complex numbers are carried out in Cartesian forms.

Let $c = a + jb$, and $h = f + jg$, then

$$c + h = (a + f) + j(b + g), \tag{4}$$

and

$$c - h = (a - f) + j(b - g), \tag{5}$$

Multiplication and Division

$$ch = (a + jb)(f + jg) = (af - bg) + j(bf + ag), \tag{6}$$

$$\frac{c}{h} = \frac{a + jb}{f + jg} = \frac{(a + jb)(f - jg)}{(f + jg)(f - jg)} = \frac{af + bg}{f^2 + g^2} + j\frac{bf - ag}{f^2 + g^2}. \tag{7}$$

Multiplication and division are more conveniently carried out in a polar form. Let

$$c = |c|\, e^{j\phi_1}, \quad h = |h|\, e^{j\phi_2}, \tag{8}$$

then

$$ch = |c|\, |h|\, e^{j(\phi_1 + \phi_2)}, \tag{9}$$

$$\frac{c}{h} = \frac{|c|}{|h|} e^{j(\phi_1 - \phi_2)}. \tag{10}$$

Square Root of a Complex Number

It is most convenient to take the square root of a complex number in ***polar form*** or by converting it to ***polar form***.

$$c = |c|\, e^{j\phi_1} = \sqrt{a^2 + b^2} e^{j \tan^{-1} \frac{b}{a}}, \tag{11}$$

$$\sqrt{c} = |c|^{\frac{1}{2}}\, e^{j\frac{\phi_1}{2}} = (a^2 + b^2)^{\frac{1}{4}} e^{j\frac{1}{2}\tan^{-1}\frac{b}{a}}. \tag{12}$$

In fact

$$c^{\frac{1}{m}} = |c|^{\frac{1}{m}}\, e^{j\frac{\phi_1}{m}} = (a^2 + b^2)^{\frac{1}{2m}} e^{j\frac{1}{m}\tan^{-1}\frac{b}{a}}. \tag{13}$$

Phasor Representation of a Time-Harmonic Scalar

If $V(t)$ is a time-harmonic signal such that

$$V(t) = V_0 \cos(\omega t + \phi), \tag{14}$$

it could also be written as

$$V(t) = \Re e\{V_0 e^{j\phi} e^{j\omega t}\}. \tag{15}$$

The term $\tilde{V} = V_0 e^{j\phi}$ is known as the phasor representation of $V(t)$.

If $U(t) = U_0 \cos(\omega t + \phi_1)$, or the phasor representation of $U(t)$ is

$$\tilde{U} = U_0 e^{j\phi_1}. \tag{16}$$

It can be shown easily that

$$V(t) + U(t) = \Re e\{[\underbrace{V_0 e^{j\phi}}_{\tilde{V}} + \underbrace{U_0 e^{j\phi_1}}_{\tilde{U}}] e^{j\omega t}\}. \tag{17}$$

Hence $\tilde{V} + \tilde{U}$ is a phasor representation of $V(t) + U(t)$.
Also

$$\frac{\partial V(t)}{\partial t} = \frac{\partial}{\partial t} \Re e\{V_0 e^{j\phi} e^{j\omega t}\} = \Re e\{j\omega \underbrace{V_0 e^{j\phi}}_{\tilde{V}} e^{j\omega t}\}. \tag{18}$$

Therefore $j\omega\tilde{V}$ is a phasor representation of $\frac{\partial}{\partial t} V(t)$. However, as a word of caution, $\tilde{V}\tilde{U}$ is not a phasor representation of $V(t)U(t)$. You can convince yourself of this.

Exercise

1) Show that,
   (a) $c + c^*$ is always real,
   (b) $c - c^*$ is always imaginary,
   (c) c/c^* has magnitude equal to 1.

2) Consider $z^2 = 1 + 2j$. It is a second order polynomial with two roots. Find the two roots.

3) Obtain the phasor representation of the following
   (a) $V(t) = 10\cos(\omega t + \frac{\pi}{3})$,
   (b) $I(t) = -8\sin(\omega t + \frac{\pi}{3})$,
   (c) $A(t) = 3\sin\omega t - 2\cos\omega t$,
   (d) $C(t) = 3\cos(\omega t + \frac{\pi}{4}) + 4\sin(\omega t + \frac{\pi}{3})$.

4) Obtain $C(t)$ in terms of ω from the following phasors:
   (a) $c = 1 + j$,

(b) $c = 4\exp(j0.8)$,
(c) $c = 3e^{j\frac{\pi}{2}} + 4e^{j0.8}$,
(d) $c = j \sin 3z$.

5) (a) Using binomial theorem, show that

$$\sqrt{1+ja} \simeq \pm\left(1 + j\frac{a}{2}\right), \qquad \text{if } |a| \ll 1.$$

(b) Show that

$$\sqrt{1+ja} \simeq \pm(1+j)\left(\frac{a}{2}\right)^{\frac{1}{2}}, \qquad \text{if } |a| \gg 1.$$