ECE 255, Frequency Response

7 December 2017

1 Introduction

In this lecture, we will study the internal capacitances and their effects on the high-frequency response of a circuit. It is based on Section 10.2 to Section 10.5 of the textbook.

2 Internal Capacitive Effects on MOSFET

Any two pieces of conductive materials can make a capacitor. Hence, when two pieces of conductors are brought to close proximity of each other, due to that unlike charges attract, charges will accumulate at these points. Then electric field is set up in between them, giving rise to electric energy stored. Electric energy stored corresponds to energy storage in a capacitor. These equivalent capacitors are called internal capacitors or parasitic capacitors. They are shown in Figure 1.

These internal capacitors gives rise to the modification to the small-signal model. This modification is shown in Figure 2(a). Here, C_{gs}, C_{gd}, C_{sb}, and C_{db} are the gate-to-source, gate-to-drain, source-to-body, and drain-to-body capacitances, respectively. But when the source terminal is connected directly to the body, then the model can be simplified as that shown in Figure 2(b). By further ignoring C_{db} the drain-to-body capacitor, which is small since drain can be quite far from the body, the model simplifies to that show in Figure 2(c) and (d).

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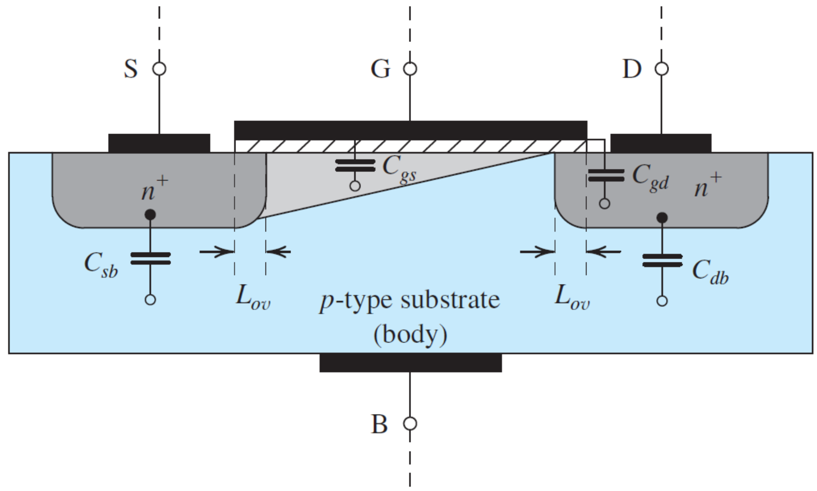


Figure 1: Internal capacitors in a MOSFET. Any two pieces of conductive materials separated by an insulator (or a region of low conductivity) will have a capacitance between them. This figure shows the internal capacitances as C_{gs}, C_{gd}, C_{sb}, and C_{db} (Courtesy of Sedra and Smith).

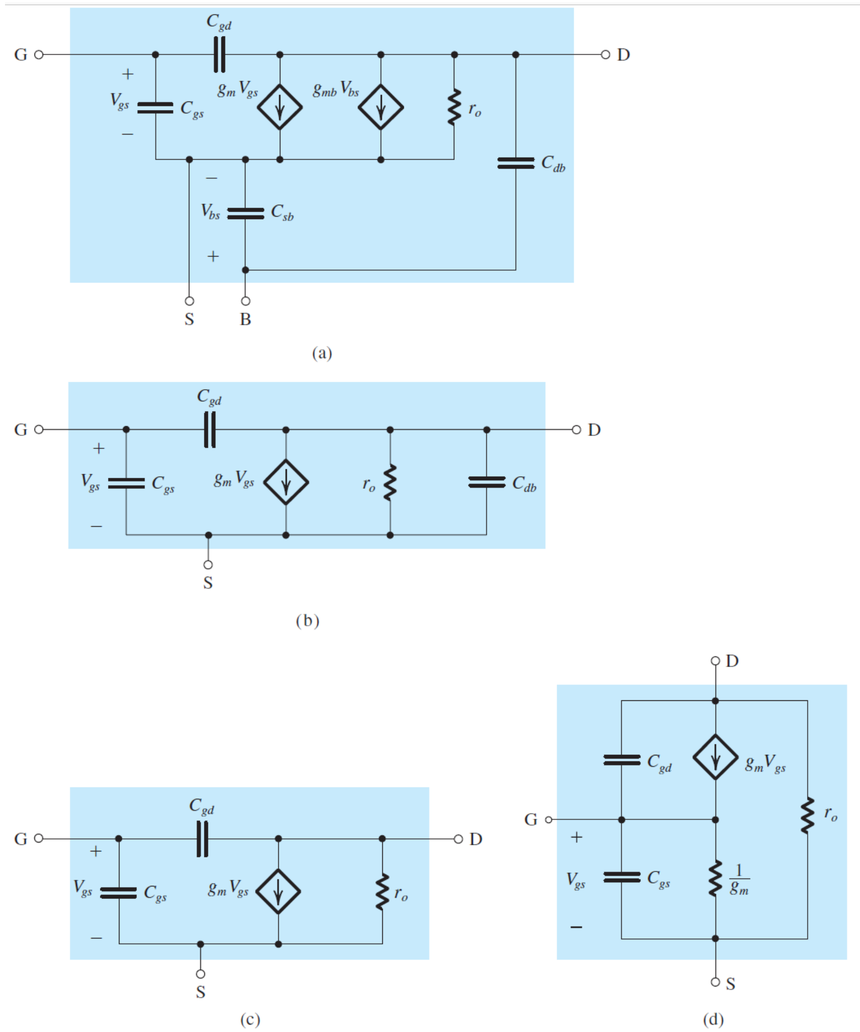


Figure 2: (a) High-frequency equivalent circuit of a MOSFET. (b) The simplified case where the source terminal is connected to the body. (c) Further simplification by ignoring C_{db}, which is usually small. (d) The simplified T model equivalent circuit (Courtesy of Sedra and Smith).

2.1 The MOSFET Unity-Gain Frequency (f_T)

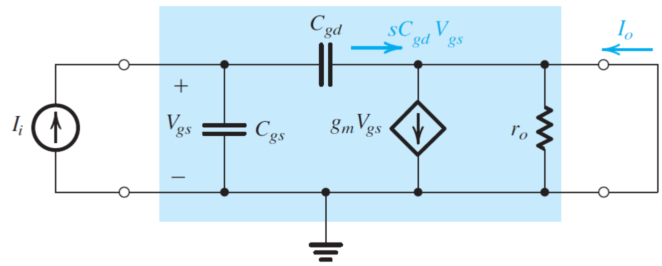


Figure 3: Model for determining the short-circuit current gain (Courtesy of Sedra and Smith).

As the frequency increases, the gain of the amplifier drops and its performance deteriorates. The amplifier becomes useless again when its gain drops below one. Therefore, it is prudent to ascertain the frequency at which the short-circuit gain becomes one. This is usually denoted as f_T, or called the **transition frequency**.

To determine the short-circuit current gain using the model shown in Figure 3, one injects a current I_i into the input port of the amplifier. Then the output current I_o, ignoring the current through r_o, is[1]

$$I_o = g_m V_{gs} - sC_{gd}V_{gs} \tag{2.1}$$

Since the capacitance C_{gd} is small, one can approximate this current as just

$$I_o \approx g_m V_{gs} \tag{2.2}$$

Furthermore, one finds V_{gs} as

$$V_{gs} = \frac{I_i}{s\left(C_{gs} + C_{gd}\right)} \tag{2.3}$$

Consequently, using (2.2) and (2.2), one gets

$$\frac{I_o}{I_i} = \frac{g_m}{s\left(C_{gs} + C_{gd}\right)} \tag{2.4}$$

By letting $s = j\omega$, one has

$$\left|\frac{I_o}{I_i}\right| = \frac{g_m}{\omega\left(C_{gs} + C_{gd}\right)} \tag{2.5}$$

[1] Using KCL which is valid for complex impedances as well.

The above becomes unity at

$$\omega = \omega_T = g_m / \left(C_{gs} + C_{gd}\right) \tag{2.6}$$

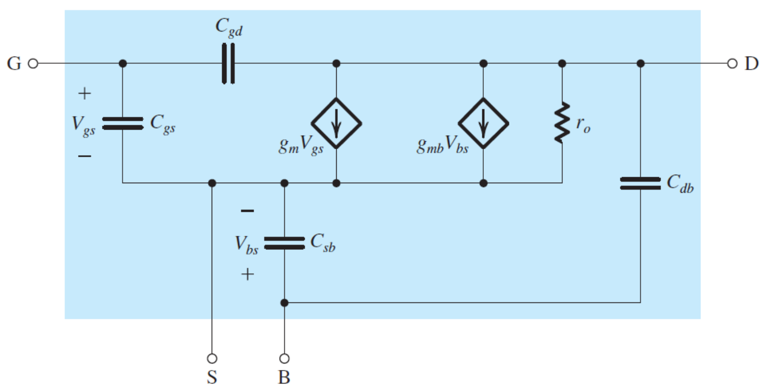
Using $f_T = \omega_T / 2\pi$ yields

$$f_T = \frac{g_m}{2\pi \left(C_{gs} + C_{gd}\right)} \tag{2.7}$$

Typically, f_T ranges from 100 MHz for older technologies (5-μm CMOS) to many GHz for newer high-speed technologies (0.13-μm CMOS).[2] The smaller the device, the smaller are the internal capacitances, since capacitance is simply given by the formula $\epsilon A/d$. Making a device 10 times smaller makes the area 100 times smaller, while the separation becomes 10 times smaller. Hence, the capacitance becomes ten times smaller.

Table 10.1 The MOSFET High-Frequency Model

Model



Model Parameters

$$g_m = \mu_n C_{ox} \frac{W}{L} |V_{OV}| = \sqrt{2\mu_n C_{ox} \frac{W}{L} I_D} = \frac{2I_D}{|V_{OV}|}$$

$$g_{mb} = \chi g_m, \quad \chi = 0.1 \text{ to } 0.2$$

$$r_o = |V_A| / I_D$$

$$C_{gs} = \frac{2}{3} WLC_{ox} + WL_{ov} C_{ox}$$

$$C_{gd} = WL_{ov} C_{ox}$$

$$C_{sb} = \frac{C_{sb0}}{\sqrt{1 + \frac{|V_{SB}|}{V_0}}}$$

$$C_{db} = \frac{C_{db0}}{\sqrt{1 + \frac{|V_{DB}|}{V_0}}}$$

$$f_T = \frac{g_m}{2\pi (C_{gs} + C_{gd})}$$

[2]Now it is possible to make transistors operating at 100 GHz.

2.2 The BJT

As before, the simple hybrid-π model of the BJT has to be modified accordingly due to the presence of internal or parasitic capacitances.

2.2.1 The Base-Charging or Diffusion Capacitance C_{de}

Parasitic capacitances slow down the switching speed of a transistor, since they have to be charged and discharged. It will be prudent to study various charge storage mechanisms in a transistor. For an *npn* transistor, the charge stored in the base region can be expressed at

$$Q_n = \tau_F i_C \tag{2.8}$$

where τ_F is the **forward base-transit time**, the average time a charge carrier (electron) takes in crossing the base. It is typically about 10 ps to 100 ps. On the average, this amount of charge will reside in the base region.

Since i_C is dependent on v_{BE}, Q_n will similarly depend on v_{BE}. And $C = Q/V$, a **small-signal diffusion capacitance** C_{de} can be derived to be

$$C_{de} = \frac{dQ_n}{dv_{BE}} = \tau_F \frac{di_C}{dv_{BE}} \tag{2.9}$$

Here, the small-signal transconductance $g_m = di_C/(dv_{BE})$ resulting in

$$C_{de} = \tau_F g_m = \tau_F \frac{I_C}{V_T} \tag{2.10}$$

We have used the formula that $g_m = I_C/V_T$.

2.2.2 The Base-Emitter Junction Capacitance C_{je}

In addition to the base-charging diffusion capacitance, there capacitance at the depletion layers at the junctions. For the base-emitter junction, this capacitance is assumed to be

$$C_{je} \approx 2C_{je0} \tag{2.11}$$

where C_{je0} is the value of C_{je} at zero EBJ voltagen or no biasing. This capacitance increases on forward biasing because the depletion layer becomes thinner.

2.2.3 The Collector-Base Junction Capacitance C_μ

Since in the active mode, the CBJ is reverse biased, there is a **depletion capacitance** C_μ is empirically given by

$$C_\mu = \frac{C_{\mu 0}}{\left(1 + \frac{V_{CB}}{V_{0c}}\right)^m} \tag{2.12}$$

Here, V_{CB} is the magnitude of the CBJ reverse-bias voltage, and V_{0c} is the CBJ built-in voltage around 0.75 V, and m is typically around $0.2 - 0.5$. It is noted that this capacitance decreases under reverse biasing due to that the depletion layer becomes thicker.

2.2.4 The High-Frequency Models

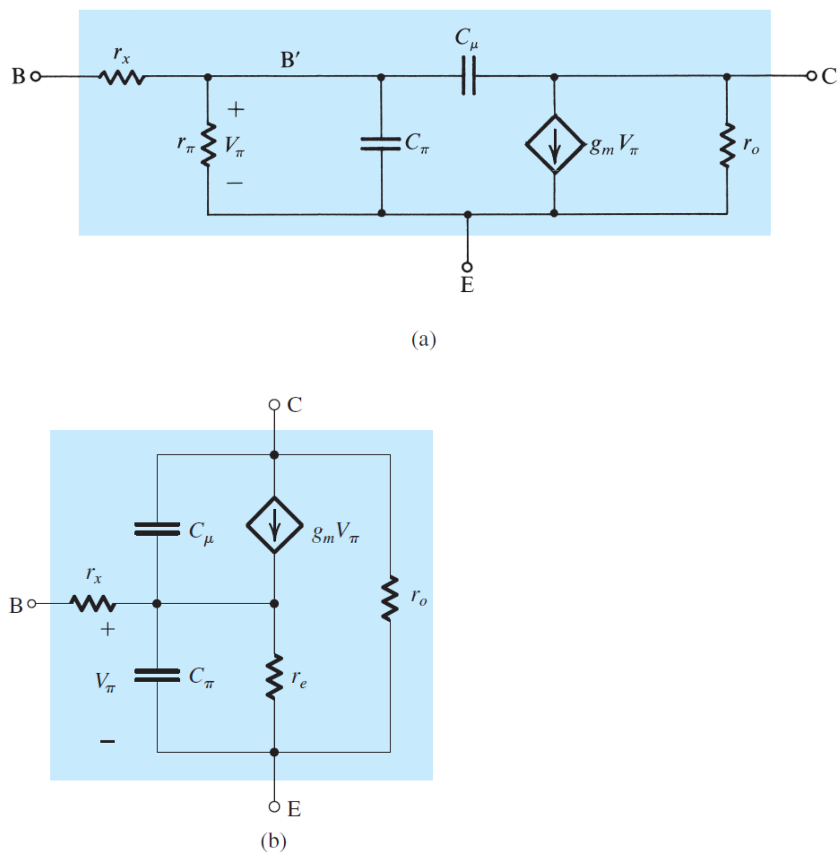


Figure 4: The high-frequency model for BJT both in hybrid-π model in (a), and the T model in (b) (Courtesy of Sedra and Smith).

Because of the internal capacitances of the BJT, the high-frequency model is shown in Figure 4 where $C_\pi = C_{de} + C_{je}$, and C_μ is as defined before. Here, C_π is a few to a few tens of picofarads, while C_μ is a fraction to a few picofarads. In general, $C_\mu \ll C_\pi$. They get increasingly smaller with progress in technology. A resistor r_x is used here to model an intrinsic silicon material resistance.

2.2.5 The BJT Unity-Gain Frequency

Figure 5 can be used to find the short-circuit current gain of a BJT at high frequency. The collector current

$$I_c = (g_m - sC_\mu)V_\pi \tag{2.13}$$

Furthermore, one can show that

$$V_\pi = I_b\,(r_\pi \parallel C_\pi \parallel C_\mu) = \frac{I_b}{1/r_\pi + sC_\pi + sC_\mu} \tag{2.14}$$

Thus, a frequency dependent β, called h_{fe}, is given as

$$h_{fe} = \frac{I_c}{I_b} = \frac{g_m - sC_\mu}{1/r_\pi + s(C_\pi + C_\pi)} \tag{2.15}$$

When $\omega C_\mu \ll g_m$, the above can be approximated as

$$h_{fe} \approx \frac{g_m r_\pi}{1 + s(C_\pi + C_\mu)r_\pi} \tag{2.16}$$

Thus,

$$h_{fe} = \frac{\beta_0}{1 + s(C_\pi + C_\mu)r_\pi} \tag{2.17}$$

where $\beta_0 = g_m r_\pi$ is the low-frequency value of β. The 3-dB point is at $\omega = \omega_\beta$ where

$$\omega_\beta = \frac{1}{(C_\pi + C_\mu)r_\pi} \tag{2.18}$$

Writing

$$h_{fe} = \frac{\beta_0}{1 + s/\omega_\beta} = \frac{\beta_0}{1 + i\omega/\omega_\beta} \tag{2.19}$$

It can be seen that when

$$\omega = \omega_T \approx \beta_0 \omega_\beta \tag{2.20}$$

the gain is approximately unity. Thus

$$\omega_T \approx \frac{g_m}{C_\pi + C_\mu} \tag{2.21}$$

and

$$f_T \approx \frac{g_m}{2\pi(C_\pi + C_\mu)} \tag{2.22}$$

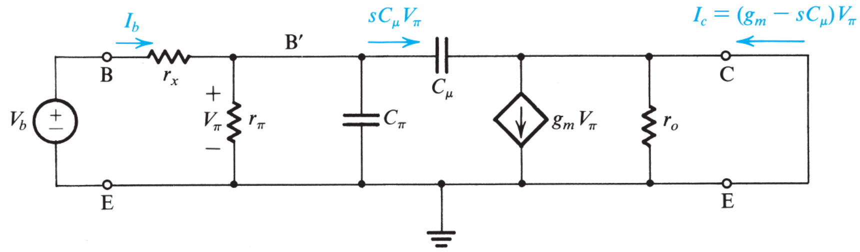


Figure 5: Circuit model for deriving h_{fe} (Courtesy of Sedra and Smith).

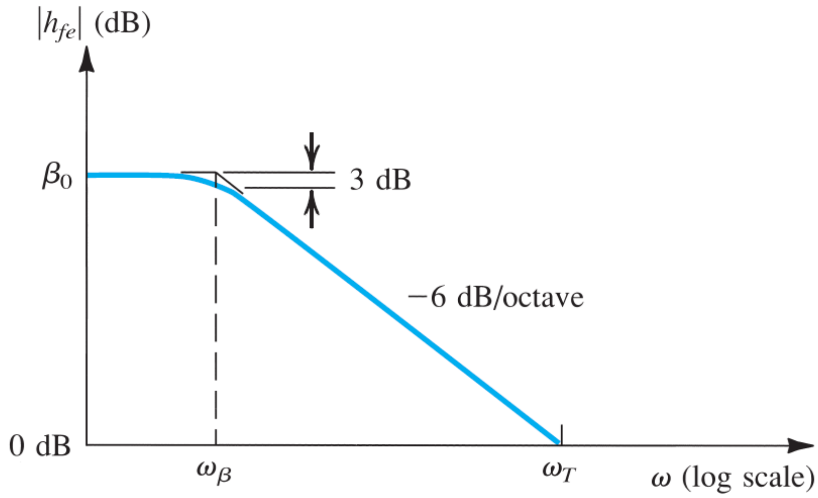
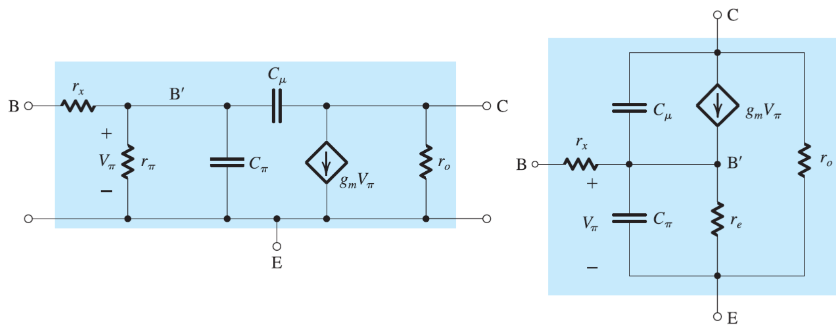


Figure 6: Bode plot for $|h_{fe}|$ (Courtesy of Sedra and Smith).

Summary

The table below summarizes the BJT high-frequency model.

Table 10.2 The BJT High-Frequency Model



$g_m = I_C/V_T$	$r_o = \lvert V_A\rvert/I_C$	$r_\pi = \beta_0/g_m$	$r_e = r_\pi/(\beta+1)$
$C_\pi + C_\mu = \dfrac{g_m}{2\pi f_T}$	$C_\pi = C_{de} + C_{je}$	$C_{de} = \tau_F g_m$	$C_{je} \simeq 2C_{je0}$
$C_\mu = C_{jc0} \Big/ \left(1 + \dfrac{\lvert V_{CB}\rvert}{V_{0c}}\right)^m$	$m = 0.3–0.5$		

3 High-Frequency Response of the CS and CE Amplifier with Miller Effect

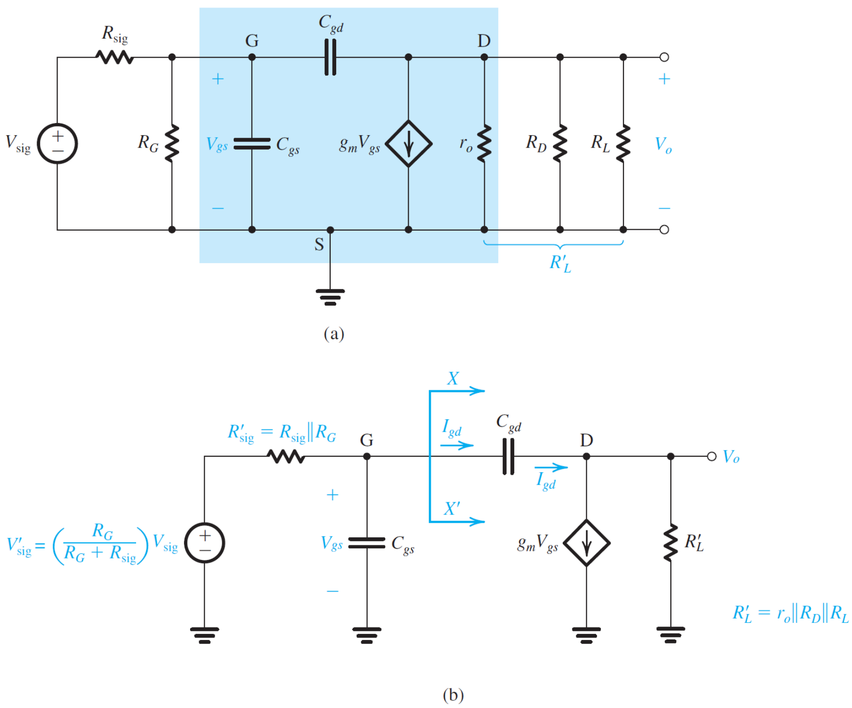


Figure 7: Models for the high-frequency response of a CS amplifier. (a) Equivalent circuit. (b) A simplified circuit by consolidation. (c) Further simplification by using C_{eq}. (d) A single-time-constant frequency response Bode plot (Courtesy of Sedra and Smith).

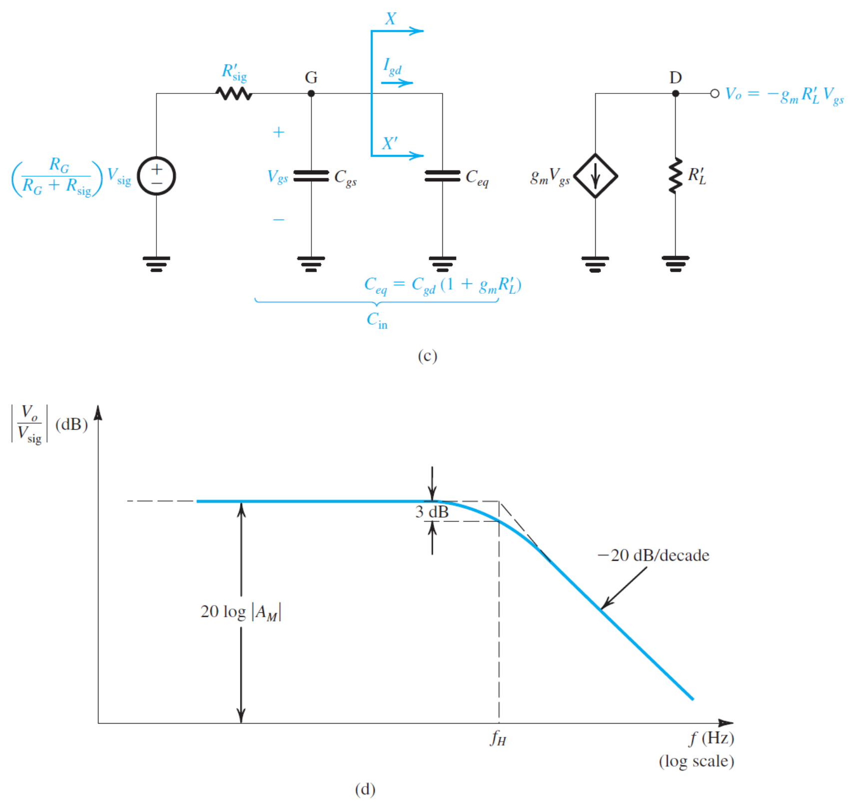


Figure 8: Continuation of the previous Figure 7 (Courtesy of Sedra and Smith).

Figure 7 shows the small-signal equivalence of a CS amplifier. The overall voltage gain is given by

$$A_M = \frac{V_o}{V_{\text{sig}}} = -\frac{R_G}{R_G + R_{\text{sig}}}(g_m R'_L) \tag{3.1}$$

In order to simplify the circuit, it can be consolidate using Thevenin theorem so that the source is modeled by only two elements as shown in Figure 7(b). Also, R'_L consolidates the three resistances at the output end.

One further simplification is to replace the capacitor with an equivalence capacitor C_{eq} as shown in Figure 7(c). We shall calculate the load current, in accordance with Figure 7(b), which is given by $(g_m V_{gs} - I_{gd})$. Then the output voltage is given by

$$V_o = (g_m V_{gs} - I_{gd})R'_L \approx -g_m R'_L V_{gs} \tag{3.2}$$

assuming that $g_m V_{gs} \gg I_{gd}$. In the above, $R'_L = r_o \parallel R_D \parallel R_L$. The current

I_{gd} can now be found as

$$I_{gd} = sC_{gd}(V_{gs} - V_o) \approx sC_{gd}\left[V_{gs} - (-g_m R'_L V_{gs})\right] = sC_{gd}(1 + g_m R'_L)V_{gs} \quad (3.3)$$

Now, one can assume that this extra current I_{gd} is due to an equivalent capacitor C_{eq} connected in parallel to C_{gs} as shown in Figure 8. The current that flows into this equivalent capacitance C_{eq} is

$$sC_{eq}V_{gs} = sC_{gd}(1 + g_m R'_L)V_{gs} \quad (3.4)$$

The above results in that

$$C_{eq} = C_{gd}(1 + g_m R'_L) \quad (3.5)$$

This equivalent capacitance C_{eq} is much larger than C_{gd}, and this effect is known as the **Miller effect**, and the factor $(1 + g_m R'_L)$ is the **Miller multiplier**. Hence, the larger the gain of the amplifier is, the larger is this effect.

The above approximation replaces a double-pole system with a single pole system or a single-time-constant (STC) circuit. Then the function can now be represented as

$$V_{gs} = \left(\frac{R_G}{R_G + R_{\text{sig}}} V_{\text{sig}}\right) \frac{1}{1 + s/\omega_0} \quad (3.6)$$

The pole frequency of the STC circuit is then

$$\omega_0 = 1/(C_{\text{in}} R'_{\text{sig}}) \quad (3.7)$$

with

$$C_{\text{in}} = C_{gs} + C_{eq} = C_{gs} + C_{gd}(1 + g_m R'_L) \quad (3.8)$$

and

$$R'_{\text{sig}} = R_{\text{sig}} \parallel R_G \quad (3.9)$$

A sanity check of (3.6) shows that is the correct formula: it reduces to the correct formula when $s = \omega = 0$. The system in Figure 8 can only have one pole with the corresponding relaxation frequency given by (3.7). Therefore, it the correct formula which can be confirmed by a longer derivation.

Then using V_{gs} given in (3.2), one has

$$\frac{V_o}{V_{\text{sig}}} = -\left(\frac{R_G}{R_G + R_{\text{sig}}}\right)(g_m R'_L)\frac{1}{1 + s/\omega_0} \quad (3.10)$$

which can be simplified as

$$\frac{V_o}{V_{\text{sig}}} = \frac{A_M}{1 + s/\omega_H} \quad (3.11)$$

where A_M is the midband gain given by (3.1), and ω_H is the upper 3-dB frequency point, or

$$\omega_H = \omega_0 = \frac{1}{C_{\text{in}} R'_{\text{sig}}}, \qquad f_H = \frac{\omega_H}{2\pi} = \frac{1}{2\pi C_{\text{in}} R'_{\text{sig}}} \quad (3.12)$$

Observations

1. The upper 3-dB frequency is determined by the interaction of $R'_{\text{sig}} = R_{\text{sig}} \parallel R_G \approx R_{\text{sig}}$ since $R_G \gg R_{\text{sig}}$, and $C_{\text{in}} = C_{gs} + C_{gd}(1 + g_m R'_L)$. Hence, a large R_{sig} will cause f_H to be lowered, decreasing the bandwidth of the amplifier.

2. The total capacitance C_{in} is increased by the Miller which magnify C_{gd} by the factor $1 + g_m R'_L$, which lowers f_H.

3. To improve the high-frequency response of MOSFET, one has to reduce the Miller effect.

4. The STC is an approximation because we are replacing a double-pole system with a single-pole system. A system with two capacitors has two poles, but replacing it with one single pole is only approximately correct.

5. The dominant high-frequency pole of the system is given by by $f_P \approx f_H$.

3.1 The Common-Emitter Amplifier

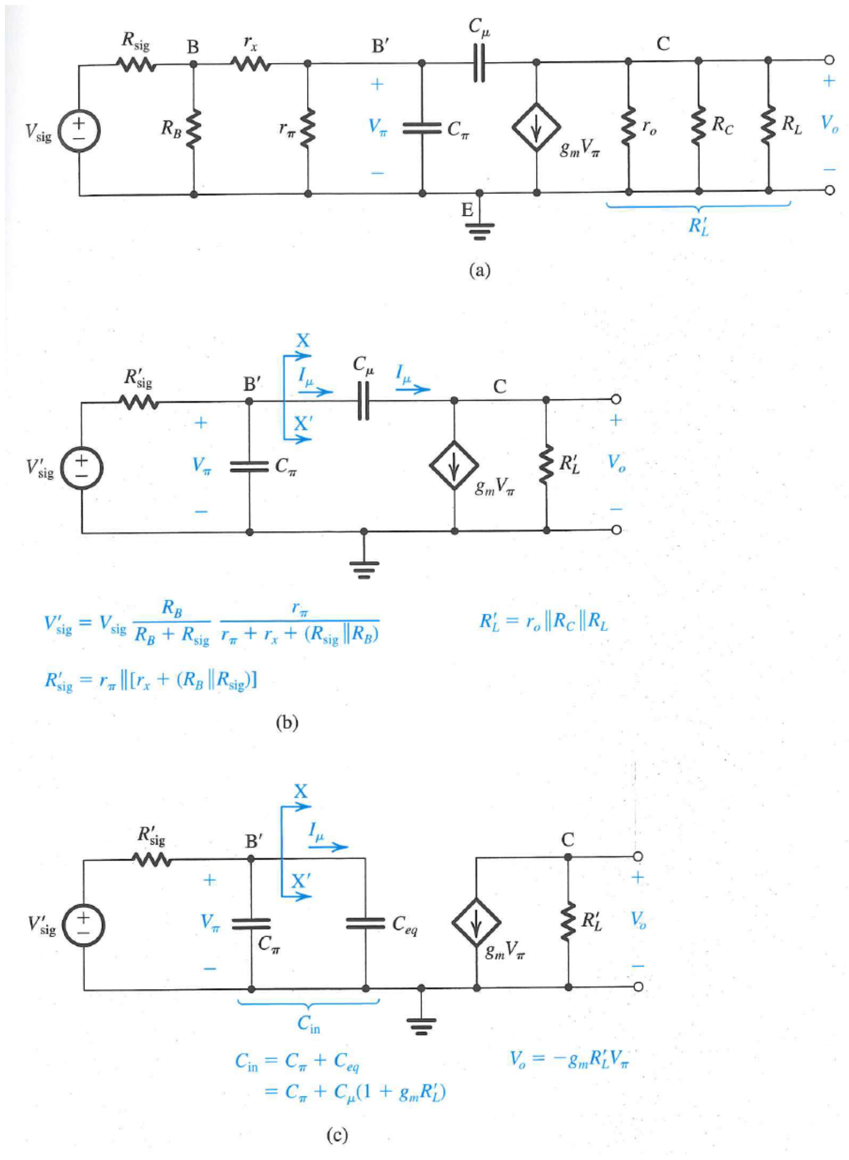


Figure 9: Models for the high-frequency response of a CE amplifier. (a) Equivalent circuit. (b) A simplified circuit by consolidation. (c) Further simplification by using C_{eq}. (d) A single-time-constant frequency response Bode plot (Courtesy of Sedra and Smith).

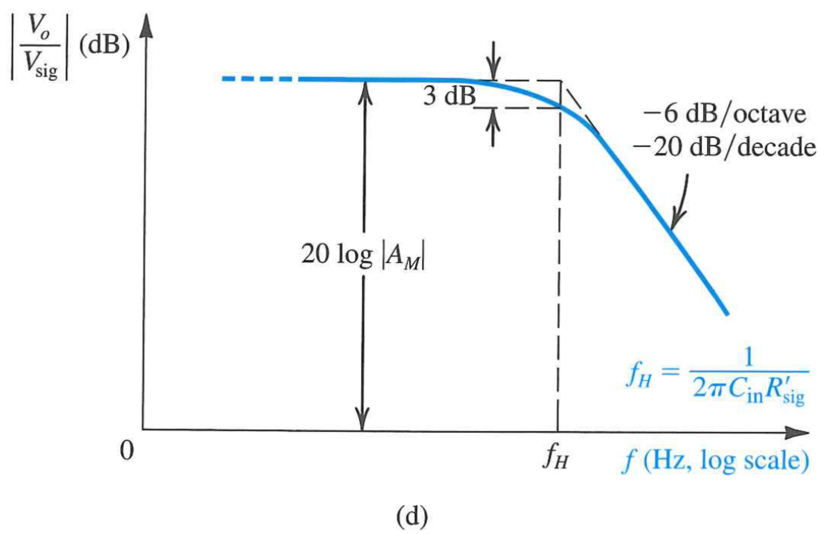


Figure 10: Continuation of the previous Figure 9 (Courtesy of Sedra and Smith).

The analysis of the CE amplifier is very similar to that of the CS amplifier as shown in Figure 9.

3.2 Miller's Theorem

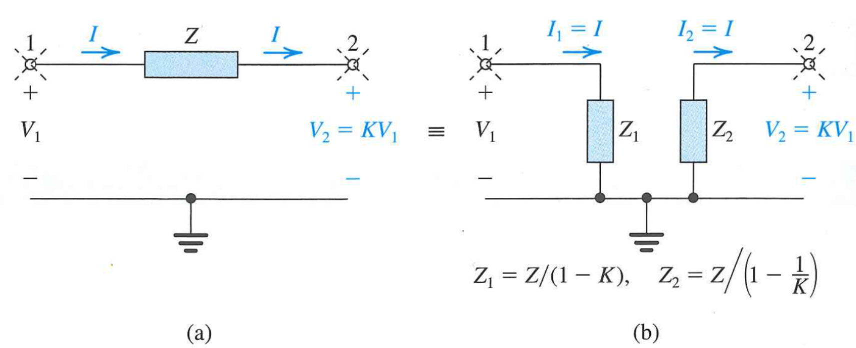


Figure 11: The Miller equivalent circuit (Courtesy of Sedra and Smith).

The **Miller's theorem** allows the replacement of a bridging capacitance by two equivalent capacitances as shown in Figure 11. This theorem relies on that

$$V_2 = KV_1 \tag{3.13}$$

in Figure 11. In this case, it can be shown that

$$Z_1 = Z/(1-K), \qquad Z_2 = Z\Big/\left(1-\frac{1}{K}\right) \tag{3.14}$$

The proof is given in the textbook and will not be reproduced here.

4 Useful Tools for High-Frequency Response of Amplifiers

When the simple analysis previously discussed fails, one may resort to more sophisticated analysis tools for the frequency response. This happens for instance, when the poles and zeros are not far apart.

4.1 High-Frequency Gain Function

The frequency gain as a function of frequency can be expressed as

$$A(s) = A_M F_H(s) \tag{4.1}$$

where

$$F_H(s) = \frac{(1+s/\omega_{Z1})(1+s/\omega_{Z2})\cdots(1+s/\omega_{Zn})}{(1+s/\omega_{P1})(1+s/\omega_{P2})\cdots(1+s/\omega_{Pm})} \tag{4.2}$$

Notice that the above function $F_H(s) \to 1$ when $s \to 0$, which is what is desired.

4.2 Determing the 3-dB Frequency f_H

As in the low-frequency case, when the above function is dominated by a single pole, then one has

$$F_H(s) \approx \frac{1}{1+s/\omega_{P1}} \tag{4.3}$$

The 3-dB point is easily shown to be

$$\omega_H \approx \omega_{P1} \tag{4.4}$$

by first letting $s = j\omega$ and then $\omega = \omega_{P1}$. The above single pole approximation is good when the next pole or zero is two octaves (4 times) further away from the dominant pole.

If a dominant pole approximation is not possible, the aggregate effects of the poles and zeros need to be considered in finding ω_H. For simplicity, one considers first a simple two-pole and two-zero system. Then

$$F_H(s) = \frac{(1+s/\omega_{Z1})(1+s/\omega_{Z2})}{(1+s/\omega_{P1})(1+s/\omega_{P2})} \tag{4.5}$$

Letting $s = j\omega$ and taking the magnitude square of the above, one gets

$$|F_H(j\omega)|^2 = \frac{(1+\omega^2/\omega_{Z1}^2)(1+\omega^2/\omega_{Z2}^2)}{(1+\omega^2/\omega_{P1}^2)(1+\omega^2/\omega_{P2}^2)} \tag{4.6}$$

By definition, at $\omega = \omega_H$, the half-power point, $|F_H(j\omega_H)|^2 = \frac{1}{2}$, and

$$\frac{1}{2} = \frac{(1+\omega_H^2/\omega_{Z1}^2)(1+\omega_H^2/\omega_{Z2}^2)}{(1+\omega_H^2/\omega_{P1}^2)(1+\omega_H^2/\omega_{P2}^2)} \approx \frac{1+\omega_H^2\left(\frac{1}{\omega_{Z1}^2}+\frac{1}{\omega_{Z2}^2}\right)+\cdots}{1+\omega_H^2\left(\frac{1}{\omega_{P1}^2}+\frac{1}{\omega_{P2}^2}\right)+\cdots} \tag{4.7}$$

where we have kept only the quadratic terms in both the numerator and denominator. The remaining terms are proportional to ω_H^4, which are negligible when ω_H is small, compared to the terms retained. The above equation can be solved approximately to yield

$$\omega_H \approx 1\Big/\sqrt{\frac{1}{\omega_{P1}^2}+\frac{1}{\omega_{P2}^2}-\frac{2}{\omega_{Z1}^2}+\frac{2}{\omega_{Z2}^2}} \tag{4.8}$$

The above expression can be generalized to a multi-pole and multi-zero system giving

$$\omega_H \approx 1\Big/\sqrt{\left(\frac{1}{\omega_{P1}^2}+\frac{1}{\omega_{P2}^2}+\cdots\right)-2\left(\frac{1}{\omega_{Z1}^2}+\frac{1}{\omega_{Z2}^2}+\cdots\right)} \tag{4.9}$$

4.3 Low-Frequency Gain Function

A similar low-frequency gain function can be defined such that the frequency gain as a function of frequency can be expressed as

$$A(s) = A_M F_L(s) \tag{4.10}$$

where

$$F_L(s) = \frac{(1+\omega_{Z1}/s)(1+\omega_{Z2}/s)\cdots(1+\omega_{Zn}/s)}{(1+\omega_{P1}/s)(1+\omega_{P2}/s)\cdots(1+\omega_{Pm}/s)} \tag{4.11}$$

Notice that the above function $F_L(s) \to 1$ when $s \to \infty$, which is what is desired.

A similar analysis shows that the half-power frequency point ω_L is

$$\omega_L \approx \sqrt{(\omega_{P1}^2+\omega_{P2}^2+\cdots)-2\,(\omega_{Z1}^2+\omega_{Z2}^2+\cdots)} \tag{4.12}$$

One can obtain the above result by comparing (4.2) and (4.11). One notices that the roles of ω's and s's are switched in the formulas. Hence, by symmetry, one arrives at the above formula by letting $s \rightleftharpoons \omega$.

4.4 The Method of Open-Circuit Time Constants

In finding f_L, when a single pole dominates and they are far from each other, one uses the short-circuit time-constant method to decouple the capacitors and find their respective time constants. The rationale is that at the highest frequency pole that decides f_L, the frequency is high enough such that the other capacitors can be considered short-circuited, and hence, the determination of the highest frequency pole is a reasonable approximation.

By the same token, when one finds f_H, it is the lowest frequency pole that dominates f_H. Again, this frequency is low enough that the other capacitors can be considered open-circuited. Then the time-constant for the lowest frequency pole is fairly accurate. With this in mind, then

$$\omega_H = 2\pi f_H \approx \frac{1}{\sum_i C_i R_i} \tag{4.13}$$

In the above sum, it will automatically be dominated by the largest RC time constant term, since it is an ordinary mean.

In contrast, in the short-circuit time-constant method,

$$\omega_L = 2\pi f_L \approx \sum_i \frac{1}{R_i C_i} \tag{4.14}$$

which is proportional to the harmonic mean of the RC time constants. The term with the shortest time constant automatically dominates this sum.