ECE 255, Terminal Resistances and Miscellaneous Topics

12 April 2018

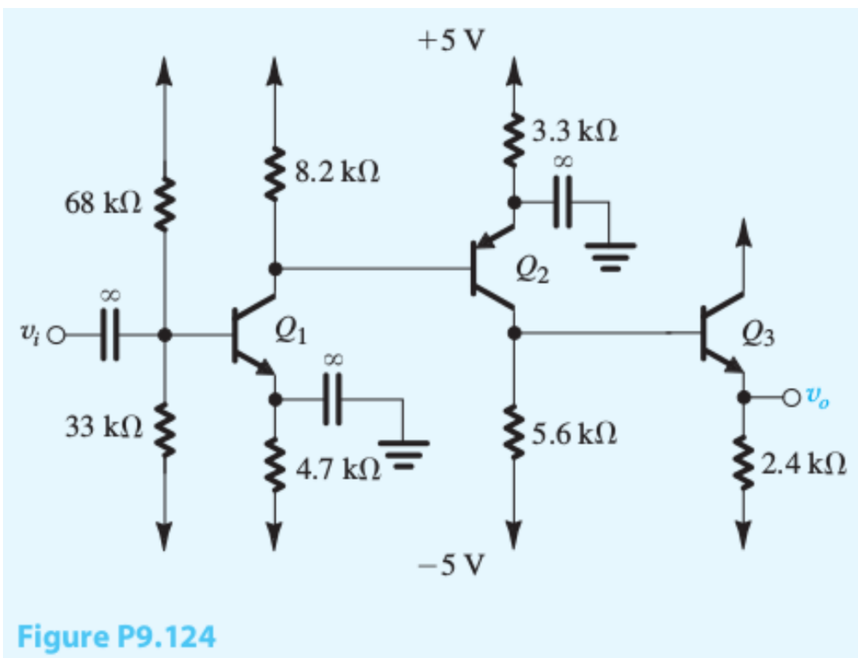

Figure 1: Example of a multi-stage amplifier by cascading a number of discrete circuit amplifiers in Problem 9.124 of the textbook (Courtesy of Jaeger and Blalock).

In this lecture, we look at some miscellaneous topics such as finding the terminal resistance of a discrete circuit amplifier. This knowledge is useful when we want to cascade a number of discrete circuit amplifiers into to obtain higher total gain as seen in Figure 1. In order to cascade them, the input resistance

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of an amplifier becomes the load of the output of the previous stage. In the small-signal linear analysis, it simplifies the circuit if the previous stages of the amplifier can be replaced with a Thévenin equivalence or a Norton equivalence. In this case, it is important to know the output resistance of the amplifier which becomes the Thévenin resistor of the equivalent circuit. Hence, it is useful to start with finding the small-signal terminal resistance of a discrete circuit amplifier.

1 Terminal Resistances

1.1 Collector Resistance

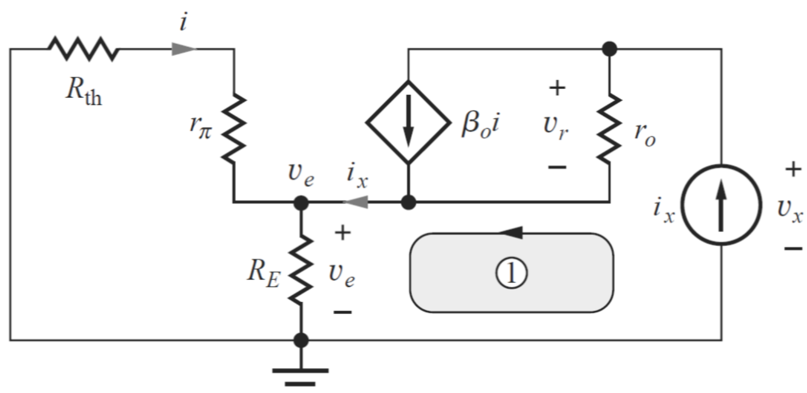

Figure 2: Collector resistance with r_o included (Courtesy of Jaeger and Blalock).

The collector resistance of an amplifier can be used to arrive at the Thévenin equivalence of the small-signal linear amplifier. To obtain the Thévenin resistor, the voltage source is set to zero, and a small test current i_x is injected into this linear circuit. This test current will cause a voltage v_x to develop across the test terminal, and from this, the Thévenin resistor can be found.

First, KVL is applied to Loop 1 as shown in Figure 2, to arrive at

$$v_x = v_r + v_e = (i_x - \beta_0 i)r_o + v_e \tag{1.1}$$

where a base current i has been assumed to flow into the base. This base current will induce a controlled-current source in the hybrid-π model for the transistor. The voltage v_e can also be obtained by Ohm's law, namely,

$$v_e = i_x \left[(R_{\text{th}} + r_\pi) \parallel R_E\right] = i_x \frac{(R_{\text{th}} + r_\pi)R_E}{R_{\text{th}} + r_\pi + R_E} \tag{1.2}$$

The current i can be found by the current-divider formula applied at the node v_e, yielding

$$i = -i_x \frac{R_E}{R_{\text{th}} + r_\pi + R_E} \tag{1.3}$$

Note that i is negative to i_x. Finally, from (1.1)

$$v_x = \left(i_x + i_x \frac{\beta_0 R_E}{R_E + r_\pi + R_{\text{th}}} \right) r_o + i_x \left[R_E \parallel (r_\pi + R_{\text{th}}) \right] \tag{1.4}$$

where all terms on the right-hand side have been expressed as a linear function of i_x. Then using

$$R_{iC} = \frac{v_x}{i_x} \tag{1.5}$$

one has

$$R_{iC} = r_o \left(1 + \frac{\beta_0 R_E}{R_{\text{th}} + r_\pi + R_E} \right) + (R_{\text{th}} + r_\pi) \parallel R_E \approx r_o \left(1 + \frac{\beta_0 R_E}{R_{\text{th}} + r_\pi + R_E} \right) \tag{1.6}$$

where it has been assumed that

$$r_o \gg R_E.$$

This inequality follows from that the Early effect gives rise to a large r_o, or that a transistor mimics almost an ideal current source.

Moreover, the following approximation and equality are to be made on the above formula, namely,

$$(r_\pi + R_E) \gg R_{\text{th}}$$

One assumes that in the design of the previous stage amplifier, R_{th} is small so as to enable maximum-power transfer to a reasonable load that justifies the above inequality. Hence,

$$R_{iC} \approx r_o \left(1 + \frac{\beta_0 R_E}{r_\pi + R_E} \right) = r_o \left(1 + \frac{\beta_0}{r_\pi} \frac{r_\pi R_E}{r_\pi + R_E} \right) \tag{1.7}$$

In the above, $\frac{\beta_0}{r_\pi} = g_m$, then

$$R_{iC} \approx r_o \left[1 + g_m (R_E \parallel r_\pi) \right] \tag{1.8}$$

Notice that R_{iC} is amplified by coupling between R_E and r_π. This is because the induced base current i is negative when i_x is positive, making the output resistance at the collector look even bigger than r_o.

Similar analysis can be applied to the MOSFET where $r_\pi = \infty$, replacing R_E by R_S, and the drain resistance is then

$$R_{iD} = r_o (1 + g_m R_S) \tag{1.9}$$

1.2 Base Resistance

The base resistance is the resistance of the base terminal looking from the base into the transistor. Using the resistance-reflection formula, this gives

$$R_{iB} = r_\pi + (\beta_0 + 1)R_E \tag{1.10}$$

The above can be understood that for every unit of current that flows in the base terminal, there is a $(\beta_0 + 1)$ unit of current that flows in the emitter terminal. Applying KVL to the base terminal and the ground of the emitter, one can arrive at the above formula. Noticing that $g_m r_\pi = \beta_0$, the above can be rewritten approximately as

$$R_{iB} \approx r_\pi(1 + g_m R_E) \tag{1.11}$$

The second form is easier to remember by some. Similar analysis can be applied to a MOSFET gate resistance, or letting $\beta_0 = \infty$, one gets that

$$R_{iG} = \infty \tag{1.12}$$

1.3 Emitter Resistance

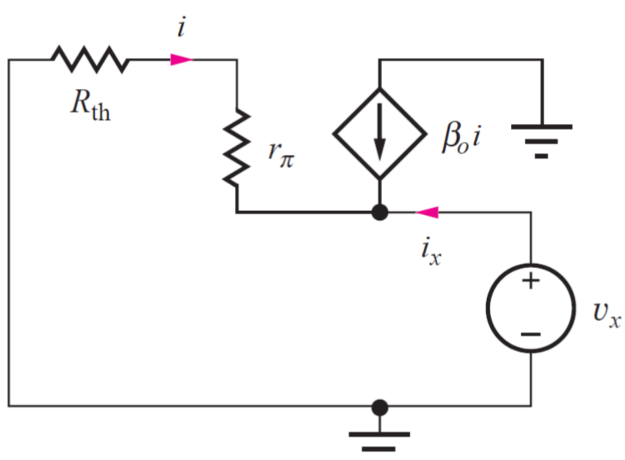

Figure 3: Emitter resistance which serves as the output resistance for the common-collector amplifier (Courtesy of Jaeger and Blalock).

The emitter resistance serves as the output resistance of the the common-collector design. It can be found using the test-current method as shown in Figure 3. Or it can be argued that for every unit of current that flows in the base, there is a $(\beta_0 + 1)$ unit of current that flows in the emitter. Hence,

$$R_{iE} = \frac{r_\pi + R_{\text{th}}}{\beta_0 + 1} \approx \frac{1}{g_m} + \frac{R_{\text{th}}}{\beta_0} \tag{1.13}$$

This is like the resistance inverse reflection formula.

Apply similar analysis to a MOSFET, or letting $\beta_0 = \infty$ yields

$$R_{iS} = \frac{1}{g_m} \tag{1.14}$$

2 Resistance Inverse Reflection Formula

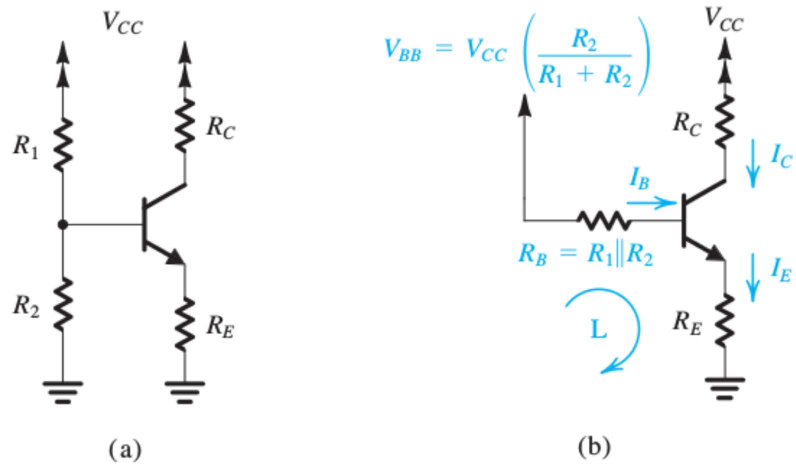

Figure 4: DC biasing of BJT using using a single power supply: (a) circuit; (b) circuit with the voltage divider supplying the base voltage replaced by its Thévenin equivalence (Courtesy of Sedra and Smith).

The fact that for every unit of current that flows in the base, there is a $(\beta_0 + 1)$ unit of current that flows in the emitter gives rise to the resistance-reflection formula. This fact remains true at DC and can be used in DC analysis as well.

Referring to Figure 4, one can write the KVL around loop L to get

$$I_B\left[(\beta_0 + 1)R_E + R_B\right] = V_{BB} - V_{BE} \tag{2.1}$$

where one notices that $I_E = (\beta_0 + 1)I_B$. The above can be written as

$$I_B(\beta_0 + 1) = \frac{V_{BB} - V_{BE}}{R_E + R_B/(\beta_0 + 1)} \tag{2.2}$$

The left-hand side is just I_E giving

$$I_E = \frac{V_{BB} - V_{BE}}{R_E + R_B/(\beta_0 + 1)} \tag{2.3}$$

One can also view the base resistance R_B appears $(\beta_0 + 1)$ times smaller because the base current is smaller by the same factor.

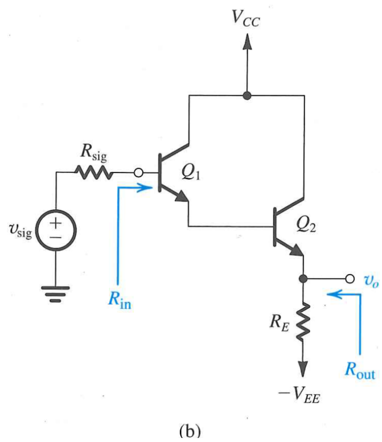

Figure 5: The Darlington pair or configuration for the analysis in Exercise 8.30 of S&S (Courtesy of Sedra and Smith).

3 Darlington Pair

In order to understand the Darlington pair, which is a frequently used multi-stage amplifier, it will be prudent to do Exercise 8.30 from Sedra and Smith. You are asked to show that (see Figure 1):

1. $R_{\rm in} = (\beta_1 + 1)\left[r_{e1} + (\beta_2 + 1)(r_{e2} + R_E)\right]$
2. $R_{\rm out} = R_E \parallel \left[r_{e2} + \frac{r_{e1}+[R_{\rm sig}/(\beta_1+1)]}{\beta_2+1}\right]$
3. $\frac{v_o}{v_{\rm sig}} = \frac{R_E}{R_E+r_{e2}+[r_{e1}+R_{\rm sig}/(\beta_1+1)]/(\beta+2+1)}$

Answers:

1. The answer is to use the resistance-reflection rule twice. If Q_1 has an emitter resistance of R_{E1}, then using formula (7.107) of S&S, one gets

   $$R_{\rm in} = (\beta_1 + 1)(r_{e1} + R_{E1})$$

   But R_{E1} is the input resistance looking into the base of Q_2. Using the reflection formula again to get R_{E1} from R_E, or

   $$R_{E1} = (\beta_2 + 1)(r_{e2} + R_E)$$

   one gets the above formula in Part 1.

   Note: Since $(\beta_1+1)r_{e1} = r_{\pi 1}$ which can be gotten by comparing the input impedance of the hybrid-π model and the T model, the above reflection formula can also be written as

   $$R_{\rm in} = (\beta_1 + 1)r_{e1}(1 + R_{E1}/r_{e1}) = r_{\pi 1}(1 + R_{E1}/r_{e1})$$

Since $g_m = \alpha / r_{e1} \approx 1/r_{e1}$, the above formula is often approximated as

$$R_{\text{in}} \approx r_{\pi 1}(1 + g_{m1} R_{E1})$$

The above formula is often used in some textbook like Jaeger and Blalock and it is easier to remember by some.

Again, using the approximate reflection formula, one can rewrite R_{E1} as

$$R_{E1} = r_{\pi 2}(1 + g_{m2} R_E)$$

2. First, one needs to find the Thévenin resistor, R_{Th} of the Thévenin equivalence of the source to the left of R_E. Then $R_{\text{out}} = R_E \parallel R_{\text{Th}}$. To find R_{Th}, one shorts out the voltage source, and uses the inverse reflection rule two times to get

$$R_{\text{Thev}} = \left[r_{e2} + \frac{r_{e1} + [R_{\text{sig}}/(\beta_1 + 1)]}{\beta_2 + 1} \right]$$

The inverse reflection formula is based on that, for a BJT, every unit of current that flows in the base, there is $\beta + 1$ unit of current that flows in the emitter to arrive at the formula in Part 2.

Using the fact that $g_m = \alpha / r_e \approx 1/r_e$, and that $\beta + 1 \approx \beta$, the above formula can also be approximated as

$$R_{\text{out}} \approx R_E \parallel \left[1/g_{m2} + \frac{1/g_{m1} + [R_{\text{sig}}/(\beta_1)]}{\beta_2} \right]$$

as is found in some textbooks.

3. For this problem, it is necessary to find the Thévenin resistor looking to the left of R_E from the emitter of Q_2. This resistor is obtained by short-circuiting the source, and looking at the impedance to the left of R_E. This Thévenin resistor is similar to that in Part 2. Next, the equivalent Thévenin voltage source needs to be found. This can be done by open-circuiting the load by letting $R_E \to \infty$. It is seen that $R_{\text{in}} \to \infty$ in this case. With the small voltage drops between the base and the emitters, it is seen that $v_{\text{Th}} \approx v_{\text{sig}}$. Hence formula in Part 3 above can be derived using the Thévenin equivalent circuit and the voltage divider formula.