ECE 255

5 October 2017

In this lecture, we will introduce small-signal analysis, operation, and models from Section 7.2 of Sedra and Smith. Unlike the text book, we will follow the historical development and start with the BJT case.

In the small-signal analysis, one assumes that the device is biased at a DC operating point, and then, a small signal is super-imposed on the DC biasing point.

1 The DC Bias Point and Linearization–The BJT Case

Before one starts, it will be prudent to refresh our memory on the salient features of the BJT from Table 6.2 of Sedra and Smith.

Table 6.2 Summary of the BJT Current–Voltage Relationships in the Active Mode

$$i_C = I_S e^{v_{BE}/V_T}$$

$$i_B = \frac{i_C}{\beta} = \left(\frac{I_S}{\beta}\right) e^{v_{BE}/V_T}$$

$$i_E = \frac{i_C}{\alpha} = \left(\frac{I_S}{\alpha}\right) e^{v_{BE}/V_T}$$

Note: For the *pnp* transistor, replace v_{BE} with v_{EB}.

$i_C = \alpha i_E$	$i_B = (1-\alpha) i_E = \dfrac{i_E}{\beta+1}$
$i_C = \beta i_B$	$i_E = (\beta+1) i_B$
$\beta = \dfrac{\alpha}{1-\alpha}$	$\alpha = \dfrac{\beta}{\beta+1}$

$$V_T = \text{thermal voltage} = \frac{kT}{q} \simeq 25\text{ mV at room temperature}$$

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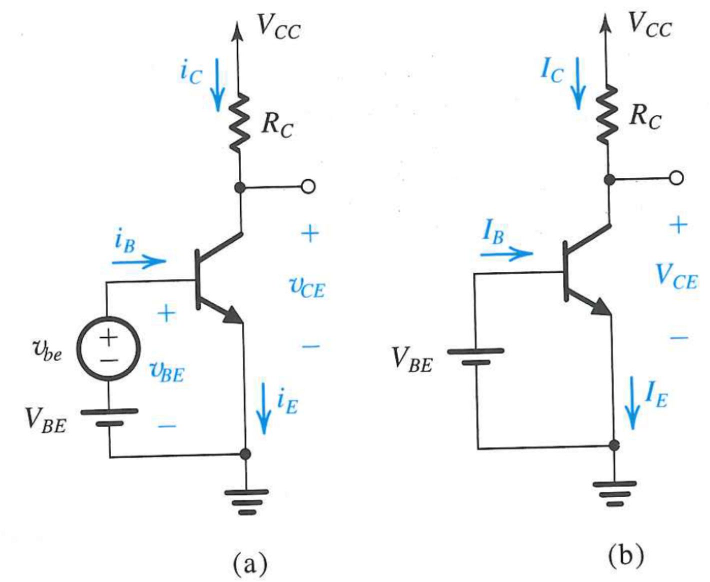


Figure 1: Circuit diagram of a transistor amplifier (a) with a small time varying signal superimposed on top of a DC voltage bias source, (b) with the small signal turned off (Courtesy of Sedra and Smith).

Figure 1 illustrates an *npn* BJT operating as an amplifier. It is being biased with a DC voltage, and a small signal is superimposed on top of the DC voltage.

Before proceeding further, one is also reminded of the *i*-*v* characteristics of a BJT.

$$i_C = I_S e^{v_{BE}/V_T} \tag{1.1}$$

$$i_E = i_C/\alpha \tag{1.2}$$

$$i_B = i_C/\beta \tag{1.3}$$

$$v_{CE} = v_{CC} - i_C R_C \tag{1.4}$$

In the above

$$i_C = I_C + i_c, \quad v_{BE} = V_{BE} + v_{be}, \quad \text{etc} \tag{1.5}$$

In the above, an uppercase variable implies a DC value, a lowercase variable with uppercase subscript implies the total value. But a lowercase variable implies a small value which can be time varying.

Hence, the left-hand side of an equation in (1.5) above is the total value, while the first term on the right-hand side is the DC value, and the second term on the right-hand side is the small time varying signal or value. Similar notation will be used for the rest of the analysis here.

As a consequence

$$i_C = I_S e^{v_{BE}/V_T} = I_S e^{(V_{BE}+v_{be})/V_T} = I_S e^{V_{BE}/V_T} e^{v_{be}/V_T} \tag{1.6}$$

By defining the DC quantity $I_C = I_S e^{V_{BE}/V_T}$, the above can be rewritten as

$$i_C = I_C e^{v_{be}/V_T} \approx I_C \left(1 + \frac{v_{be}}{V_T}\right) = I_C + \frac{I_C}{V_T} v_{be} \tag{1.7}$$

where it is assumed that $v_{be} \ll V_T$. The above is the essence of Taylor series expansion, as implicitly, it has been used in the aforementioned approximation.

From the above, one gathers that the small signal collector current is

$$i_c \approx \frac{I_C}{V_T} v_{be} = g_m v_{be} \tag{1.8}$$

where $g_m = \frac{I_C}{V_T}$ is the **transconductance**. The above is the essence of linearization: A nonlinear relation between i_C and v_{BE} in (1.6) is now reduced to a linear relation between the small signals, i_c and v_{be}.

Due to the exponential relation of the *i*-*v* characteristics for BJT, as oppose to the algebraic relation in MOSFET, the transconductance of a BJT is much larger than that of a MOSFET. It is noted that a high transconductance is good, as a small v_{be} gives rise to a larger i_c. The transconductance can be increased by increasing I_C.

The transconductance, is related to the incremental conductance, and hence is also given by

$$g_m = \left. \frac{\partial i_C}{\partial v_{BE}} \right|_{i_C = I_C} \tag{1.9}$$

This relationship is also shown graphically in Figure 2.

2 Small Signal Analysis with Base Current

By the same token, one can perform a small signal analysis of the base current, using the approximation for i_C in (1.7), to arrive at

$$i_B = I_B + i_b = \frac{i_C}{\beta} \approx \frac{I_C}{\beta} + \frac{1}{\beta} \frac{I_C}{V_T} v_{be} \tag{2.1}$$

where $I_B = I_C/\beta$ and consequently,

$$i_b \approx \frac{1}{\beta} \frac{I_C}{V_T} v_{be} = \frac{I_B}{V_T} v_{be} = \frac{g_m}{\beta} v_{be} \tag{2.2}$$

Similarly, one can define

$$r_\pi = \frac{v_{be}}{i_b} = \frac{\beta}{g_m} = \frac{V_T}{I_B} \tag{2.3}$$

In the above, r_π is the incremental resistance seen by a small signal driving the small base current from the base to the emitter.

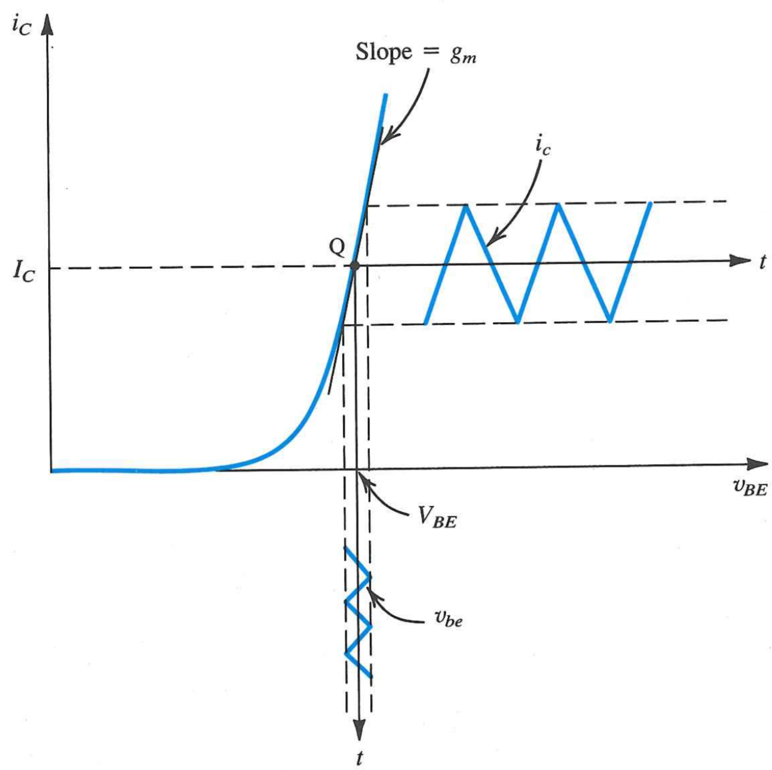


Figure 2: Graphical depiction of the small signal analysis for BJT (Courtesy of Sedra and Smith).

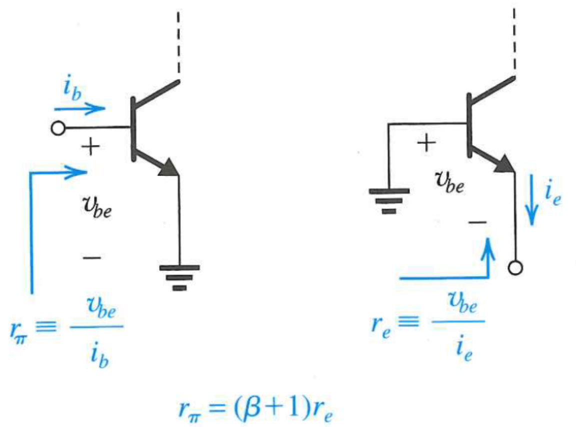


Figure 3: The definition of r_π and r_e (Courtesy of Sedra and Smith).

3 Small Signal Analysis of the Emitter Current

Applying small signal analysis to the emitter current, one gets

$$i_E = \frac{i_C}{\alpha} = \frac{I_C}{\alpha} + \frac{i_c}{\alpha} = I_E + i_e \tag{3.1}$$

where

$$i_e = \frac{i_C}{\alpha} \approx \frac{I_C}{\alpha V_T} v_{be} = \frac{I_E}{V_T} v_{be} \tag{3.2}$$

One can define **emitter resistance** from the above to be given by

$$r_e = \frac{v_{be}}{i_e} = \frac{V_T}{I_E} = \frac{\alpha}{g_m} \approx \frac{1}{g_m}, \quad \text{for large } \beta. \tag{3.3}$$

It is to be reminded that the formula $g_m = I_C/V_T$ has been used above. By noticing that $v_{be} = i_b r_\pi = i_e r_e$, one deduces that

$$r_\pi = (i_e/i_b) r_e = (\beta + 1) r_e \tag{3.4}$$

Figure 3 illustrates the definition of r_π and r_e.

4 Voltage Gain

The total collector voltage v_{CE} is

$$v_{CE} = V_{CC} - i_C R_C = V_{CC} - (I_C + i_c) R_C = (V_{CC} - I_C R_C) - i_c R_C = V_{CE} - i_c R_C \tag{4.1}$$

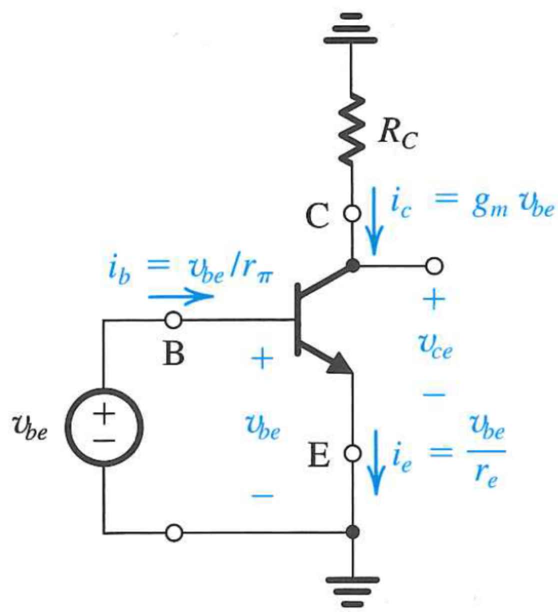


Figure 4: The small signal circuit of the original amplifier circuit (Courtesy of Sedra and Smith).

Thus the small collector voltage is

$$v_{ce} = -i_c R_C \approx -g_m v_{be} R_C = (-g_m R_C) v_{be} = A_v v_{be} \tag{4.2}$$

where

$$A_v = \frac{v_{ce}}{v_{be}} = -g_m R_C \tag{4.3}$$

Finally, the voltage gain can be expressed as

$$A_v = -\frac{I_C R_C}{V_T} \tag{4.4}$$

The negative sign comes about because if v_{BE} increases, i_C increases, and the voltage drop across R_C increases, giving rise to a decrease in the voltage drop v_{CE}. Hence, the small signals are of opposite polarity.

5 Hybrid-π Model

The name comes about because the circuit model looks like a Π symbol, and that both voltage and current are used in defining the model. The hybrid-π

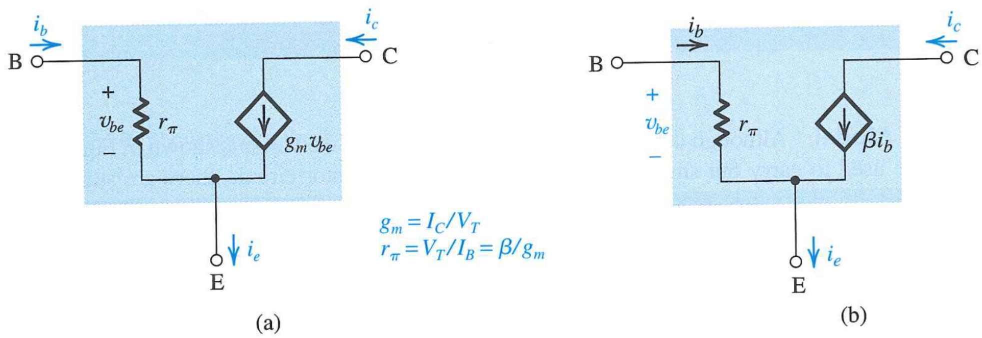


Figure 5: Two different version of the hybrid-π model for small signals (a) voltage controlled current source (VCCS), and (b) current controlled current source (CCCS) (Courtesy of Sedra and Smith).

model does correctly predict the collector current $i_c = g_m v_{be}$ and $i_b = v_{be}/r_\pi$. But it also correctly predicts the correct value for i_e, namely

$$i_e = \frac{v_{be}}{r_\pi} + g_m v_{be} = \frac{v_{be}}{r_\pi}(1 + g_m r_\pi) = \frac{v_{be}}{r_\pi}(1+\beta) = v_{be} \Bigg/ \left(\frac{r_\pi}{1+\beta}\right) = v_{be}/r_e \tag{5.1}$$

where (2.3) and (3.4) have been used in the above.

Alternatively, one can relate the voltage v_{be} to the current i_b, and use the controlled current source model instead. To this end, one writes

$$g_m v_{be} = g_m (i_b r_\pi) = (g_m r_\pi) i_b = \beta i_b \tag{5.2}$$

These models are shown in Figure 5(a) and (b).

The Early effect can be accounted for by adding a resistor $r_o = V_A/I_C$ where V_A is the negative intercept of the Early effect, and I_C is the collector current if the Early effect is not there. These equivalent circuit model for small signal is shown in Figure 6.

It is seen that in the above model, the incremental current i_c does not change with respect to the incremental voltage v_{ce}, without the output resistor r_o. With the output resistance r_o, the incremental current i_c now changes with respect to incremental voltage v_{ce}.

6 T Model

The T models are shown in Figure 7 for the voltage controlled current source and the current controlled current source models. They can be shown to be equivalent to the hybrid-π model. The model that accounts for the Early effect is shown in Figure 8.

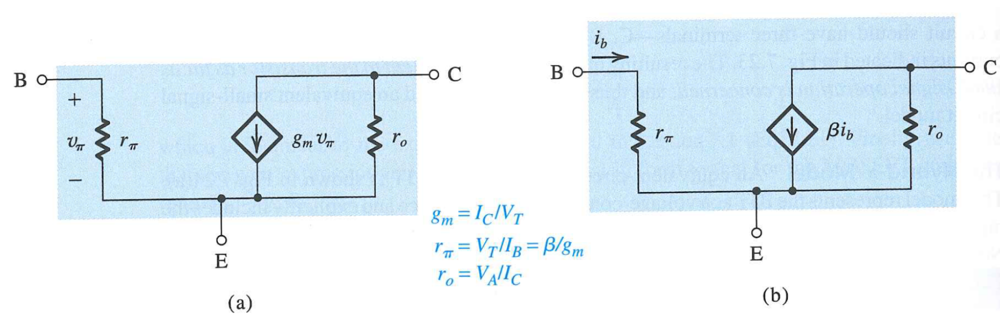


Figure 6: Two different version of the hybrid-π model for small signals with the Early effect included (a) VCCS, (b) CCCS (Courtesy of Sedra and Smith).

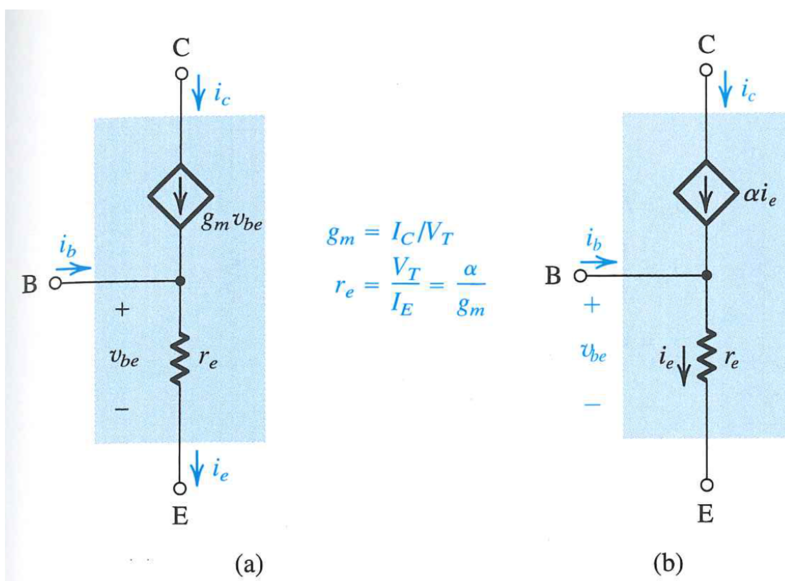


Figure 7: Two different version of the T model for small signals, (a) VCCS, (b) CCCS (Courtesy of Sedra and Smith).

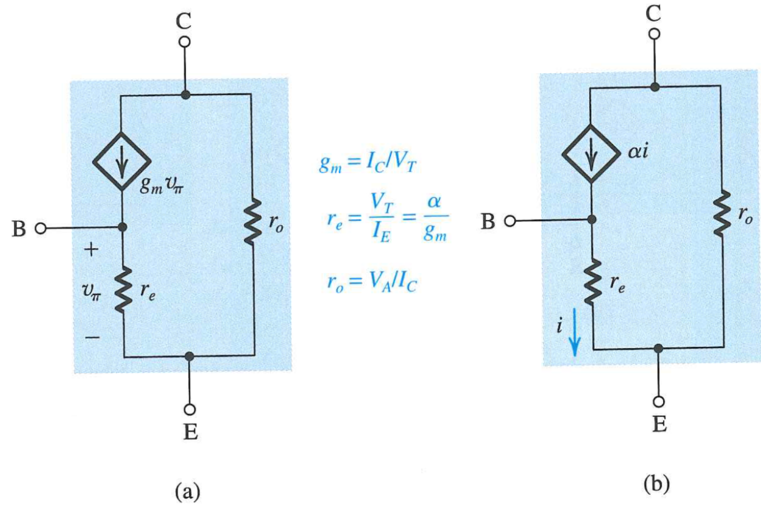


Figure 8: Two different version of the T model for small signals where the Early effect is included (Courtesy of Sedra and Smith).

7 Small Signal Analysis on Circuit Diagram

The important point to notice about small signal analysis is that a small signal is riding on top of a larger DC signal. Then by linearization approximation, the small signals are actually linearly related to each other, even though the DC components are nonlinearly related. The procedure outlined in Figure 9 shows the way to derive these linear relationship.

8 Summary Tables

The main points of this lecture can be summarized in summary tables.

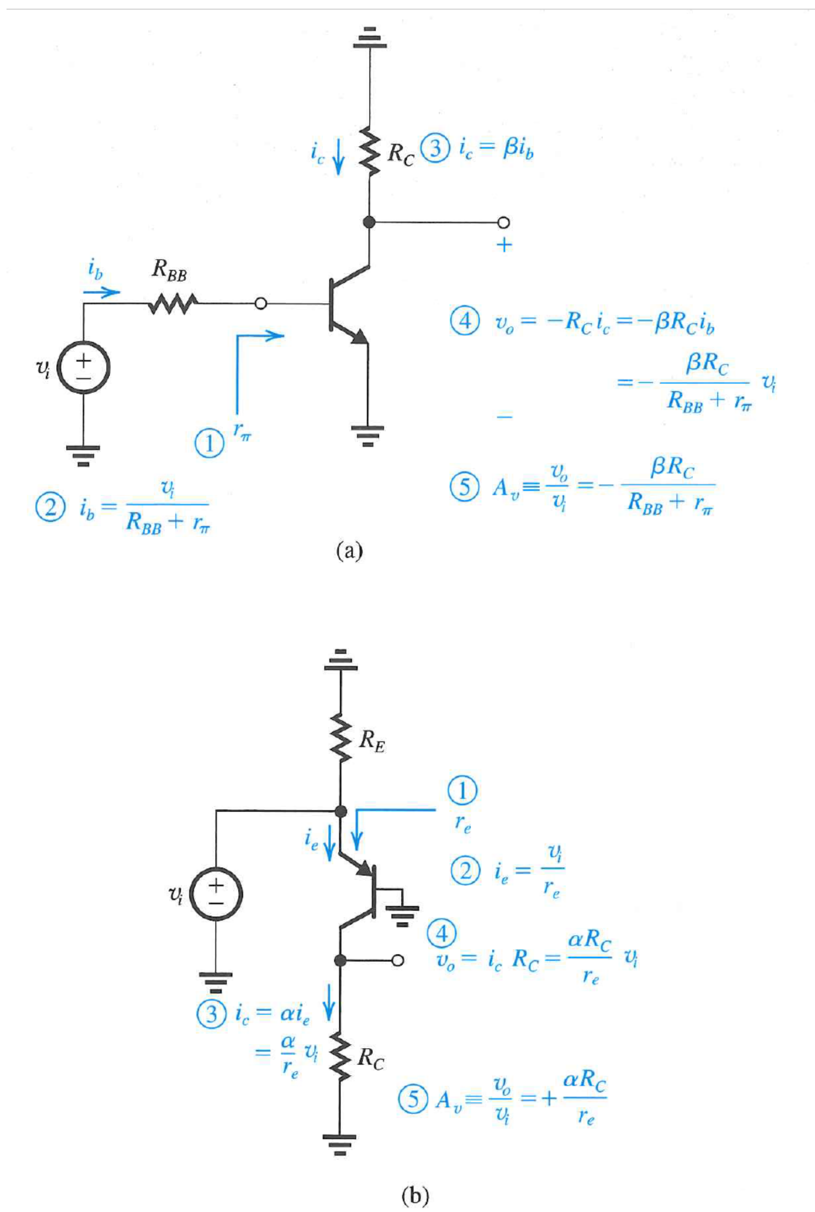


Figure 9: Step for performing the small signal analysis for two examples (Courtesy of Sedra and Smith).

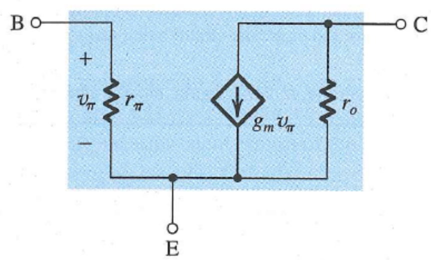
Table 7.1 Systematic Procedure for the Analysis of Transistor Amplifier Circuits

1. Eliminate the signal source and determine the dc operating point of the transistor.
2. Calculate the values of the parameters of the small-signal model.
3. Eliminate the dc sources by replacing each dc voltage source by a short circuit and each dc current source by an open circuit.
4. Replace the transistor with one of its small-signal, equivalent-circuit models. Although any of the models can be used, one might be more convenient than the others for the particular circuit being analyzed. This point will be made clearer in the next section.
5. Analyze the resulting circuit to determine the required quantities (e.g., voltage gain, input resistance).

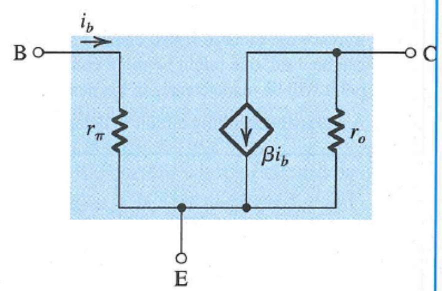
Table 7.3 Small-Signal Models of the BJT

Hybrid-π Model

■ ($g_m v_\pi$) Version

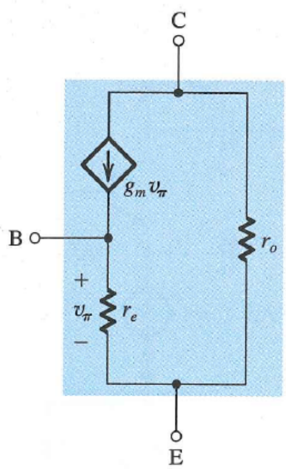


■ (βi_b) Version

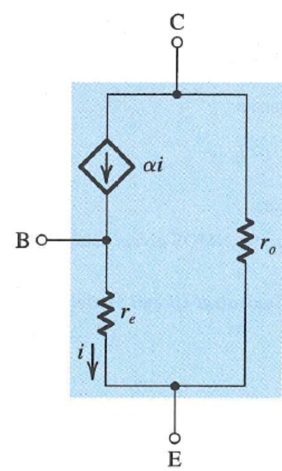


T Model

■ ($g_m v_\pi$) Version



■ (αi) Version



Model Parameters in Terms of DC Bias Currents

$g_m = \frac{I_C}{V_T}$ $\quad r_e = \frac{V_T}{I_E} = \alpha \frac{V_T}{I_C}$ $\quad r_\pi = \frac{V_T}{I_B} = \beta \frac{V_T}{I_C}$ $\quad r_o = \frac{|V_A|}{I_C}$

In Terms of g_m

$r_e = \frac{\alpha}{g_m}$ $\quad r_\pi = \frac{\beta}{g_m}$

In Terms of r_e

$g_m = \frac{\alpha}{r_e}$ $\quad r_\pi = (\beta + 1) r_e$ $\quad g_m + \frac{1}{r_\pi} = \frac{1}{r_e}$

Relationships between α and β

$\beta = \frac{\alpha}{1 - \alpha}$ $\quad \alpha = \frac{\beta}{\beta + 1}$ $\quad \beta + 1 = \frac{1}{1 - \alpha}$

Example 7.5

We wish to analyze the transistor amplifier shown in Fig. 7.28(a) to determine its voltage gain v_o/v_i. Assume $\beta = 100$ and neglect the Early effect.

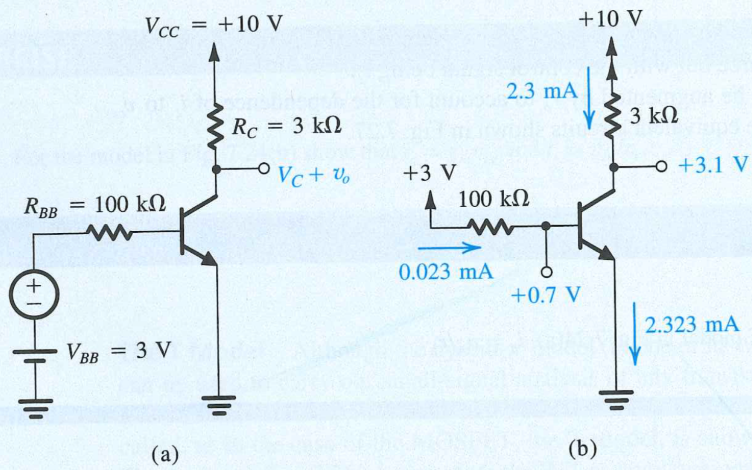


Figure 7.28 Example 7.5: (**a**) amplifier circuit; (**b**) circuit for dc analysis; (**c**) amplifier circuit with dc sources replaced by short circuits; (**d**) amplifier circuit with transistor replaced by its hybrid-π, small-signal model.

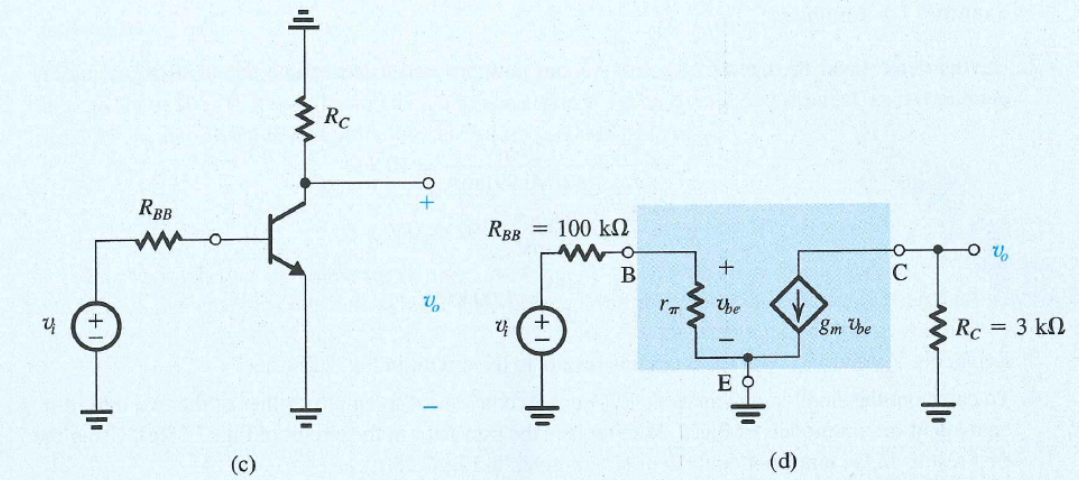


Figure 7.28 *continued*

Solution

We shall follow a five-step process:

1. The first step in the analysis consists of determining the quiescent operating point. For this purpose we assume that $v_i = 0$ and thus obtain the dc circuit shown in Fig. 7.28(b). The dc base current will be

$$I_B = \frac{V_{BB} - V_{BE}}{R_{BB}}$$
$$\simeq \frac{3 - 0.7}{100} = 0.023 \text{ mA}$$

The dc collector current will be

$$I_C = \beta I_B = 100 \times 0.023 = 2.3 \text{ mA}$$

The dc voltage at the collector will be

$$V_C = V_{CC} - I_C R_C$$
$$= +10 - 2.3 \times 3 = +3.1 \text{ V}$$

Since V_B at +0.7 V is less than V_C, it follows that in the quiescent condition the transistor will be operating in the active mode. The dc analysis is illustrated in Fig. 7.28(b).

Example 7.5 *continued*

2. Having determined the operating point, we can now proceed to determine the small-signal model parameters:

$$r_e = \frac{V_T}{I_E} = \frac{25\text{ mV}}{(2.3/0.99)\text{ mA}} = 10.8\ \Omega$$

$$g_m = \frac{I_C}{V_T} = \frac{2.3\text{ mA}}{25\text{ mV}} = 92\text{ mA/V}$$

$$r_\pi = \frac{\beta}{g_m} = \frac{100}{92} = 1.09\text{ k}\Omega$$

3. Replacing V_{BB} and V_{CC} with short circuits results in the circuit in Fig. 7.28(c).
4. To carry out the small-signal analysis it is equally convenient to employ either of the two hybrid-π, equivalent-circuit models of Fig. 7.24 to replace the transistor in the circuit of Fig. 7.28(c). Using the first results in the amplifier equivalent circuit given in Fig. 7.28(d).
5. Analysis of the equivalent circuit in Fig. 7.28(d) proceeds as follows:

$$v_{be} = v_i \frac{r_\pi}{r_\pi + R_{BB}}$$

$$= v_i \frac{1.09}{101.09} = 0.011 v_i \qquad (7.81)$$

The output voltage v_o is given by

$$v_o = -g_m v_{be} R_C$$

$$= -92 \times 0.011 v_i \times 3 = -3.04 v_i$$

Thus the voltage gain will be

$$A_v = \frac{v_o}{v_i} = -3.04\text{ V/V} \qquad (7.82)$$