ECE 255, Miller Effect, Etc.

26 April 2018

1 Introduction

In this lecture, we will focus on the study of the Miller effect and other high-frequency analysis tools for transistors.

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2 High-Frequency Response of the CS and CE Amplifier with Miller Effect

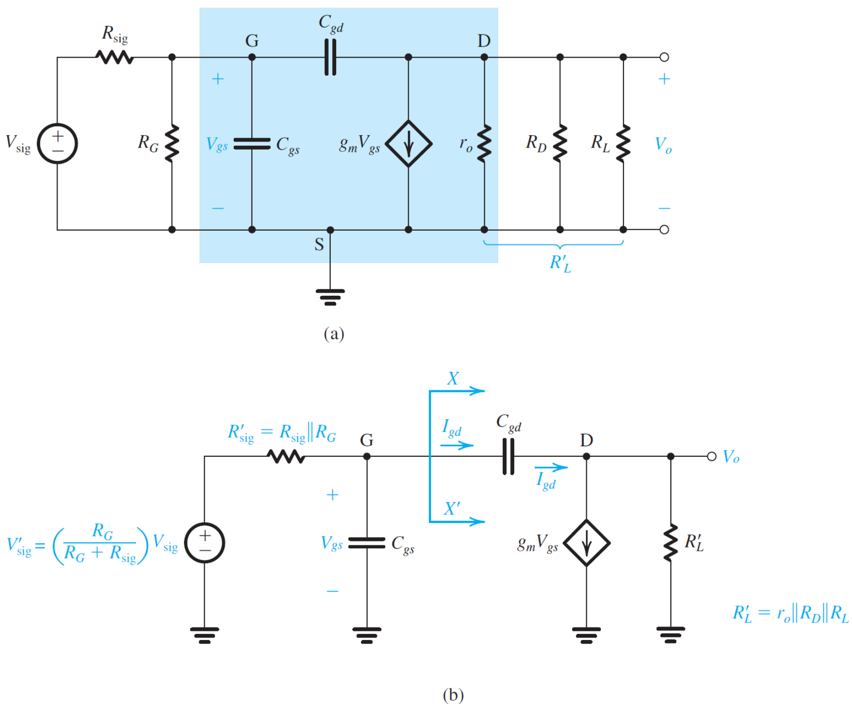

Figure 1: Models for the high-frequency response of a CS amplifier. (a) Equivalent circuit. (b) A simplified circuit by consolidation: The source signal is replaced by a Thévenin equivalence with effective source V'_{sig} and Thévenin resistor R'_{sig}, and the load is replaced by an effective load R'_L (Courtesy of Sedra and Smith).

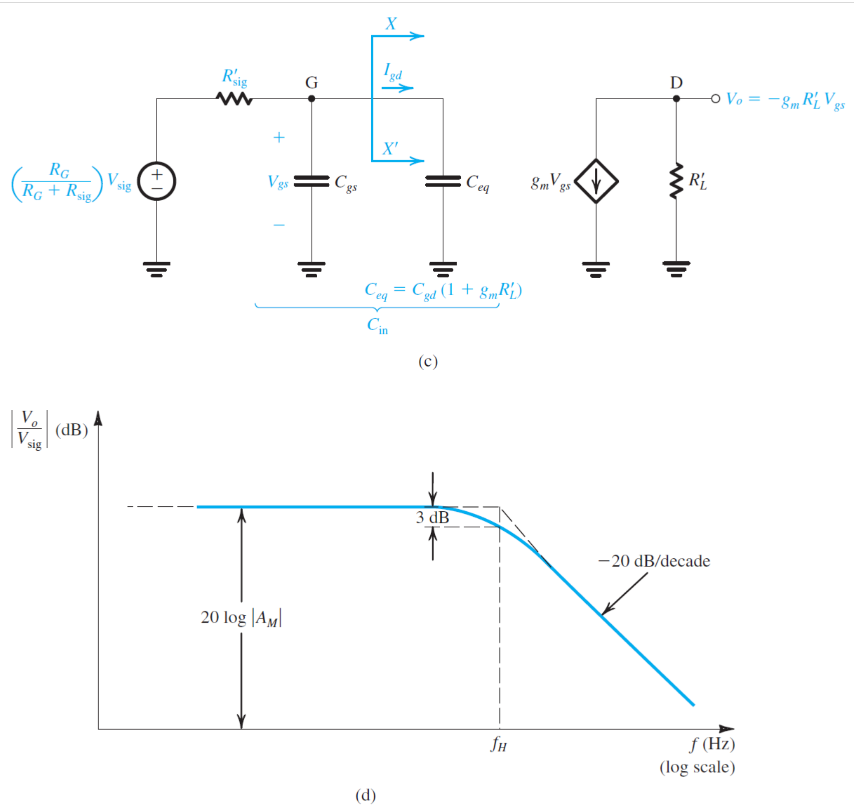

Figure 2: Continuation of the previous Figure 1. (c) Further simplification by using C_{eq}. (d) A single-time-constant frequency response Bode plot for a single-pole system (Courtesy of Sedra and Smith).

Figure 1 shows the small-signal equivalence of a CS amplifier. The overall voltage gain is given by

$$A_M = \frac{V_o}{V_{\text{sig}}} = -\frac{R_G}{R_G + R_{\text{sig}}}(g_m R'_L) \tag{2.1}$$

In order to simplify the circuit, it can be consolidated using Thévenin theorem so that the source is modeled by only two elements as shown in Figure 1(b). Also, R'_L consolidates the three resistances at the output end.

One further simplification is to replace the capacitor C_{gd} with an equivalence capacitor C_{eq} as shown in Figure 1(c). To this end, we shall calculate the load current, in accordance with Figure 1(b), which is given by $(-g_m V_{gs} + I_{gd})$ via KCL at node D. Then using generalized Ohm's law, the output voltage is given by

$$V_o = (-g_m V_{gs} + I_{gd})R'_L \approx -g_m R'_L V_{gs} \tag{2.2}$$

One assumes that $|g_m V_{gs}| \gg |I_{gd}|$ in the last approximation. In the above, $R'_L = r_o \parallel R_D \parallel R_L$. The current I_{gd} can now be found using V_o from (2.2) as

$$I_{gd} = sC_{gd}(V_{gs} - V_o) \approx sC_{gd}\left[V_{gs} - (-g_m R'_L V_{gs})\right] = sC_{gd}(1 + g_m R'_L)V_{gs} \quad (2.3)$$

Next, one can assume that this extra current I_{gd} is due to an equivalent capacitor C_{eq} connected in parallel to C_{gs} as shown in Figure 2(c). The current that flows into this equivalent capacitance C_{eq} is given by I_{gd}, which is given by

$$sC_{eq}V_{gs} = sC_{gd}(1 + g_m R'_L)V_{gs} \quad (2.4)$$

The above results in that

$$C_{eq} = C_{gd}(1 + g_m R'_L) \quad (2.5)$$

This equivalent capacitance C_{eq} is much larger than C_{gd}, and this effect is known as the **Miller effect**, and the factor $(1 + g_m R'_L)$ is the **Miller multiplier**. Hence, the larger the gain of the amplifier is, the larger this effect is.

One can see that C_{eq} is much larger than C_{gd} because when a positive V_{gs} is applied at the gate, the negative voltage V_o is generated at the output node according to (2.2). This negative V_o siphons more current from the gate to the drain as result, increasing I_{gd} in accordance with (2.3).

One has to be reminded that the above derivation is predicated on the assumption that $|g_m V_{gs}| \gg |I_{gd}|$. However, it is seen that the Miller effect amplifies I_{gd} as shown in (2.3). Hence, the $|g_m V_{gs}| \gg |I_{gd}|$ has to be checked when one uses the above simplified formula for the equivalence capacitance C_{eq}.

2.1 Single Pole Approximation

The above approximation replaces a double-pole system with a single pole system or a single-time-constant (STC) circuit. Then the frequency dependent function with one pole and no zero can now be represented as

$$V_{gs} = \left(\frac{R_G}{R_G + R_{\text{sig}}} V_{\text{sig}}\right) \frac{1}{1 + s/\omega_0} \quad (2.6)$$

The effective capacitance of this system is $C_{\text{in}} = C_{gs} + C_{eq}$. This capacitance relaxes its charge via the effective resistor R'_{sig}. Hence, the pole frequency of this STC circuit is then

$$\omega_0 = 1/(C_{\text{in}} R'_{\text{sig}}) \quad (2.7)$$

with

$$C_{\text{in}} = C_{gs} + C_{eq} = C_{gs} + C_{gd}(1 + g_m R'_L) \quad (2.8)$$

and

$$R'_{\text{sig}} = R_{\text{sig}} \parallel R_G \quad (2.9)$$

A sanity check of (2.6) shows that this in fact is the correct formula: it reduces to the correct formula when $s = \omega = 0$. The system in Figure 2(c) can only have one pole with the corresponding relaxation frequency given by (2.7). The zero of the system is at $s = \infty$, or unimportant. Therefore, it is the correct formula which can be confirmed by a longer derivation.

Then using V_{gs} from (2.6) in (2.2), one arrives at

$$\frac{V_o}{V_{\mathrm{sig}}} = -\left(\frac{R_G}{R_G + R_{\mathrm{sig}}}\right)(g_m R'_L)\frac{1}{1 + s/\omega_0} \tag{2.10}$$

which can be simplified as

$$\frac{V_o}{V_{\mathrm{sig}}} = \frac{A_M}{1 + s/\omega_H} \tag{2.11}$$

where A_M is the midband gain given by (2.1), and ω_H is the upper 3-dB frequency point, or

$$\omega_H = \omega_0 = \frac{1}{C_{\mathrm{in}} R'_{\mathrm{sig}}}, \qquad f_H = \frac{\omega_H}{2\pi} = \frac{1}{2\pi C_{\mathrm{in}} R'_{\mathrm{sig}}} \tag{2.12}$$

2.1.1 Validity of Single-Pole Approximation

The single-pole approximation is valid when the second pole is far away from the first pole. It can be shown that with exact analysis, when the inequality $|g_m V_{gs}| \gg |I_{gd}|$ is satisfied, the second pole is indeed far away. Then one has to check if the zeros of the transfer function are also far away.

The zeros of the amplifier can be obtained by looking at Figure 1(a). The zero occurs as $s = \infty$ as C_{gs} becomes short circuit. This is the unimportant zero. Also, another possible zero occurs when $V_o = 0$. In this case, the voltage across the capacitor C_{gd} is V_{gs} and the current through it is $sC_{gd}V_{gs}$. Then if $g_m V_{gs}$ and $sC_{gd}V_{gs}$ are equal to each other, no current flows to the output node giving rise to $V_o = 0$. This is only possible if

$$g_m = sC_{gd} \tag{2.13}$$

or when $s = g_m/C_{gd}$ which is the location of this zero. But C_{gd} is usually small, making this zero far away compared to ω_0 in (2.10).

2.2 Miller Effect of the Common-Emitter Amplifier

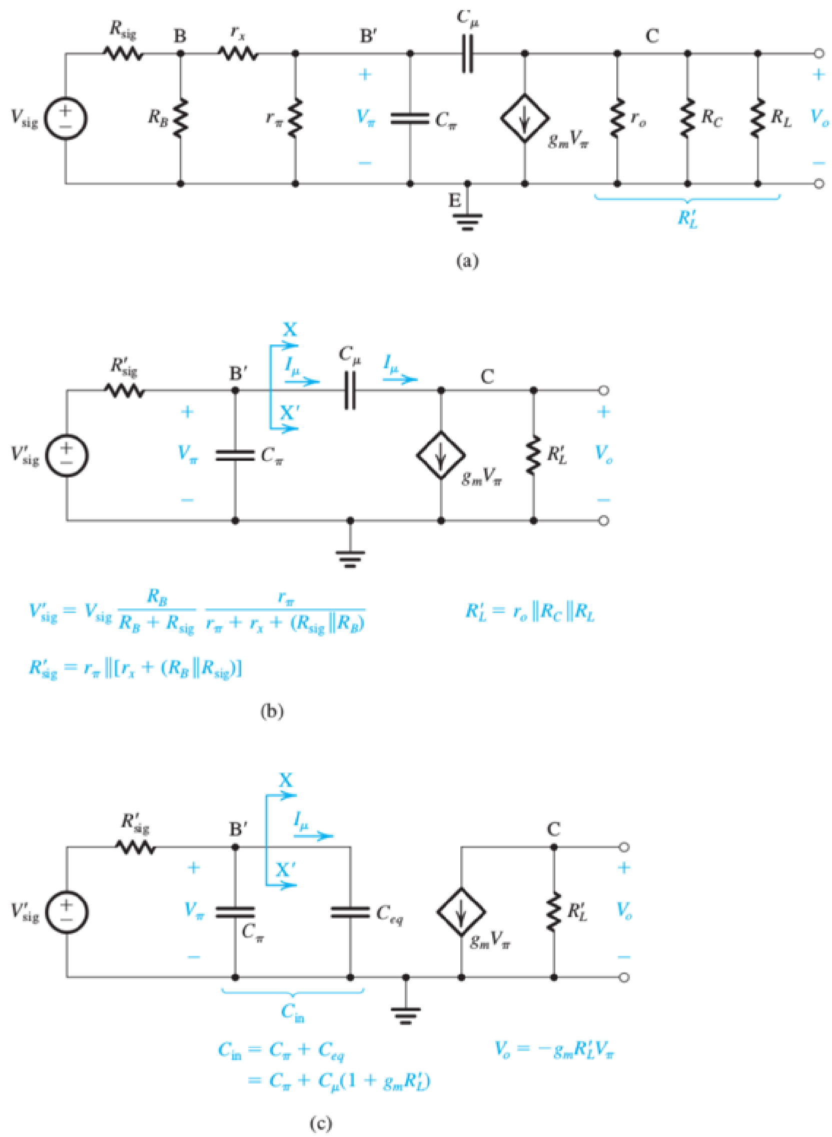

Figure 3: Models for the high-frequency response of a CE amplifier. (a) Equivalent circuit. (b) A simplified circuit by consolidation. (c) Further simplification by using C_{eq} (Courtesy of Sedra and Smith).

The analysis of the CE amplifier is very similar to that of the CS amplifier as shown in Figure 3.

2.3 Some Observations

1. The upper 3-dB frequency is determined by the interaction of R'_{sig} with $C_{\text{in}} = C_{gs} + C_{gd}(1 + g_m R'_L)$. Further, $R_{\text{sig}} \parallel R_G \approx R_{\text{sig}}$ when $R_G \gg R_{\text{sig}}$. Hence, a large R_{sig} will cause f_H to be lowered, decreasing the bandwidth of the amplifier.

2. The total capacitance C_{in} is increased by the Miller effect which magnifies C_{gd} by the factor $1 + g_m R'_L$, which lowers f_H.

3. To improve the high-frequency response of MOSFET, one has to reduce the Miller effect.

4. The STC is an approximation because we are replacing a double-pole system with a single-pole system. A system with two capacitors has two poles, but replacing it with one capacitor or one single pole is only approximately correct.

5. The dominant high-frequency pole of the system is given by by $f_P \approx f_H$.

2.4 Miller's Theorem

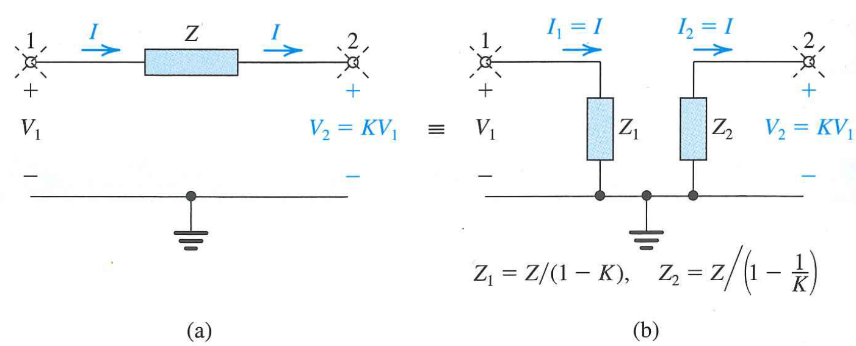

Figure 4: The Miller equivalent circuit (Courtesy of Sedra and Smith).

The **Miller's theorem** allows the replacement of a bridging capacitance by two equivalent capacitances as shown in Figure 4. This theorem relies on that

$$V_2 = K V_1 \tag{2.14}$$

in Figure 4. In this case, it can be shown that

$$Z_1 = Z/(1 - K), \qquad Z_2 = Z \Big/ \left(1 - \frac{1}{K}\right) \tag{2.15}$$

The proof is given in the textbook and will not be reproduced here.[1]

3 Useful Tools for High-Frequency Response of Amplifiers

Next, some more sophisticated and useful tools for high-frequency analysis of transistor amplifiers will be given. When the simple analysis previously discussed fails, one may resort to these more sophisticated tools. This happens for instance, when the poles and zeros are not far apart.

3.1 High-Frequency Gain Function

The frequency gain as a function of frequency can be expressed as

$$A(s) = A_M F_H(s) \tag{3.1}$$

where in general,

$$F_H(s) = \frac{(1+s/\omega_{Z1})(1+s/\omega_{Z2})\cdots(1+s/\omega_{Zn})}{(1+s/\omega_{P1})(1+s/\omega_{P2})\cdots(1+s/\omega_{Pm})} \tag{3.2}$$

Notice that the above function $F_H(s) \to 1$ when $s \to 0$, which is what is desired.

3.2 Determining the 3-dB Frequency f_H

3.2.1 Single-Pole Case

As in the low-frequency case, when the above function is dominated by a single pole, then one has

$$F_H(s) \approx \frac{1}{1+s/\omega_{P1}} \tag{3.3}$$

The 3-dB point is easily shown to be

$$\omega_H \approx \omega_{P1} \tag{3.4}$$

by first letting $s = j\omega$ and then $\omega = \omega_{P1}$. The above single pole approximation is good when the next pole or zero is two octaves (4 times) further away from the dominant pole.

3.2.2 Double-Pole Case

If a dominant pole approximation is not possible, the aggregate effects of the poles and zeros need to be considered in finding ω_H. For simplicity, one considers first a simple two-pole and two-zero system. Then

$$F_H(s) = \frac{(1+s/\omega_{Z1})(1+s/\omega_{Z2})}{(1+s/\omega_{P1})(1+s/\omega_{P2})} \tag{3.5}$$

[1] An elegant proof is also given by Davidović in 1999 in IEEE Trans. Education.

Letting $s = j\omega$ and taking the magnitude square of the above,[2] one gets

$$|F_H(j\omega)|^2 = \frac{(1+\omega^2/\omega_{Z1}^2)(1+\omega^2/\omega_{Z2}^2)}{(1+\omega^2/\omega_{P1}^2)(1+\omega^2/\omega_{P2}^2)} \tag{3.6}$$

By definition, at $\omega = \omega_H$, the half-power point, $|F_H(j\omega_H)|^2 = \frac{1}{2}$, and

$$\frac{1}{2} = \frac{(1+\omega_H^2/\omega_{Z1}^2)(1+\omega_H^2/\omega_{Z2}^2)}{(1+\omega_H^2/\omega_{P1}^2)(1+\omega_H^2/\omega_{P2}^2)} \approx \frac{1+\omega_H^2\left(\frac{1}{\omega_{Z1}^2}+\frac{1}{\omega_{Z2}^2}\right)+\cdots}{1+\omega_H^2\left(\frac{1}{\omega_{P1}^2}+\frac{1}{\omega_{P2}^2}\right)+\cdots} \tag{3.7}$$

where we have kept only the quadratic terms in both the numerator and denominator. The remaining terms are proportional to ω_H^4, which are negligible when ω_H is small, compared to the terms retained. The above equation can be solved approximately to yield[3]

$$\omega_H \approx 1\Bigg/\sqrt{\left(\frac{1}{\omega_{P1}^2}+\frac{1}{\omega_{P2}^2}\right)-2\left(\frac{1}{\omega_{Z1}^2}+\frac{1}{\omega_{Z2}^2}\right)} \tag{3.8}$$

Another statement that the above formula is making is that the half-power frequency point ω_H exists only if

$$2\left(\frac{1}{\omega_{Z1}^2}+\frac{1}{\omega_{Z2}^2}\right) > \left(\frac{1}{\omega_{P1}^2}+\frac{1}{\omega_{P2}^2}\right) \tag{3.9}$$

Otherwise, this frequency point is pure imaginary. This implies that the pole locations have to be small enough compared to the zero locations so that the above inequality can be satisfied.

3.2.3 Multi-Pole Case

The above expression can be generalized to a multi-pole and multi-zero system giving

$$\omega_H \approx 1\Bigg/\sqrt{\left(\frac{1}{\omega_{P1}^2}+\frac{1}{\omega_{P2}^2}+\cdots\right)-2\left(\frac{1}{\omega_{Z1}^2}+\frac{1}{\omega_{Z2}^2}+\cdots\right)} \tag{3.10}$$

Again, the half-power frequency point, ω_H exists only if

$$\left(\frac{1}{\omega_{P1}^2}+\frac{1}{\omega_{P2}^2}+\cdots\right) > 2\left(\frac{1}{\omega_{Z1}^2}+\frac{1}{\omega_{Z2}^2}+\cdots\right) \tag{3.11}$$

[2] One has used that $|A/B| = |A|/|B|$, $|AB| = |A||B|$, and $|x+jy|^2 = x^2+y^2$ to arrive at this.

[3] Without this approximation, the quartic equation is otherwise not easily solvable.

3.3 Low-Frequency Gain Function

A similar low-frequency gain function can be defined such that the frequency gain as a function of frequency can be expressed as

$$A(s) = A_M F_L(s) \tag{3.12}$$

where

$$F_L(s) = \frac{(1+\omega_{Z1}/s)(1+\omega_{Z2}/s)\cdots(1+\omega_{Zn}/s)}{(1+\omega_{P1}/s)(1+\omega_{P2}/s)\cdots(1+\omega_{Pm}/s)} \tag{3.13}$$

Notice that the above function $F_L(s) \to 1$ when $s \to \infty$, which is what is desired.

A similar analysis shows that the half-power frequency point ω_L is[4]

$$\omega_L \approx \sqrt{(\omega_{P1}^2 + \omega_{P2}^2 + \cdots) - 2\,(\omega_{Z1}^2 + \omega_{Z2}^2 + \cdots)} \tag{3.14}$$

Again, this half-power frequency point ω_L exists only if

$$\left(\omega_{P1}^2 + \omega_{P2}^2 + \cdots\right) > 2\left(\omega_{Z1}^2 + \omega_{Z2}^2 + \cdots\right) \tag{3.15}$$

One can obtain the above result by comparing (3.2) and (3.13). One notices that the roles of ω's and s's are switched in the formulas. Hence, by symmetry, one arrives at the above formula by letting $s \rightleftharpoons \omega$.

3.4 The Method of Open-Circuit Time Constants

In finding f_L, when a single pole dominates and they are far from each other, one uses the short-circuit time-constant method to decouple the capacitors and find their respective time constants. The rationale is that at the highest frequency pole that decides f_L, the frequency is high enough such that the other capacitors can be considered short-circuited, and hence, the determination of the highest frequency pole is a reasonable approximation (see Figure 5).

[4] This formula is given without derivation in the textbook in equation (10.21). Here's the derivation for it.

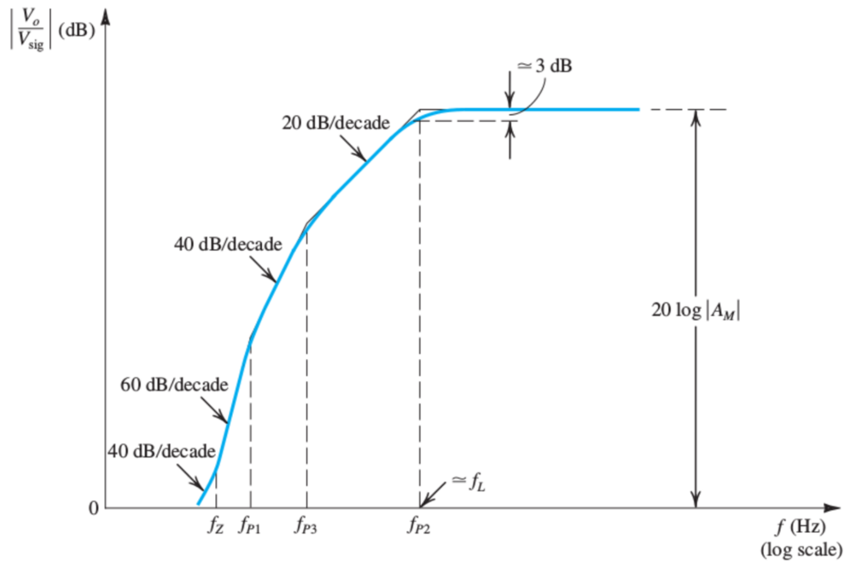

Figure 5: Reminder figure for the Bode plot of the low-frequency response of a transistor amplifier for defining ω_L (f_L) (Courtesy of Sedra and Smith).

By the same token, when one finds f_H, it is the lowest frequency pole that dominates f_H (see Figure 6). Again, this frequency is low enough that the other capacitors can be considered open-circuited. Then the time-constant for the lowest frequency pole is fairly accurate. With this in mind, then

$$\omega_H = 2\pi f_H \approx \frac{1}{\sum_i C_i R_i} \tag{3.16}$$

In the above sum, it will automatically be dominated by the largest RC time constant term, since it is an ordinary mean.

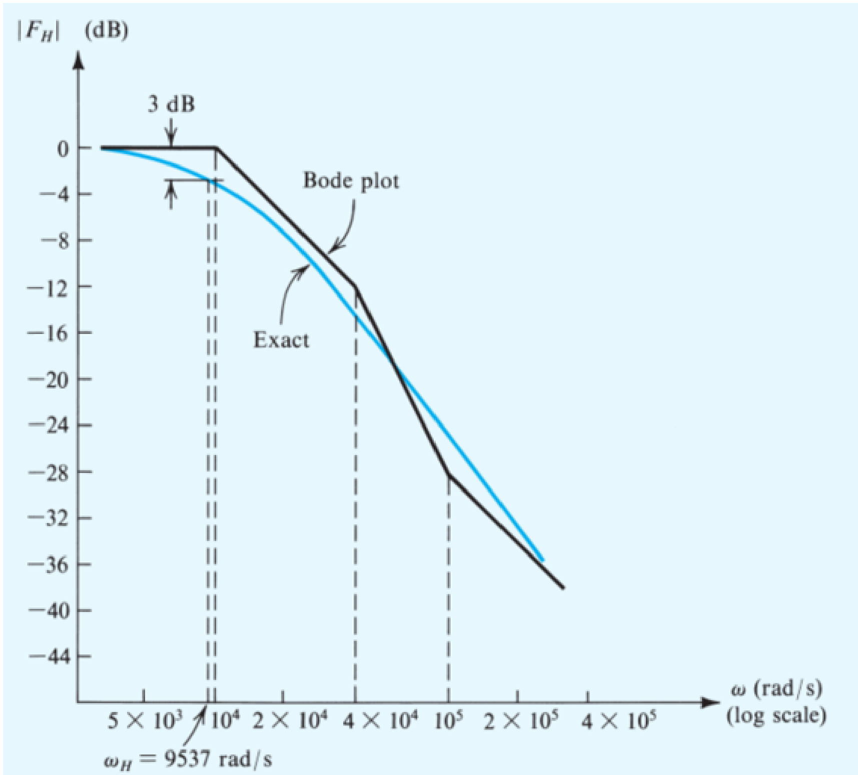

Figure 6: Reminder figure for the Bode plot of the high-frequency response of a transistor amplifier for defining ω_H (f_H) (Courtesy of Sedra and Smith).

In contrast, in the short-circuit time-constant method,

$$\omega_L = 2\pi f_L \approx \sum_i \frac{1}{R_i C_i} \tag{3.17}$$

which is proportional to the harmonic mean of the RC time constants. The term with the shortest time constant automatically dominates this sum.