ECE 255, Frequency Response

19 April 2018

1 Introduction

In this lecture, we address the frequency response of amplifiers. This was touched upon briefly in our previous lecture in Section 7.5 of the textbook. Here, we delve into more details on this topic. It will be from Sections 10.1 and 10.2 of the textbook.

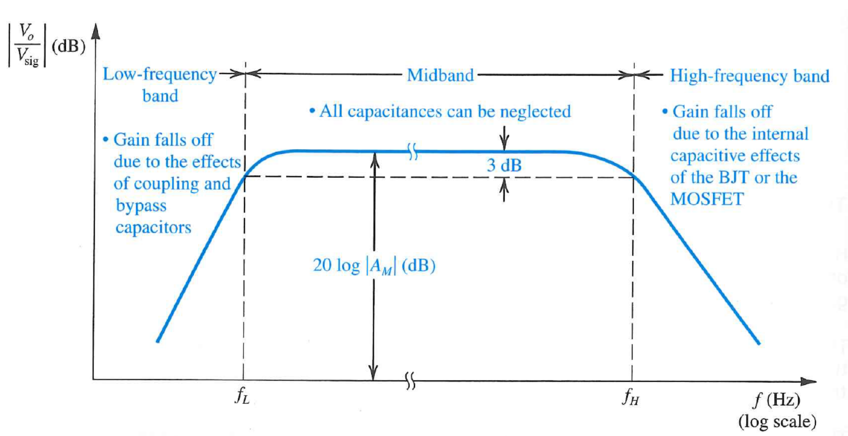

Figure 1: The frequency response of a discrete circuit is affected by the coupling capacitors and bypass capacitors at the low frequency end. At the high-frequency end, it is affected by the internal capacitors (or parasitic capacitances) of the circuit (Courtesy of Sedra and Smith).

Printed on April 19, 2018 at 15 : 33: W.C. Chew and S.K. Gupta.

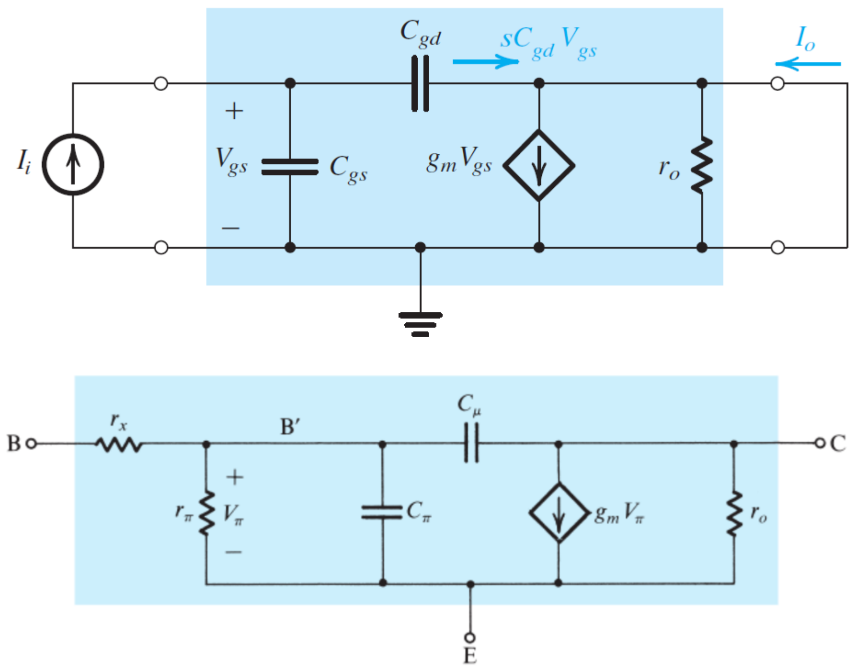

Figure 2: Internal or parasitic capacitances become important at high frequency for both the MOSFET transistor (top) and the BJT transistor (bottom) (Courtesy of Sedra and Smith).

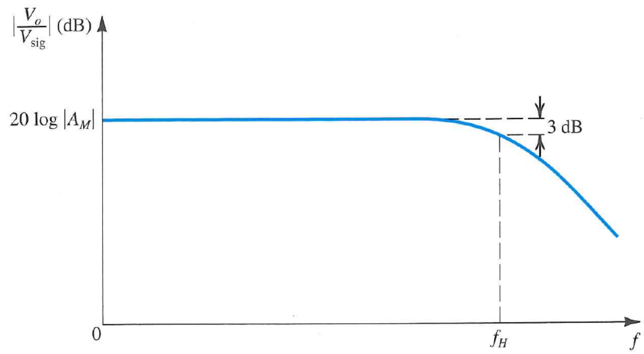

Figure 3: For integrated circuits (IC) where there are no coupling and bypass capacitor needed, e.g., in differential amplifiers, there is no low-frequency cut-off, and the circuits work down to very low frequency or DC. Then only internal capacitors affect the high-frequency response (Courtesy of Sedra and Smith).

As shown in Figure 1, the gain of the amplifier falls off at low frequency because the coupling capacitors and the bypass capacitors become open circuit or they have high impedances. Hence, they have non-negligible effect at lower frequencies as treating them as short-circuits is invalid.

At high frequencies, as shown in Figure 2, the simple hybrid-π model or T model is insufficient to describe the internal physics of the devices. Any two pieces of metal close to each other become a capacitance at higher frequencies. Internal capacitances need to be added, to account for capacitive coupling between these metal pieces. For instance, for the MOSFET, they give rise to a capacitance C_{gs} between the gate and the source, and a capacitance C_{gd} between the gate and the drain. Similar parasitic capacitances need to be added for the BJT, causing these devices to depart from their ideal model. Again, this causes the amplifier gain to fall off at higher frequency.

The operational frequency range or bandwidth of the amplifier is delineated by f_L, at the lower end of the operating frequency range, and f_H, at the upper end. These frequencies are defined as the frequencies when the gain drops below 3 dB[1] of the midband gain of the amplifier. For IC's as shown in Figure 3 when no coupling capacitors are used, e.g., for differential amplifier design, then f_L is zero. Hence, bandwidth BW is defined as

$$BW = f_H - f_L, \text{ (discrete-circuit amplifiers)}, \tag{1.1}$$

$$BW = f_H, \text{ (integrated-circuit amplifiers)} \tag{1.2}$$

A figure of merit for amplifier design is the **gain-bandwidth product**, defined as

$$GB = |A_M|BW \tag{1.3}$$

[1]This is also the half-power point.

where $|A_M|$ is the midband gain. The gain-bandwidth product is often a constant for many amplifiers. It can be shown to be a constant when the amplifier has only one pole for example. In other words, $|A_M|$ increases when BW decreases, and vice versa causing GB to remain constant.

2 Low-Frequency Response of Discrete-Circuit Common-Source and Common-Emitter Amplifiers

We will study the effects of the coupling and bypass capacitors on the low-frequency performance of the common-source (CS) and the common-emitter (CE) discrete-circuit amplifiers in this lecture.

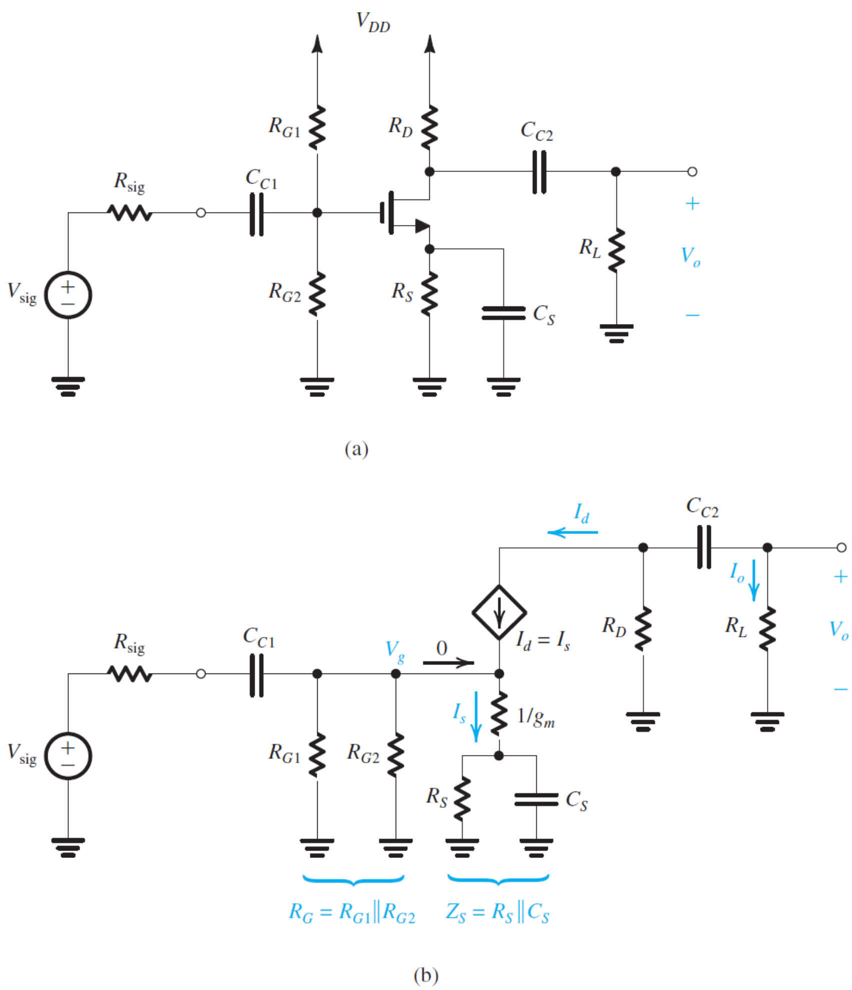

Figure 4: (a) A discrete-circuit common source (CS) amplifier. (b) The small-signal model with the coupling and bypass capacitor in place (Courtesy of Sedra and Smith).

2.1 The CS Amplifier

Figure 4(a) shows a CS (common source) amplifier with coupling capacitors and bypass capacitors. First, we replace the discrete-circuit amplifier with its small-signal model, but keeping the capacitors in the circuit as shown in Figure 4(b). To determine the amplifier gain V_o/V_{sig}, we write it as

$$\frac{V_o}{V_{\text{sig}}} = \left(\frac{V_g}{V_{\text{sig}}}\right)\left(\frac{I_d}{V_g}\right)\left(\frac{V_o}{I_d}\right) \tag{2.1}$$

1. Fist we invoke the voltage-divider formula which also works for complex impedances. To this end, writing $R_G = R_{G1} \parallel R_{G2}$, one gets

$$\frac{V_g}{V_{\text{sig}}} = \frac{R_G}{R_G + \frac{1}{sC_{C1}} + R_{\text{sig}}} = \frac{R_G}{R_G + R_{\text{sig}}} \left(\frac{s}{s + \frac{1}{C_{C1}(R_G + R_{\text{sig}})}} \right) \tag{2.2}$$

In the above, s is the Laplace transform variable, and is related to the frequency ω by $s = j\omega$. Thus, one can see that a pole exists at in the complex ω plane at

$$\omega_{P1} = j/[C_{C1}(R_{\text{sig}} + R_G)] \tag{2.3}$$

and there is a zero at $\omega = s = 0$. This is due to the DC blocking effect of the coupling capacitor.

2. In addition, the drain current I_d and the source current I_s are the same, namely,

$$I_d = I_s = \frac{V_g}{\frac{1}{g_m} + Z_S} = V_g \frac{g_m Y_S}{g_m + Y_S} \tag{2.4}$$

where Z_S is the source impedance, which is the analog of the source resistance in previous discrete model in Chapter 7, and Y_S is the source admittance. The above yields I_d/V_g, an important factor in (2.1), viz.,

$$\frac{I_d}{V_g} = \frac{g_m Y_S}{g_m + Y_S} \tag{2.5}$$

Moreover,

$$Y_S = \frac{1}{Z_S} = \frac{1}{R_S} + sC_S \tag{2.6}$$

Hence, by letting $s = j\omega$ again, the bypass capacitor introduces a pole in the complex ω-plane at

$$\omega_{P2} = j \frac{g_m + 1/R_S}{C_S} \tag{2.7}$$

A zero occurs at

$$s_Z = -\frac{1}{C_S R_S} \tag{2.8}$$

or at

$$\omega_Z = \frac{j}{C_S R_S} \tag{2.9}$$

Usually, $|\omega_{P2}| \gg |\omega_Z|$.

3. To complete the analysis, using the current-divider formula,

$$I_o = -I_d \frac{R_D}{R_D + \frac{1}{sC_{C2}} + R_L} = -I_d \frac{R_D}{R_D + R_L} \left(\frac{s}{s + \frac{1}{C_{C2}(R_D + R_L)}} \right) \tag{2.10}$$

Since $V_o = I_o R_L$, one gets, from (2.10), that

$$\frac{V_o}{I_d} = R_L \frac{I_o}{I_d} = -\frac{R_D R_L}{R_D + R_L} \frac{s}{s + \frac{1}{C_{C2}(R_D+R_L)}} \tag{2.11}$$

A pole is introduced at

$$\omega_{P3} = \frac{j}{C_{C2}(R_D + R_L)} \tag{2.12}$$

while a zero occurs at $s = 0$.

In aggregate,

$$\frac{V_o}{V_{\text{sig}}} = A_M \left(\frac{s}{s + \omega_{P1}}\right) \left(\frac{s + \omega_Z}{s + \omega_{P2}}\right) \left(\frac{s}{s + \omega_{P3}}\right) \tag{2.13}$$

where

$$A_M = -\frac{R_G}{R_G + R_{\text{sig}}} g_m \left(R_D \parallel R_L\right) \tag{2.14}$$

Here, A_M is the overall midband gain.

3 Bode Plots

To simplify the plotting of the frequency response, it is best to do it with Bode plots. They are log versus log or log-log plots or dB versus log-of-the-frequency plots. Then amplitudes are converted to dB with the formula that

$$\text{GAIN in dB} = 20 \log_{10}(\text{GAIN}) \tag{3.1}$$

where GAIN is the voltage gain. To this end, one takes the log of (2.13) to arrive at that

$$\log \left|\frac{V_o}{V_{\text{sig}}}\right| = \log |A_M| + \log \left|\left(\frac{s}{s + \omega_{P1}}\right)\right| + \log \left|\left(\frac{s + \omega_Z}{s + \omega_{P2}}\right)\right| + \log \left|\left(\frac{s}{s + \omega_{P3}}\right)\right| \tag{3.2}$$

One sees that product of three functions now becomes sum of three functions in the logarithm world. Therefore, if one can figure out the salient feature of the plot of each of the terms, then one can add them up to see the aggregate plot.

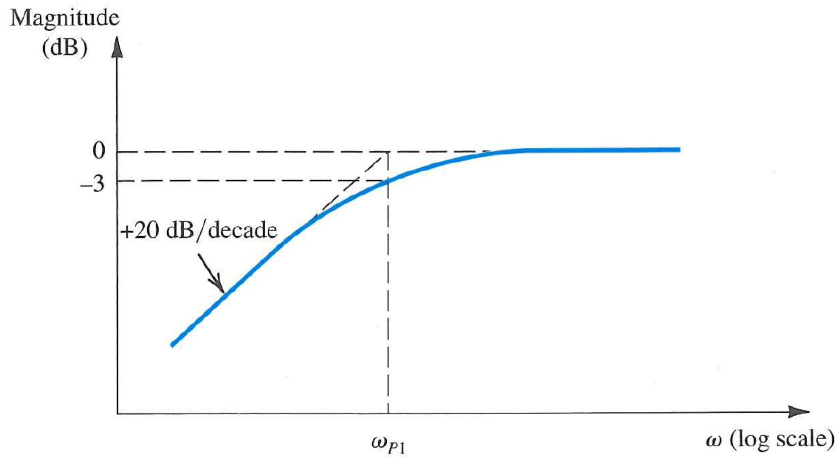

Figure 5: Bode plot of the function $f_1(s) = \frac{s}{s+\omega_{P1}}$ (Courtesy of Sedra and Smith).

For simplicity, one looks at the first function in (2.13), namely, the function $f_1(s) = \frac{s}{s+\omega_{P1}}$. One sees that at the high-frequency end, $f_1(s) \to 1$ as $s \to \infty$. Thus, on a log scale, it tends to zero as $s \to \infty$ or $\omega \to \infty$ since $s = j\omega$.

At the low-frequency end, however, $f_1(s) \sim s/\omega_{P1}$ as $s \to 0$. On a dB-log scale, we have in dB,

$$|f_1(s)|_{\text{dB}} \sim 20\log_{10}|\omega/\omega_{P1}| = 20\log_{10}|\omega| + 20\log_{10}|\omega_{P1}|,$$

$\omega \to 0$, after letting $s = j\omega$. The above means that it is a straight line which has a slope of 20 on a dB versus log-frequency plot as shown in Figure 5.

One can see that when $\omega = \omega_{P1}$, or $s = j\omega_{P1}$, then

$$|f_1(s = j\omega_{P1})| = \left|\frac{j\omega_{P1}}{j\omega_{P1} + \omega_{P1}}\right| = \left|\frac{j}{j+1}\right| = \frac{1}{|j+1|} = \frac{1}{\sqrt{2}} \tag{3.3}$$

The above implies that

$$|f_1(s = j\omega_{P1})|^2 = \frac{1}{2} \tag{3.4}$$

or that $\omega = \omega_{P1}$ is the half-power point or on the dB scale, the 3 dB point.

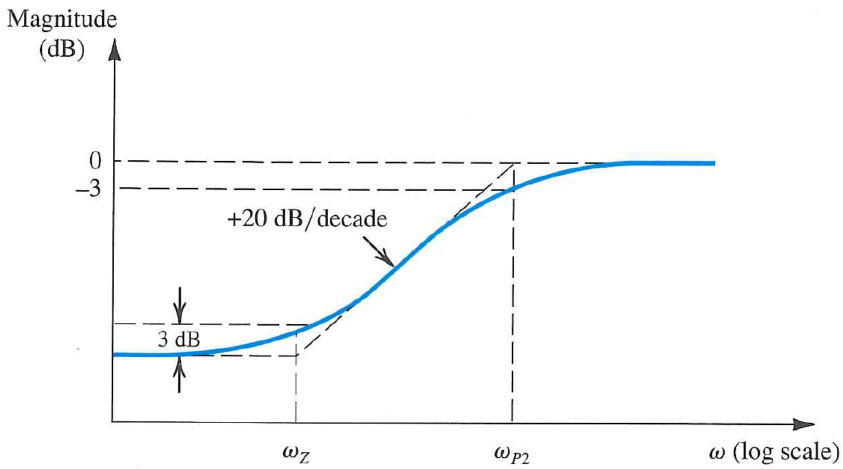

Figure 6: Bode plot of the function $f_2(s) = \frac{s+\omega_Z}{s+\omega_{P2}}$ (Courtesy of Sedra and Smith).

Now one looks at the second function in (2.13), namely, the function

$$f_2(s) = \frac{s + \omega_Z}{s + \omega_{P2}}.$$

By the same token, it can be shown that the Bode plot of the function $f_2(s)$ is as shown in Figure 6. If the two points ω_Z and ω_{P2} are far apart, then when $|s| \gg \omega_Z$,

$$f_2(s) \approx \frac{s}{s + \omega_{P2}}$$

and the Bode plot of $f_2(s)$ resembles that of $f_1(s)$ as shown in the right-hand side of Figure 6. When $|s| \ll \omega_{P2}$, then

$$f_2(s) \approx \frac{s + \omega_Z}{\omega_{P2}} = \frac{\omega_Z}{\omega_{P2}} \frac{s + \omega_Z}{\omega_Z}$$

The above implies that $f_2(s)$ grows with increasing ω as shown on the left-hand side of Figure 6. The 3 dB point, however, is at $\omega = \omega_Z$, and the slope of the straight line connecting the pole and the zero points is 20 dB/decade, as shown in Figure 6.

Given the characteristic of the Bode plot of a simple rational function, the aggregate Bode plot can be obtained by adding the individual plots together as shown in Figure 7. The Bode plots can be approximated by staggered piecewise constant and linear functions when we add them. This is possible when the poles and zeros are far apart. In this figure, f_L is an important number since it decides when the mid-band gain of the amplifier starts to deteriorate when the frequency drops.

It is to be noted that if we add two straight line plots, $y_1 = a_1 x + c_1$ and $y_2 = a_2 x + c_2$, then the resultant plot of $y = y_1 + y_2 = (a_1 + a_2)x + c_1 + c_2$. Hence, one just adds the slopes of the two straight line plots in Bode plots. So if two lines have a slope of 20 dB/decade each, the resultant aggregate slope is 40 dB/decade.

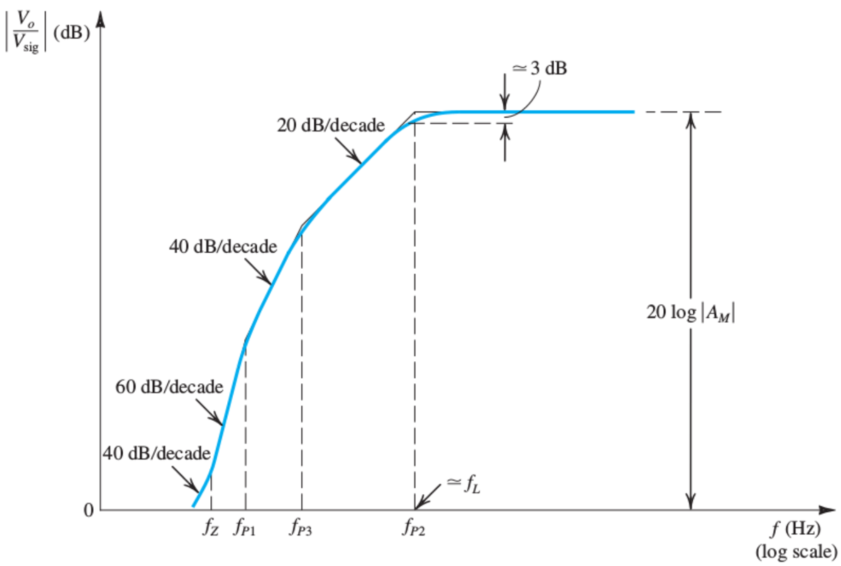

Figure 7: Additive sum of the Bode plot of a rational function with three poles and one zero away from the origin (Courtesy of Sedra and Smith).

4 Rational Functions, Poles and Zeros

In general, the gain function is a rational function of s. A rational function is a function which is the ratio of two polynomial functions, namely,

$$A(s) = \frac{a_0 + a_1 s + \ldots + a_M s^M}{b_0 + b_1 s + \ldots + b_N s^N} \tag{4.1}$$

By simple expansion,

$$a_N (s - s_1)(s - s_2) \ldots (s - s_N) = a_0 + a_1 s + \ldots + a_N s^N \tag{4.2}$$

which is a polynomial. The above also implies that an N-th order polynomial (right-hand side) can be written in a factorized form (left-hand side) in terms of factors of the roots. It also says that an N-th order polynomial has N zeros. In other words, a rational function of two polynomials, an M-th order polynomial in the numerator, and an N-th order polynomial in the denominator, will have

M zeros and N poles. More important, it turns out that many functions can be well approximated by rational functions. Therefore, it is possible to write a voltage gain function as

$$A(s) = A_0 F(s) = A_0 \frac{(s - s_1)(s - s_2)\ldots(s - s_M)}{(s - s_1)(s - s_2)\ldots(s - s_N)} \tag{4.3}$$

4.1 The Meanings of the Zeros and Poles

The deeper meanings of the zeros and poles can be contemplated by looking at (2.13) more carefully.

4.1.1 Meanings of Zeros

At the zeros of the right-hand side, it is seen that $V_o = 0$ when $V_{\text{sig}} \neq 0$. It implies that no signal can be sent from the input to the output. Something is blocking this signal! This happens when the coupling capacitors have infinite impedances, which happens at $s = 0$ for both C_{C1} and C_{C2}. In other words at $\omega = 0$, they are open circuited, and the gain of the amplifier is zero.

The other zero is obtained when $Z_S = \infty$ or when the source impedance in Figure 4 is infinite. This is precisely what happens when $\omega = \omega_Z$ in (2.13). At this frequency, $I_S = 0$, because of the infinite source impedance. When this is the case, the gain of the amplifier is again zero.

4.1.2 Meanings of Poles

To understand the poles, note that at these frequencies, the right-hand side of (2.13) becomes infinite. That means that $V_o \neq 0$ even if $V_{\text{sig}} \to 0$. Or non-zero signal exists in the circuit even if $V_{\text{sig}} = 0$. This is possible if the capacitors are charged, and their charges are relaxing or discharging via the resistors connected to them even when $V_{\text{sig}} = 0$. These are called the **natural solutions** or the **resonant solutions** of a system. In mathematics, these are the **homogeneous solutions** of a system of equations.

An example of a natural resonant system is the tank circuit as shown in Figure 8. One can connect a voltage source V_{sig} to the loop of this tank circuit. The voltage source will charge the capacitor in the loop. The output can be taken to be the voltage drop across either the inductor or the capacitor. When one switches off the voltage source replacing it with a short circuit, the capacitor discharges, giving rise to non-zero voltage and current in the loop. Thus, the output will be nonzero even when the input is zero because of the resonant solution! The resonant solutions form the poles of the system and on the complex s plane they are at

$$s = \pm \frac{j}{\sqrt{LC}} \tag{4.4}$$

or they are along the imaginary axis. A Laplace component of a signal has time dependence of the form $\exp(st)$, and hence, the natural solution will have time dependence of the form $\exp(st) = \exp(\pm jt/(LC))$. They are sinusoidal signals.

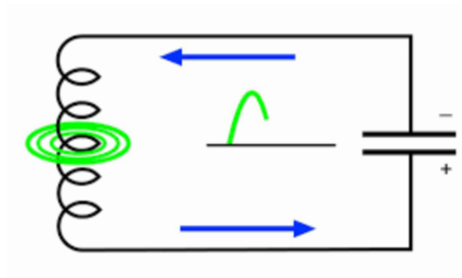

Figure 8: An LC tank circuit resonating at its natural frequency via its natural solution.

One can repeat this gedanken experiment (thought experiment) with an RC circuit as shown in Figure 9, and the same meaning will prevail but instead, the pole is at

$$s = -\frac{1}{RC} \tag{4.5}$$

namely, on the negative real axis. A Laplace component of a signal has time dependence of the form $\exp(st)$, and hence, the natural solution will have time dependence of the form $\exp(st) = \exp(-t/(RC))$ which are exponentially decaying.

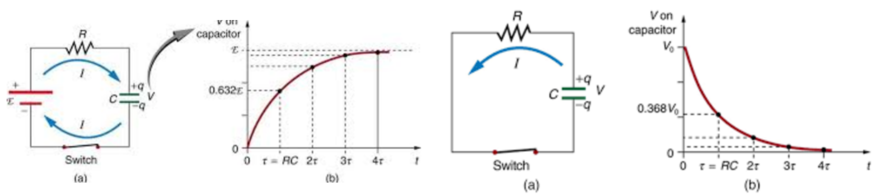

Figure 9: A capacitor charging and discharging. When it is discharging, it does so via its natural solution.

The charging and discharging of the capacitor is shown in Figure 9. When the capacitor is discharging, the input signal is zero, and it is discharging via the natural solution of the circuit.

5 Rigorous Method for Finding Time Constants

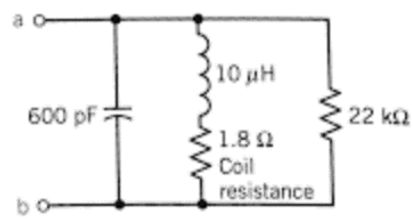

Figure 10: A capacitor charging and discharging via a general circuit.

When a capacitor is connected to a complicated circuit, it can discharge into this complicated circuit, or relax its charge into this complicated circuit. Hence, finding its relaxation time can be rather daunting. Figure 10 shows a capacitor connected to a somewhat general circuit. The general circuit can be a lot more complicated that what has been shown!

Imaging now that the capacitor and the rest of the circuit form a loop from which one can write a KVL. Assuming that a voltage V_{sig} is inserted into this loop, and a current I is flowing. Then the KVL equation is

$$I\left[1/(sC) + Z(s)\right] = V_{\mathrm{sig}} \tag{5.1}$$

One can take I as the output of this system, and that V_{sig} as the input. The question is that when $V_{\mathrm{sig}} = 0$, or the input turned off, can I, the output be nonzero. Yes, it can be if

$$1/(sC) + Z(s) = 0 \tag{5.2}$$

From this equation, one can solve for the frequency s at which a natural solution can exist. In general, $Z(s)$ can be a very complicated function, and solving (5.2) is non-trivial, unless the circuit is very simple. Here, $Z(s)$, for example, can be found by the test current method, making it a very complicated function of s.

5.1 The Relaxation Times—Simple Case

It turns out that the relaxation times of the capacitors in Figure 4 can be found rather easily, because the relaxation of the charge of the capacitors happens independently of each other. The discharge from one capacitor is not received by another capacitor, due the the configuration of the circuit.

As aforementioned, when a capacitor C is connected in series to a resistor R, the charge in the capacitor relaxes via the time constant $\tau = RC$, or its RC time constant. In other words, charges in the capacitor decay according to $\exp(-t/\tau) = \exp(-t/(RC))$.

The relaxation solutions are depicted in Figure 11. There are three such solutions for this case, each associated with the relaxation of each capacitor. Hence, in this circuit that is being studied, the number of poles is equal to the number of capacitors in the circuit, because each capacitor has its own relaxation time.

If only C_{C1} is charged and the other capacitors are uncharged, then the only way for the capacitor C_{C1} to discharge is via the series connected resistors R_{sig} and R_G. Hence the relaxation time is as such, and that is the location of the pole ω_{P1} as expressed in (2.3). This discharging circuit is shown in Figure 11(a). The capacitor can be viewed to be connected in series with the resistors.

If only C_S is charged, then $V_g = 0$ always since no current can flow through the node at V_g, and $V_{\mathrm{sig}} = 0$. So the other end of the $1/g_m$ resistor is connected to a virtual ground.

Then C_S is discharging via the circuit shown in Figure 11(b), giving rise to the second pole ω_{P2} as given (2.7).

If only C_{C2} is charged, then it can only discharge via the circuit shown in Figure 11(c), giving rise to the third pole ω_{P3} shown in (2.12). Again, the capacitor is discharging via the resistors that are connected in series to it.

5.2 Selecting the Bypass and Coupling Capacitors

The relaxation time of the circuit is determined by the RC time constants. Since usually, C_S sees the smallest resistor which is $(1/g_m \parallel R_S)$, and hence it has the smallest time constant or the highest frequency pole, it usually determines f_L in the discrete circuit. Then the coupling capacitors C_{C1} and C_{C2} are picked to make the other poles far enough so that the simple analysis previously discussed can be used.

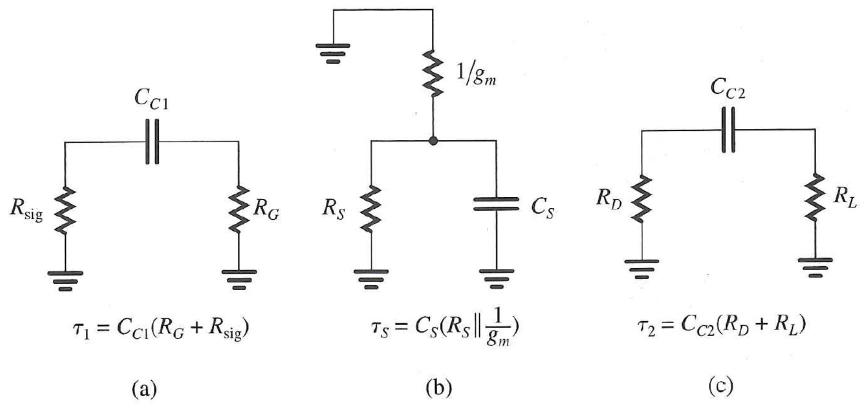

Figure 11: The three circuits that determine the relaxation time associated with the three capacitors for the MOSFET CS amplifier of Figure 4 (Courtesy of Sedra and Smith).

6 The Method of Short-Circuit Time Constants

For a general circuit, the relaxation times of the capacitors are difficult to find. However, an approximate method can be used to unscramble the coupling between the capacitors. One is usually interested in determining f_L, which corresponds to the highest-frequency pole of the system. Then one can go through the following procedure to find the highest-frequency pole approximately.

1. One makes the assumption that at this highest frequency f_L, all the capacitors are short-circuited except for the capacitor of interest.
2. This approximation definitely removes the inter-capacitor coupling.
3. It is seen that at the highest frequency pole, the approximation is a good one.
4. One then steps through the capacitors and find their respective relaxation frequencies that are associated with them with this approximation.
5. Even though at the other lower frequency poles, the approximation is not a good one, in practice, f_L is calculated using

$$\omega_L \approx \sum_{i=1}^{n} \frac{1}{C_i R_i} \tag{6.1}$$

6. The above sum will be dominated by the smallest relaxation time or the highest frequency pole, which is best approximated by the short-circuit time-constant calculation.

The above sum gives a rough estimate of the relaxation frequency of the highest frequency pole.

6.1 The CE Amplifier

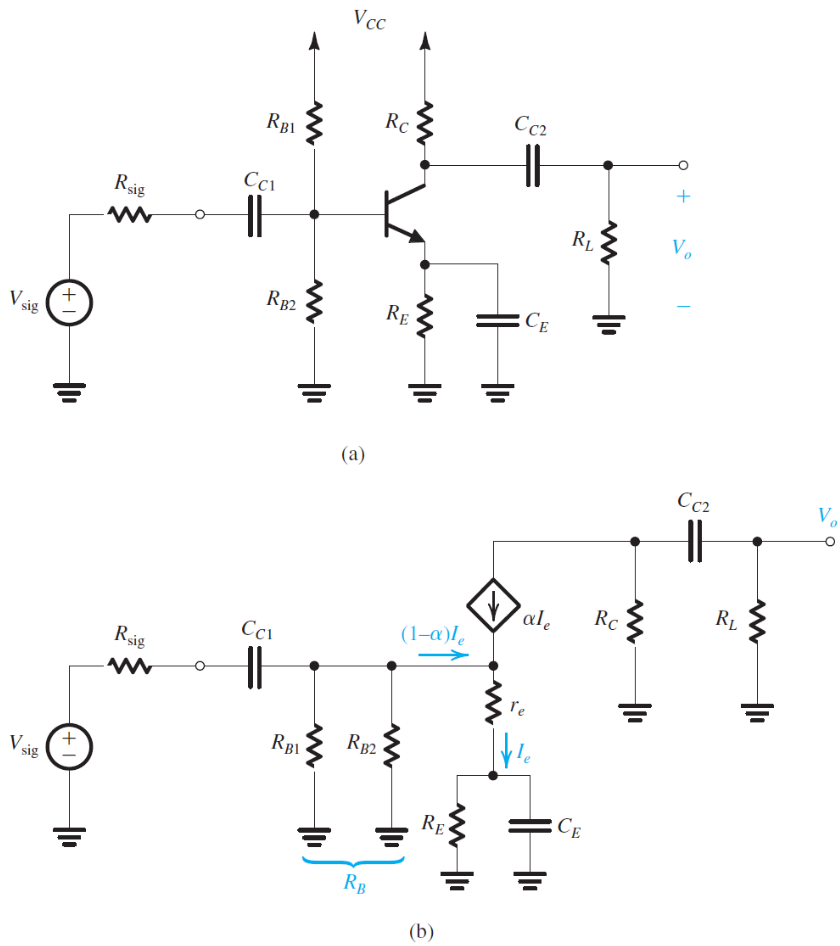

Figure 12: (a) A discrete-circuit common-emitter amplifier. (b) The corresponding small-signal equivalent circuit (Courtesy of Sedra and Smith).

The way to analyze MOSFET discrete-circuit amplifier at the low frequency end can be used to analyze a BJT discrete-circuit amplifier. This circuit is shown in Figure 12. The analysis is very similar to before, except that there is a difference when one finds the relaxation times of the capacitors. Here, the relaxation circuits cannot be easily divided into three independent ones, because of the inter-coupling effect between the capacitors C_{C1} and C_E. The base current is related to the emitter current, or the discharge of capacitor C_{C1} is received by capacitor C_{CE}.

To remove the coupling, the short-circuit time-constant method is used to find their time constants. These relaxation circuits are shown in Figure 13. Their relaxation times can be easily found, and then used in (6.1) to estimate the approximate ω_L or f_L.

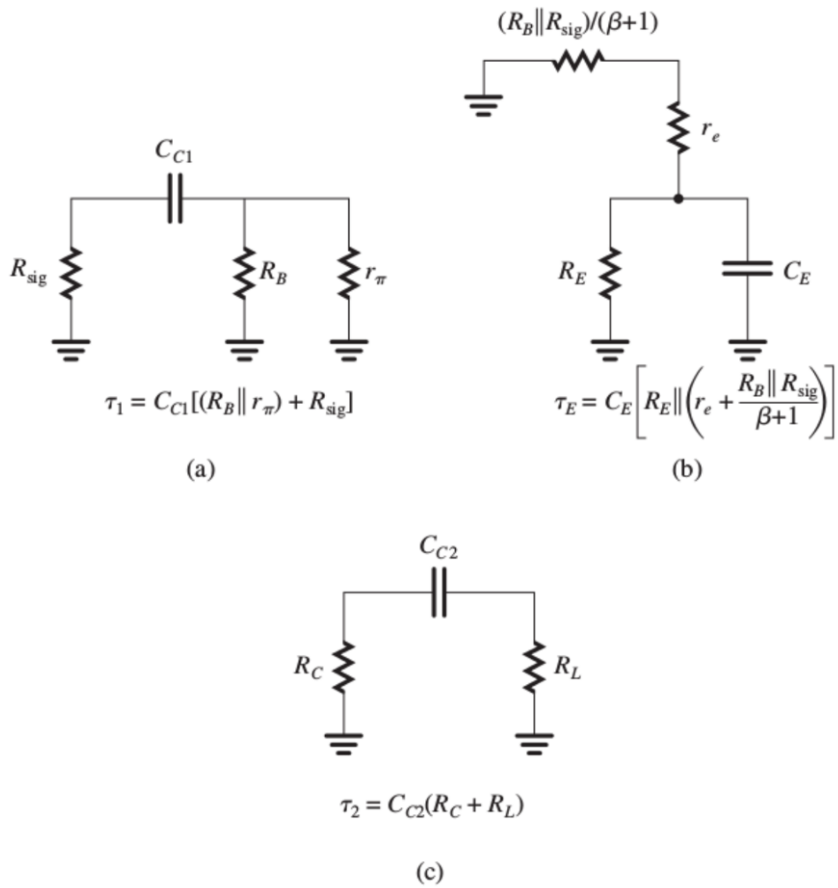

Figure 13: Circuits for determining the short-circuit time constants for the CE amplifier shown in Figure 12 in this case (Courtesy of Sedra and Smith).