

19. Representation of a Plane Wave.

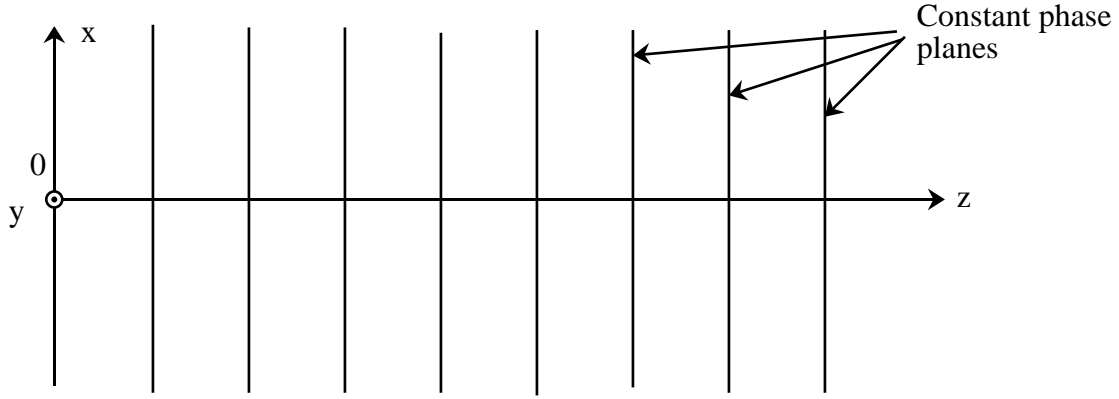
When $\nabla \cdot \mathbf{E} = 0$, the electric field satisfies the wave equation

$$\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0, \quad (1)$$

where $\beta^2 = \omega^2 \mu \epsilon$. We have learnt that one of the many possible solutions to the above equation is

$$\mathbf{E} = \hat{x} E_0 e^{-j\beta z}. \quad (2)$$

The expression $e^{-j\beta z}$, when viewed in three dimensions, has constant phase planes or wave fronts which are orthogonal to the z -axis.



To denote a plane wave propagating in other directions, we write it as

$$\mathbf{E} = \hat{a} E_0 e^{-j\beta_x x - j\beta_y y - j\beta_z z}, \quad (3)$$

where \hat{a} is a constant unit vector, and E_0 a constant. If we substitute (3) into (1), we obtain

$$[-\beta_x^2 - \beta_y^2 - \beta_z^2 + \beta^2] E_0 = 0. \quad (4)$$

In order for (3) to satisfy (1) and that $E_0 \neq 0$, we require that

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2 \mu \epsilon. \quad (5)$$

If we define a vector $\boldsymbol{\beta} = \hat{x}\beta_x + \hat{y}\beta_y + \hat{z}\beta_z$, and $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$, then (3) can be written as

$$\mathbf{E} = \hat{a} E_0 e^{-j\boldsymbol{\beta} \cdot \mathbf{r}}, \quad (6)$$

where the magnitude of $\boldsymbol{\beta}$ is

$$|\boldsymbol{\beta}| = [\beta_x^2 + \beta_y^2 + \beta_z^2]^{\frac{1}{2}} = \beta. \quad (7)$$

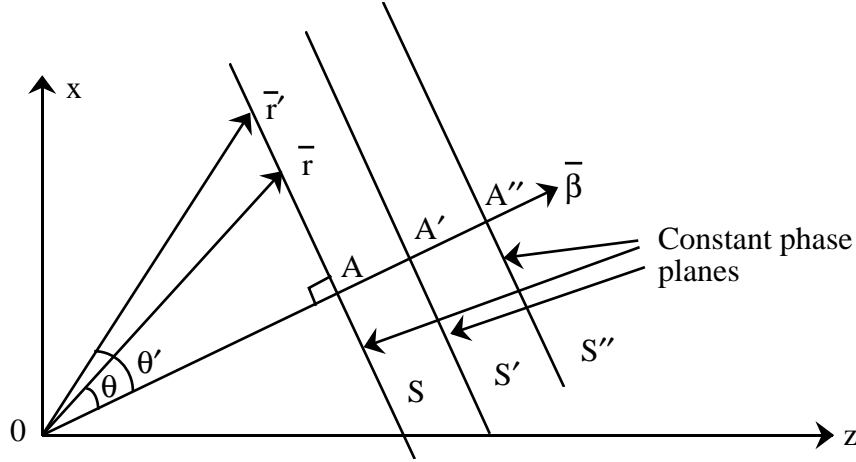
Equation (6) is a concise way to write a solution to (1). Since $\nabla \cdot \mathbf{E} = 0$ using (3), we note that

$$\nabla \cdot \mathbf{E} = -j[\hat{x}\beta_x + \hat{y}\beta_y + \hat{z}\beta_z] \cdot \hat{a}E_0e^{-j\boldsymbol{\beta} \cdot \mathbf{r}}. \quad (8)$$

Therefore, in order for $\nabla \cdot \mathbf{E} = 0$, we require that

$$\boldsymbol{\beta} \cdot \hat{a} = 0. \quad (9)$$

To explore further how the function $e^{-j\boldsymbol{\beta} \cdot \mathbf{r}}$ look like, we assume $\boldsymbol{\beta}$ to be pointing in a direction as shown in the figure. The value of $\boldsymbol{\beta} \cdot \mathbf{r}$ is constant on a plane that is orthogonal to $\boldsymbol{\beta}$.



That is

$$\boldsymbol{\beta} \cdot \mathbf{r} = |\boldsymbol{\beta}| |\mathbf{r}| \cos \theta = \beta(OA), \quad (10)$$

for all \mathbf{r} on the plane S that is orthogonal to $\boldsymbol{\beta}$. Hence, S is the constant phase plane of $e^{-j\boldsymbol{\beta} \cdot \mathbf{r}} = e^{-j\beta(OA)}$. As one moves progressively in the $\boldsymbol{\beta}$ direction, the function $e^{-j\boldsymbol{\beta} \cdot \mathbf{r}}$ has a phase that is linearly decreasing with distance. Therefore, $e^{-j\boldsymbol{\beta} \cdot \mathbf{r}}$ denotes a plane wave that is propagating in the $\boldsymbol{\beta}$ direction. When $\boldsymbol{\beta}$ is pointing in the z -direction, such that $\boldsymbol{\beta} = \hat{z}\beta$, then $e^{-j\boldsymbol{\beta} \cdot \mathbf{r}} = e^{-j\beta z}$, which is our familiar solution of a plane wave propagating in the z -direction.

An example of a plane wave electric field satisfying Maxwell's equations is

$$\mathbf{E} = \hat{y}E_0e^{-j\beta_x x - j\beta_z z}, \quad (11)$$

where $\beta_x^2 + \beta_z^2 = \beta^2$. The corresponding magnetic field can be derived using Maxwell's equations.

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}. \quad (12)$$

Hence,

$$\begin{aligned} \mathbf{H} &= \frac{-1}{j\omega\mu} \left(\hat{z} \frac{\partial}{\partial x} E_y - \hat{x} \frac{\partial}{\partial z} E_y \right) \\ &= (\hat{z}\beta_x - \hat{x}\beta_z) \frac{E_0}{\omega\mu} e^{-j\beta_x x - j\beta_z z}. \end{aligned} \quad (13)$$

In general, when ∇ operates on a plane wave phasor described by $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$, it transforms into $-j\boldsymbol{\beta}$. This is obvious also from Equation (8). Therefore, from (12), we can express

$$\mathbf{H} = \frac{1}{\omega\mu}\boldsymbol{\beta} \times \mathbf{E}. \quad (14)$$

Therefore, \mathbf{H} is orthogonal to both \mathbf{E} and $\boldsymbol{\beta}$, or that $\mathbf{H} \cdot \mathbf{E} = 0$, and that $\mathbf{H} \cdot \boldsymbol{\beta} = 0$, in addition to $\mathbf{E} \cdot \boldsymbol{\beta} = 0$. Furthermore, $\mathbf{E} \times \mathbf{H}$ points in the direction of $\boldsymbol{\beta}$. Therefore, for a plane electromagnetic wave, \mathbf{E} , \mathbf{H} , and $\boldsymbol{\beta}$ form a right-handed orthogonal system. It is also a transverse electromagnetic (TEM) wave.

