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ECE 350 Lecture Notes

9. Complex Power on a Transmission Line

Complex Power

Since we are dealing with phasors, it is convenient to define a complex power which has an imaginary part as well as a real part. We shall define the meaning of complex power.

A complex power is defined as

$$\tilde{P} = \tilde{V}\tilde{I}^*, \tag{1}$$

i.e. the product of a voltage phasor and a current phasor at a given point. If

$$\tilde{V} = |\tilde{V}|e^{j\phi_V}, \quad \tilde{I} = |\tilde{I}|e^{j\phi_I}, \tag{2}$$

then

$$\tilde{P} = |\tilde{V}||\tilde{I}|[\cos(\phi_V - \phi_I) + \sin(\phi_V - \phi_I)]. \tag{3}$$

The corresponding real time voltage and current are

$$V(t) = |\tilde{V}|\cos(\omega t + \phi_V), \quad I(t) = |\tilde{I}|\cos(\omega t + \phi_I). \tag{4}$$

Then, the instantaneous power is

$$\begin{aligned} P(t) = V(t)I(t) &= |\tilde{V}||\tilde{I}|\cos(\omega t + \phi_V)\cos(\omega t + \phi_V + \phi_I - \phi_V) \\ &= |\tilde{V}||\tilde{I}|[\cos^2(\omega t + \phi_V)\cos(\phi_I - \phi_V) \\ &\qquad - \cos(\omega t + \phi_V)\sin(\omega t + \phi_V)\sin(\phi_I - \phi_V)]. \end{aligned} \tag{5}$$

The time average of $P(t)$, defined as

$$\begin{aligned} \langle P(t)\rangle = \langle V(t)I(t)\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_0^T dt P(t) \\ &= |\tilde{V}||\tilde{I}|[\langle\cos^2(\omega t + \phi_V)\rangle\cos(\phi_I - \phi_V) \\ &\qquad - \langle\cos(\omega t + \phi_V)\sin(\omega t + \phi_V)\rangle\sin(\phi_I - \phi_V)]. \end{aligned} \tag{6}$$

Since

$$\langle\cos^2(\omega t + \phi_V)\rangle = \frac{1}{2}, \quad \langle\cos(\omega t + \phi_V)\sin(\omega t + \phi_V)\rangle = 0, \tag{7}$$

we have

$$\langle P(t)\rangle = \frac{1}{2}|\tilde{V}\tilde{I}|\cos(\phi_I - \phi_V). \tag{8}$$

Comparing with (3), we see that

$$\langle P(t)\rangle = \frac{1}{2}\Re e[\tilde{P}]\,. \tag{9}$$

The imaginary part of the complex power is proportional to the second term in (5), and hence, the imaginary part of the complex power is proportional to a part of the instantaneous power that averages to zero. Consequently, the imaginary part of the complex power is called ***reactive power***. For example, a purely reactive device dissipates no power on the average, but instantaneous power is being constantly absorbed and released by a reactive device. The current and voltage through a reactive device is 90° out-of-phase, and the complex power is purely imaginary or purely reactive.

Complex Power on a Transmission Line

The voltage on a transmission line could be written as

$$\begin{aligned}\tilde{V}(z) &= V_0\left(e^{-j\beta z} + \rho_v e^{j\beta z}\right)\\ &= V_0 e^{-j\beta z}[1+\Gamma(z)]\,.\end{aligned} \tag{10}$$

The current on the line could be written as

$$\tilde{I}(z) = \frac{V_0}{Z_0}e^{-j\beta z}\left[1-\Gamma(z)\right]\,. \tag{11}$$

The complex power is given by

$$\tilde{P} = \tilde{V}\tilde{I}^* = \frac{|V|^2}{Z_0}[1+\Gamma(z)][1-\Gamma(z)^*]\,, \tag{12}$$

which reduces to

$$\tilde{P} = \frac{|V|^2}{Z_0}[1-|\Gamma(z)|^2+\Gamma(z)-\Gamma(z)^*]\,, \tag{13}$$

or

$$\tilde{P} = \tilde{V}\tilde{I}^* = \frac{|V|^2}{Z_0}[1-|\rho_v|^2+j2\Im m\Gamma(z)]\,. \tag{14}$$

The time average power, defined to be

$$\langle P(z,t)\rangle = \frac{1}{2}\Re e[\tilde{P}(z)] = \frac{|V|^2}{2Z_0}(1-|\rho_v|^2)\,, \tag{15}$$

for a lossless transmission line. If $\rho_v = 0$, or when the load is ***matched*** to the transmission line, (i.e., $Z_L = Z_0$), all the power carried in the forward going wave is dumped into the load. Otherwise, part of the power is reflected. The power carried by the forward going wave is

$$\langle P_+\rangle = \frac{|V|^2}{2Z_0}, \tag{16}$$

and the power carried by the backward going wave is

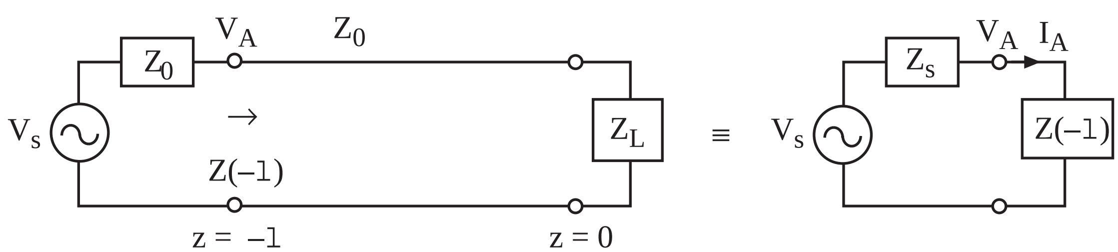
$$\langle P_-\rangle = \frac{|V|^2}{2Z_0}|\rho_v|^2 . \tag{17}$$

Note that $\langle P(z,t)\rangle$ is independent of z because of energy conservation.

$$\langle P\rangle = \langle P_+\rangle - \langle P_-\rangle, \tag{18}$$

is everywhere the same on the lossless transmission line because the total power leaving the source all arrive at the load end with no loss on the lossless transmission line. The transmission line can only absorb reactive power. Hence, the reactive power in (14) is not a constant of position.

Power Delivered to the Load on a Transmission Line



To find the power delivered to the load on a lossless transmission line, we can first find $Z(-l)$ using formula (6.11). Then, we can replace the transmission line circuit with the equivalent circuit for finding V_A, and I_A. The real power delivered to $Z(-l)$ would be the same as the real power delivered to Z_L.

$$\tilde{P} = V_A I_A^* = \frac{|V_A|^2}{Z^*(-l)} = \left|\frac{Z(-l)}{Z_s + Z(-l)}\right|^2 \frac{|V_S|^2}{Z^*(-l)} = \frac{Z(-l)|V_S|^2}{|Z_s + Z(-l)|^2} . \tag{19}$$

The time-average power delivered to the load is

$$\langle P\rangle = \frac{1}{2}\Re e[\tilde{P}] = \frac{1}{2}\frac{R(-l)|V_S|^2}{|R_s + jX_S + R(-l) + jX(-l)|^2}, \tag{20}$$

where we have assumed that $Z_S = R_S + jX_S$, and $Z(-l) = R(-l) + jX(-l)$. To optimize $\langle P\rangle$, with respect to $X(-l)$, we choose $X(-l) = -X_S$, hence,

$$\langle P\rangle = \frac{1}{2}\frac{R(-l)|V_S|^2}{|R_s + R(-l)|^2}. \tag{21}$$

The above is maximum when $R(-l) = R_S$. Hence, maximum power is delivered to the load when

$$Z(-l) = Z_S^* . \tag{22}$$