W.C.Chew
ECE 350 Lecture Notes

13. Properties of Fields in a Transmission Line.

The field or wave in a transmission line is TEM (Transmission Electro-Magnetic) because both the $\mathbf{H}$-field and the $\mathbf{E}$-field are transverse to the direction of propagation. If the wave is propagating in the $\hat{z}$-direction, then both E_z and H_z are zero for such a wave. In such a case, the fields are

$$\mathbf{E} = \mathbf{E}_s, \mathbf{H} = \mathbf{H}_s, \tag{1}$$

where we have used the subscript s to denote fields transverse to the direction of propagation. We can also define a del operation such that

$$\nabla = \nabla_s + \hat{z}\frac{\partial}{\partial z}, \tag{2}$$

where ∇_s is transverse to the $\hat{z}$-direction, and in Cartesian coordinate, it is $\nabla_s = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$. From

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{3}$$

or

$$\left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \times \mathbf{H}_s = \epsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{4}$$

Since $\nabla_s \times \mathbf{H}_s$ points in the $\hat{z}$-direction, $\hat{z}\frac{\partial}{\partial z} \times \mathbf{H}_s$ is $\hat{z}$-directed, we have

$$\nabla_s \times \mathbf{H}_s = 0, \tag{5}$$

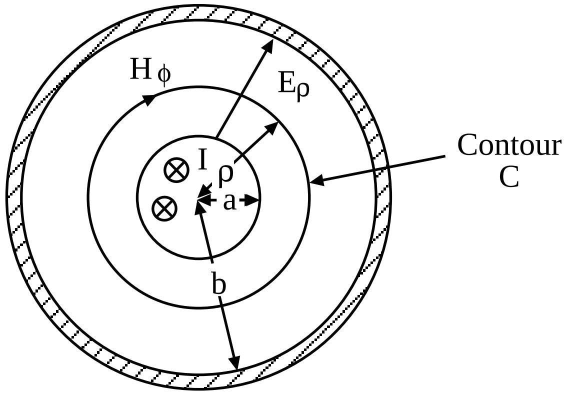
$$\frac{\partial}{\partial z}(\hat{z} \times \mathbf{H}_s) = \epsilon \frac{\partial \mathbf{E}_s}{\partial t}. \tag{6}$$

Similarly, from $\nabla_s \times \mathbf{E}_s = -\mu \frac{\partial \mathbf{H}_s}{\partial t}$, we can show that

$$\nabla_s \times \mathbf{E}_s = 0, \tag{7}$$

$$\frac{\partial}{\partial z}(\hat{z} \times \mathbf{E}_s) = -\mu \frac{\partial \mathbf{H}_s}{\partial t}. \tag{8}$$

Equations (5) and (7) shows that the transverse curl of the fields are zero. This implies that the fields in the transverse directions of a transmission line resembles that of the electrostatic fields. Furthermore, Equations (6) and (8) couple the $\mathbf{E}_s$ and $\mathbf{H}_s$ fields together. These two equations are the electromagnetic field analogues of the telegrapher's equations.



A current in a coaxial cable will produce a magnetic field polarized in the ϕ direction. From Ampere's Law, we have

$$\oint_C \mathbf{H}_s \cdot dl = \int_A \mathbf{J} \cdot ds = I, \tag{9}$$

or

$$\int_0^{2\pi} \rho \, d\phi H_\phi = I. \tag{10}$$

Hence,

$$H_\phi(\rho, z, t) = \frac{I(z,t)}{2\pi\rho}. \tag{11}$$

If we assume that the inner conductor in the coaxial line is charged up with the line charge Q in coulomb/m, then from $\oint \epsilon \mathbf{E} \cdot \hat{n} \, ds = Q$, we have

$$2\pi\rho\epsilon E_\rho = Q, \tag{12}$$

or

$$E_\rho = \frac{Q}{2\pi\rho\epsilon}. \tag{13}$$

Since the potential between a and b is $\int_a^b E_\rho \, d\rho$, we have

$$V = \int_a^b E_\rho \, d\rho = \frac{Q}{2\pi\epsilon} \ln\left(\frac{b}{a}\right). \tag{14}$$

Hence,

$$E_\rho(\rho, z, t) = \frac{V(z,t)}{\rho \ln(\frac{b}{a})} = \frac{Q(z,t)}{2\pi\epsilon\rho}. \tag{15}$$

The ratio $\frac{Q}{V}$ is the capacitance per unit length, and it is

$$C = \frac{2\pi\epsilon}{\ln(\frac{b}{a})}. \tag{16}$$

If $\mathbf{E}_s = \hat{\rho} E_\rho$, $\mathbf{H}_s = \hat{\phi} H_\phi$, equations (6) and (8) become

$$\frac{\partial}{\partial z} H_\phi = -\epsilon \frac{\partial E_\rho}{\partial t}, \tag{17}$$

$$\frac{\partial}{\partial z} E_\rho = -\mu \frac{\partial H_\phi}{\partial t}. \tag{18}$$

Substituting (11) for H_ϕ and (15) for E_ρ, we get

$$\frac{\partial}{\partial z} I(z,t) = -\frac{2\pi\epsilon}{\ln(\frac{b}{a})} \frac{\partial V}{\partial t}, \tag{19}$$

and

$$\frac{\partial}{\partial z} V(z,t) = -\frac{\mu \ln(\frac{b}{a})}{2\pi} \frac{\partial I}{\partial t}. \tag{20}$$

This is just the telegrapher's equations derived from Maxwell's equations. C is given by (16) while the inductance per unit length L is obtained by comparing (20) with the telegrapher's equations

$$L = \mu \frac{\ln(\frac{b}{a})}{2\pi}. \tag{21}$$

Note that the velocity of the wave on a transmission line is

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}}, \tag{22}$$

which is independent of the dimensions of the line. This is because all TEM waves have velocity given by $\frac{1}{\sqrt{\mu\epsilon}}$.