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ECE 350 Lecture Notes

25. Vector Potential - Introduction to Antennas & Radiations

Maxwell's equations are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}, \tag{1}$$
$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J}, \tag{2}$$
$$\nabla \cdot \mu\mathbf{H} = 0, \tag{3}$$
$$\nabla \cdot \epsilon\mathbf{E} = \rho. \tag{4}$$

Since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, we can let

$$\mu\mathbf{H} = \nabla \times \mathbf{A}, \tag{5}$$

so that equation (3) is automatically satisfied. Substituting (5) into (1), we have

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0. \tag{6}$$

Since $\nabla \times \nabla\phi = 0$, we have

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\phi. \tag{7}$$

Hence, knowing $\mathbf{A}$ and ϕ uniquely determines $\mathbf{E}$ and $\mathbf{H}$. We shall relate $\mathbf{A}$ and ϕ to the **sources** $\mathbf{J}$ and ρ of Maxwell's equations. Substituting (5) and (7) into (2), we have

$$\nabla \times \nabla \times \mathbf{A} = j\omega\mu\epsilon[-j\omega\mathbf{A} - \nabla\phi] + \mu\mathbf{J}, \tag{8}$$

or

$$\nabla^2\mathbf{A} + \omega^2\mu\epsilon\mathbf{A} = -\mu\mathbf{J} + j\omega\mu\epsilon\nabla\phi + \nabla\nabla \cdot \mathbf{A}. \tag{9}$$

Using (7) in (4), we have

$$\nabla \cdot (j\omega\mathbf{A} + \nabla\phi) = -\frac{\rho}{\epsilon}. \tag{10}$$

The above could be simplified for the following observation. Equations (5) and (7) give the same $\mathbf{E}$ and $\mathbf{H}$ fields under the transformation

$$\mathbf{A}' = \mathbf{A} + \nabla\psi, \tag{11}$$
$$\phi' = \phi - j\omega\psi. \tag{12}$$

The above are known as the **Gauge Transformation**. With the new $\mathbf{A}'$ and ϕ', we can substitute into (5) and (7) and they give the same $\mathbf{E}$ and $\mathbf{H}$ fields, i.e.

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times \nabla\psi = \nabla \times \mathbf{A} = \mu\mathbf{H}, \tag{13}$$
$$-j\omega\mathbf{A}' - \nabla\phi' = -j\omega\mathbf{A} - j\omega\nabla\psi - \nabla\phi + j\omega\nabla\psi = \mathbf{E}. \tag{14}$$

It implies that $\mathbf{A}$ and ϕ are not unique. The vector field $\mathbf{A}$ is not unique unless we specify both its curl and its divergence. Hence, in order to make $\mathbf{A}$ unique, we have to specify its divergence. If we specify the divergence of $\mathbf{A}$ such that

$$\nabla \cdot \mathbf{A} = -j\omega\mu\epsilon\phi, \tag{15}$$

then (9) and (10) become

$$\nabla^2\mathbf{A} + \omega^2\mu\epsilon = -\mu\mathbf{J}, \tag{16}$$

$$\nabla^2\phi + \omega^2\mu\epsilon\phi = -\frac{\rho}{\epsilon}. \tag{17}$$

The condition in (15) is also known as the **Lorentz gauge**. Equations (16) and (17) represent a set of four inhomogeneous wave equations driven by the sources of Maxwell's equations. Hence given the sources ρ and $\mathbf{J}$, we may find $\mathbf{A}$ and ϕ. $\mathbf{E}$ and $\mathbf{H}$ may in turn be found using (5) and (7). However, as a consequence of the Lorentz gauge, we need only to find $\mathbf{A}$; ϕ follows directly from equation (15).

Let us consider the relation due to an elemental current that can be described by

$$\mathbf{J} = \hat{z}Il\delta(\mathbf{r}) \quad A/m^2, \tag{18}$$

where Il denotes the strength of this current, and $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$. Equation (16) becomes

$$\nabla^2 A_z + \omega^2\mu\epsilon A_z = -\mu Il\delta(\mathbf{r}). \tag{19}$$

Taking advantage of the spherical symmetry of the problem, ∇^2 has only r dependence in spherical coordinates, we have

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}A_z + \beta^2 A_z = -\mu Il\delta(\mathbf{r}), \tag{20}$$

where $\beta^2 = \omega^2\mu\epsilon$. Equations (19) and (20) are similar in form to Poisson's equation with a point charge Q at the origin,

$$\nabla^2\phi = -\frac{Q}{\epsilon}\delta(\mathbf{r}). \tag{21}$$

We know that (21) has the solution of the form

$$\phi = \frac{Q}{4\pi\epsilon r}. \tag{22}$$

Hence, we guess that the solution to (20) is of the form

$$A_z = \frac{\mu Il}{4\pi r}C(r). \tag{23}$$

It can be shown that

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}f(r) = \frac{1}{r}\frac{d^2}{dr^2}rf(r). \tag{24}$$

Outside the origin, the RHS of (20) is zero, and after using (23) and (24) in (20), we have

$$\frac{d^2}{dr^2}C(r) + \beta^2 C(r) = 0. \tag{25}$$

This gives

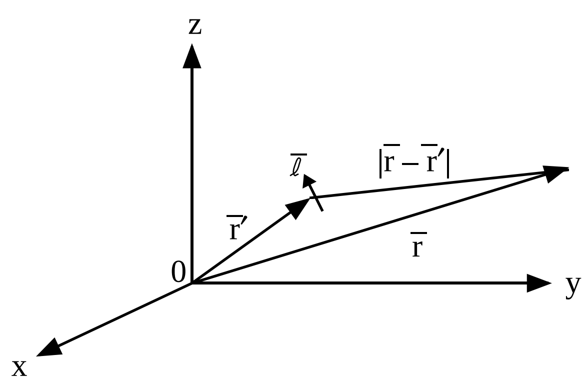
$$C(r) = e^{\pm j\beta r}. \tag{26}$$

Since we are looking for a solution that radiates energy to infinity, we choose an outgoing solution in (26). Hence,

$$A_z(r) = \frac{\mu I l}{4\pi r} e^{-j\beta r}, \tag{27}$$

for a source directed at a $\hat{z}$-direction. From (16), we note that $\mathbf{A}$ and $\mathbf{J}$ always point in the same direction. Therefore, for a point source directed at $\mathbf{l}$ and located at $\mathbf{r}'$ instead of the origin, the vector potential $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu I \mathbf{l}}{4\pi\,|\mathbf{r}-\mathbf{r}'|} e^{-j\beta|\mathbf{r}-\mathbf{r}'|}. \tag{28}$$



By linear superposition, the vector potential due to an arbitrary source $\mathbf{J}$ is

$$\mathbf{A} = \frac{\mu}{4\pi} \iiint d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{-j\beta|\mathbf{r}-\mathbf{r}'|}. \tag{29}$$

Similarly, we can show that

$$\phi = \frac{1}{4\pi\epsilon} \iiint d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{-j\beta|\mathbf{r}-\mathbf{r}'|}. \tag{30}$$