ECE 255, Frequency Response

5 December 2017

1 Introduction

In this lecture, we address the frequency response of amplifiers. This was touched upon briefly in our previous lecture in Section 7.5 of the textbook. Here, we delve into more details on this topic. It will be from Sections 10.1 and 10.2 of the textbook.

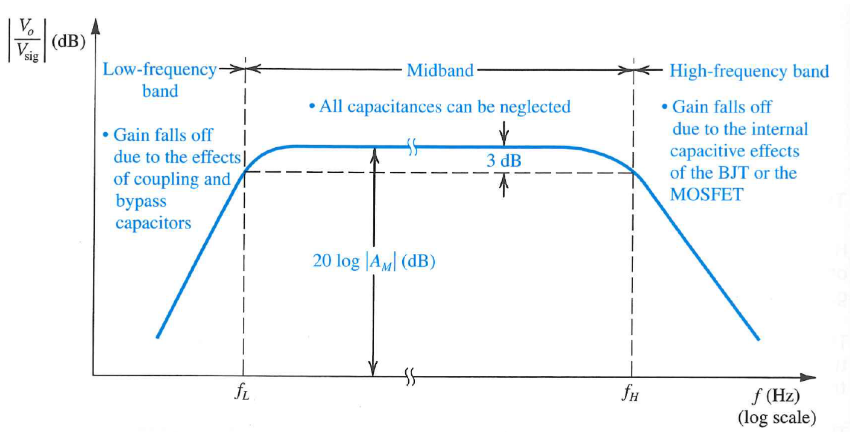

Figure 1: The frequency response of a discrete circuit is affected by the coupling capacitors and bypass capacitors at the low frequency end. At the high-frequency end, it is affected by the internal capacitors (or parasitic capacitances) of the circuit (Courtesy of Sedra and Smith).

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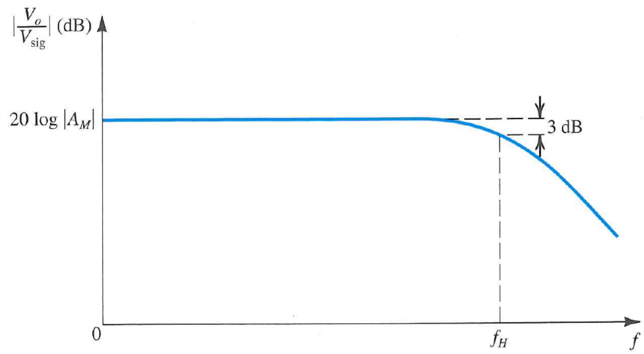

Figure 2: For integrated circuits (IC) where there are no coupling and bypass capacitor needed, there is no low-frequency cut-off, and the circuits work down to very low frequency or DC. Then only internal capacitors affect the high-frequency response (Courtesy of Sedra and Smith).

As shown in Figure 1, the gain of the amplifier falls off at low frequency because the coupling capacitors and the bypass capacitors become open circuit or they have high impedances. Hence, they have non-negligible effect at lower frequencies as treating them as short-circuits is invalid. At high frequencies, the simple hybrid-π model or T model is insufficient to describe the internal physics of the devices. Internal capacitances need to be added, causing these devices to depart from their ideal model. Again, this causes the gain to fall off at higher frequency.

The operational frequency range or bandwidth of the amplifier is delineated by f_L, at the lower end of the operating frequency range, and f_H, at the upper end. These frequencies are defined as the frequencies when the gain drops below 3 dB[1] of the midband gain of the amplifier. For IC's when no coupling capacitors are used, then f_L is zero. Hence, bandwidth BW is defined as

$$BW = f_H - f_L, \text{ (discrete-circuit amplifiers)}, \tag{1.1}$$

$$BW = f_H, \text{ (integrated-circuit amplifiers)} \tag{1.2}$$

A figure of merit for amplifier design is the **gain-bandwidth product**, defined as

$$GB = |A_M| BW \tag{1.3}$$

where $|A_M|$ is the midband gain. The gain-bandwidth product is often a constant for many amplifiers. It can be shown to be a constant when the amplifier has only one pole for example.

[1] This is also the half-power point.

2 Low-Frequency Response of Discrete-Circuit Common-Source and Common-Emitter Amplifiers

We will study the effects of the coupling and bypass capacitors on the low-frequency performance of the common-source (CS) and the common-emitter (CE) discrete-circuit amplifiers

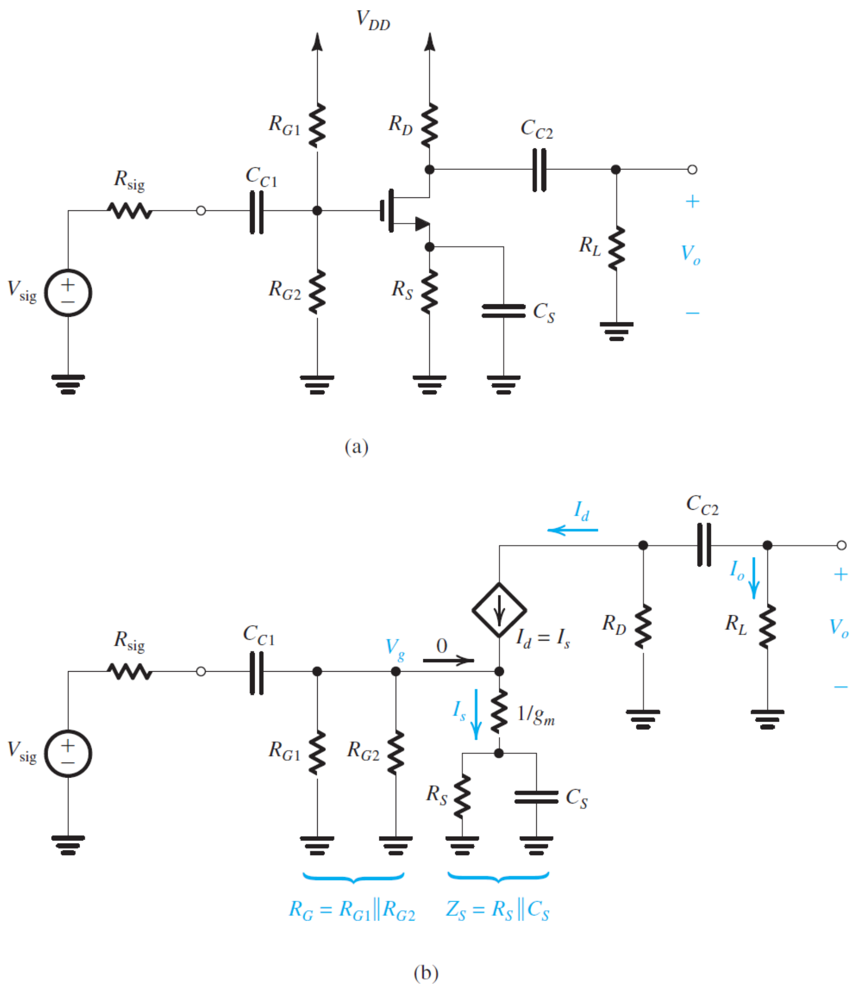

Figure 3: (a) A discrete-circuit common source (CS) amplifier. (b) The small-signal model by with the coupling and bypass capacitor in place (Courtesy of Sedra and Smith).

2.1 The CS Amplifier

Figure 3(a) shows a CS (common source) amplifier with coupling capacitors and bypass capacitors. First, we replace the discrete-circuit amplifier with its small-signal model, but keeping the capacitors in the circuit as shown in Figure 3(b). To determine the amplifier gain V_o/V_{sig}, we write it as

$$\frac{V_o}{V_{\text{sig}}} = \left(\frac{V_g}{V_{\text{sig}}}\right)\left(\frac{I_d}{V_g}\right)\left(\frac{V_o}{I_d}\right) \tag{2.1}$$

1. Fist we invoke the voltage-divider formula which also works for complex impedances. To this end, writing $R_G = R_{G1} \parallel R_{G2}$, one gets

$$\frac{V_g}{V_{\text{sig}}} = \frac{R_G}{R_G + \frac{1}{sC_{C1}} + R_{\text{sig}}} = \frac{R_G}{R_G + R_{\text{sig}}}\left(\frac{s}{s + \frac{1}{C_{C1}(R_G+R_{\text{sig}})}}\right) \tag{2.2}$$

   In the above, s is the Laplace transform variable, and is related to the frequency ω by $s = j\omega$. Thus, one can see that a pole exists at in the complex ω plane at

$$\omega_{P1} = j/[C_{C1}(R_{\text{sig}} + R_G)] \tag{2.3}$$

   and there is a zero at $\omega = s = 0$. This is due to the DC blocking effect of the coupling capacitor.

2. In addition, the drain current I_d and the source current I_s are the same, namely,

$$I_d = I_s = \frac{V_g}{\frac{1}{g_m} + Z_S} = V_g \frac{g_m Y_S}{g_m + Y_S} \tag{2.4}$$

   where Z_S is the source impedance, which is the analog of the source resistance in previous discrete model in Chapter 7, and Y_S is the source admittance. The above yields I_d/V_g, an important factor in (2.1). Hence,

$$Y_S = \frac{1}{Z_S} = \frac{1}{R_S} + sC_S \tag{2.5}$$

   Hence, by letting $s = j\omega$ again, the bypass capacitor introduces a pole in the complex ω-plane at

$$\omega_{P2} = j\frac{g_m + 1/R_S}{C_S} \tag{2.6}$$

   A zero occurs at

$$s_Z = -\frac{1}{C_S R_S} \tag{2.7}$$

   or at

$$\omega_Z = \frac{j}{C_S R_S} \tag{2.8}$$

   Usually, $|\omega_{P2}| \gg |\omega_Z|$.

3. To complete the analysis, using the current-divider formula,

$$I_o = -I_d \frac{R_D}{R_D + \frac{1}{sC_{C2}} + R_L} = -I_d \frac{R_D}{R_D + R_L} \left(\frac{s}{s + \frac{1}{C_{C2}(R_D+R_L)}} \right) \tag{2.9}$$

Since $V_o = I_o R_L$, one gets, from (2.9), that

$$\frac{V_o}{I_d} == R_L \frac{I_o}{I_d} = -\frac{R_D R_L}{R_D + R_L} \frac{s}{s + \frac{1}{C_{C2}(R_D+R_L)}} \tag{2.10}$$

A pole is introduced at

$$\omega_{P3} = \frac{j}{C_{C2}(R_D + R_L)} \tag{2.11}$$

while a zero occurs at $s = 0$.

In aggregate,

$$\frac{V_o}{V_{\text{sig}}} = A_M \left(\frac{s}{s + \omega_{P1}} \right) \left(\frac{s + \omega_Z}{s + \omega_{P2}} \right) \left(\frac{s}{s + \omega_{P3}} \right) \tag{2.12}$$

where

$$A_M = -\frac{R_G}{R_G + R_{\text{sig}}} g_m \left(R_D \parallel R_L \right) \tag{2.13}$$

Here, A_M is the overall midband gain.

2.2 Bode Plots

To simplify the plotting of the frequency response, it is best to do it with Bode plots. They are plots with log-log plots or dB versus log of the frequency. Then amplitudes are converted to dB with the formula the

$$\text{GAIN in dB} = 20 \log_{10}(\text{GAIN}) \tag{2.14}$$

To this end, one takes the log of (2.12) to arrive at that

$$\log \left| \frac{V_o}{V_{\text{sig}}} \right| = \log |A_M| + \log \left| \left(\frac{s}{s + \omega_{P1}} \right) \right| + \log \left| \left(\frac{s + \omega_Z}{s + \omega_{P2}} \right) \right| + \log \left| \left(\frac{s}{s + \omega_{P3}} \right) \right| \tag{2.15}$$

One sees that product of three functions now becomes sum of three functions in the logarithm world. Therefore, if one can figure out the salient feature of the plot of each of the terms, then one can add them up to see the aggregate plot.

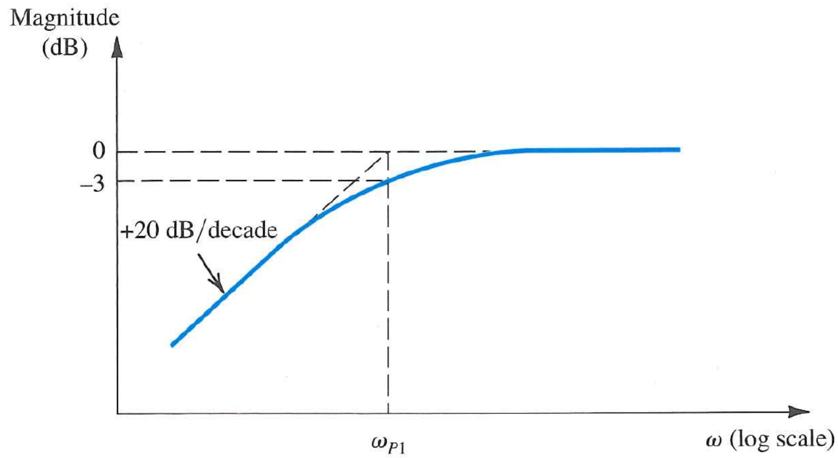

Figure 4: Bode plot of the function $\frac{s}{s+\omega_{P1}}$ (Courtesy of Sedra and Smith).

For simplicity, we look at one of the functions in (2.12), the function $f_1(s) = \frac{s}{s+\omega_{P1}}$. One sees that $f_1(s) \to 1$ as $s \to \infty$. Thus, on a log scale, it tends to zero as $s \to \infty$ or $\omega \to \infty$ since $s = j\omega$. Moreover, $f_1(s) \sim s/\omega_{P1}$ as $s \to 0$. On a dB-log scale, we have in dB, $|f_1(s)|_{\text{dB}} \sim 20\log_{10}|\omega/\omega_{P1}|$, $\omega \to 0$, after letting $s = j\omega$, meaning that it has a slope of 20 on a dB versus log-frequency plot as shown in Figure 4.

One can see that when $\omega = \omega_{P1}$, or $s = j\omega_{P1}$, then

$$|f_1(s = j\omega_{P1})| = \left|\frac{j\omega_{P1}}{j\omega_{P1} + \omega_{P1}}\right| = \left|\frac{j}{j+1}\right| = \frac{1}{|j+1|} = \frac{1}{\sqrt{2}} \tag{2.16}$$

The above implies that

$$|f_1(s = j\omega_{P1})|^2 = \frac{1}{2} \tag{2.17}$$

or that at $\omega = \omega_{P1}$, it is the half-power point or on the dB scale, the 3 dB point.

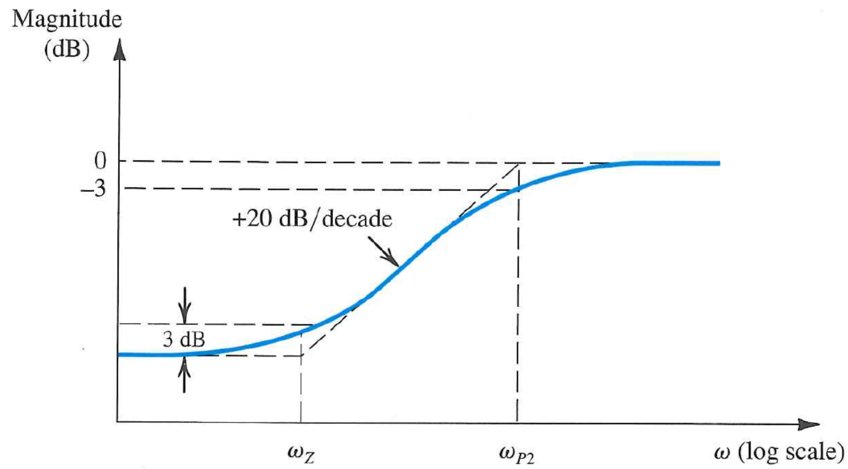

Figure 5: Bode plot of the function $\frac{s+\omega_Z}{s+\omega_{P2}}$ (Courtesy of Sedra and Smith).

By the same token, it can be shown that the Bode plot of the function $\frac{s+\omega_Z}{s+\omega_{P2}}$ is as shown in Figure 5. If the two points ω_Z and ω_{P2} are far apart, one can see that the turning points or the 3 dB points are as shown in Figure 5 connected by the 20 dB/decade straight line.

Given the characteristic of the Bode plot of a simple rational function, the aggregate Bode plot can be obtained by adding the individual plots together as shown in Figure 6. The Bode plots can be approximated by piecewise constant functions when we add them. This is possible when the poles and zeros are far apart.

It is to be noted that if we add two straight line plots, $y_1 = a_1x + c_1$ and $y_2 = a_2x + c_2$, then the resultant plot of $y = y_1 + y_2 = (a_1 + a_2)x + c_1 + c_2$. Hence, one just adds the slopes of the two straight line plots in Bode plots.

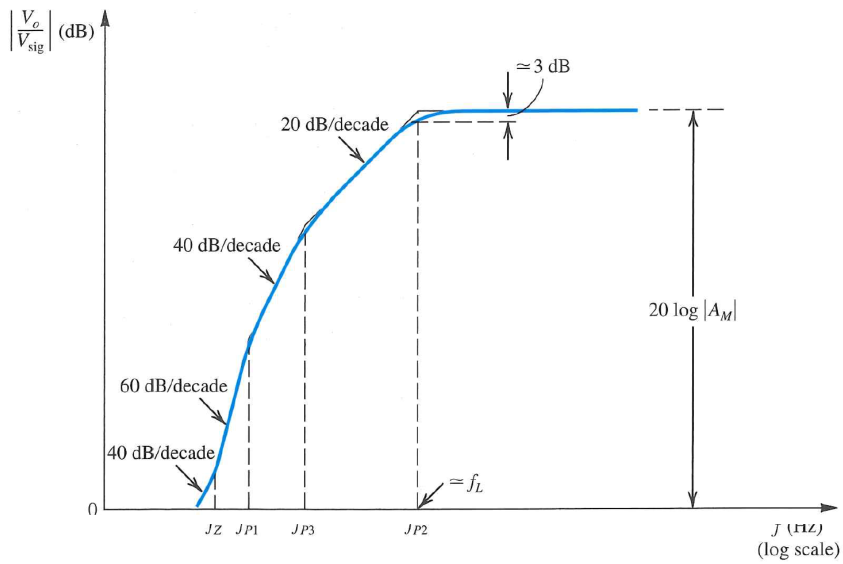

Figure 6: Additive sum of the Bode plot of a rational function with three poles and one zero away from the origin (Courtesy of Sedra and Smith).

2.2.1 Rational Functions, Poles and Zeros

In general, the gain function is a rational function of s. A rational function is a function which is the ratio of two polynomial functions. By simple expansion,

$$(s - s_1)(s - s_2)\ldots(s - s_N) = a_0 + a_1 s + \ldots + a_N s^N \tag{2.18}$$

which is a polynomial. The above also implies that an N-th order polynomial (right-hand side) can be written in a factorized form (left-hand side) in terms of factors of the roots. It also says that an N-th order polynomial has N zeros. In other words, a rational function of two polynomials, an M-th order polynomial in the numerator, and N-th order polynomial in the denominator, will have M zeros and N poles. More important, it turns out that many functions can be well approximated by rational functions.

2.2.2 The Meanings of the Zeros and Poles

The deeper meanings of the zeros and poles can be obtained by looking at (2.12) more carefully. At the zeros of the right-hand side, it is seen that $V_o = 0$ when $V_{\text{sig}} \neq 0$. It implies that no signal can be sent from the input to the output. This happens when the coupling capacitors have infinite impedances, which happens at $s = 0$ for both C_{C1} and C_{C2}. In other words at $\omega = 0$ they are open circuited, and the gain of the amplifier is zero.

The other zero is obtained when $Z_S = \infty$ and is precisely what happens when $\omega = \omega_Z$ in (2.8). At this frequency, $I_S = 0$, because of the infinite source impedance. When this is the case, the gain of the amplifier will again be zero.

To understand the poles, note that at these frequencies, the right-hand side of (2.12) becomes infinite. That means that $V_o \neq 0$ even if $V_{\text{sig}} \to 0$. Or non-zero signal exists in the circuit even if $V_{\text{sig}} = 0$. This is possible if the capacitors are charged, and their charges are relaxing or discharging via the resistors connected to them even when $V_{\text{sig}} = 0$. These are called the **natural solutions** or the **resonant solutions** of a system. In mathematics, these are the **homogeneous solutions** of a system of equations.

2.2.3 The Relaxation Times

When a capacitor C is connected in series to a resistor R, the charges in the capacitor relaxes via the time constant $\tau = RC$, or its RC time constant. In other words, charges in the capacitor decay according to $\exp(t/\tau) = \exp(t/(RC))$.

The relaxation solutions are depicted in Figure 7. There are three such solutions for this case, each associated with the relaxation of each capacitor. Hence, in this circuit that is being studied, the number of poles is equal to the number of capacitors in the circuit.

If only C_{C1} is charged and the other capacitors are uncharged, then the only way for the capacitor C_{C1} to discharge is via the series connected resistors R_{sig} and R_G. Hence the relaxation time is as such, and that is the location of the pole ω_{P1} as shown in (2.3). This discharging circuit is shown in Figure 7(a). The capacitor can be viewed to be connected in series with the resistors.

If only C_S is charged, then $V_g = 0$ always since no current can flow through the node at V_g, and the other end of the $1/g_m$ resistor is connected to a virtual ground. Then C_S is discharging via the circuit shown in Figure 7(b), giving rise to the second pole ω_{P2} as given (2.6).

If only C_{C2} is charged, then it can only discharge via the circuit shown in Figure 7(c), giving rise to the third pole ω_{P3} shown in (2.11). Again, the capacitor is discharging via the resistors that are connected in series to it.

These relaxation solutions can happen even if $V_{\text{sig}} = 0$, and are the natural solutions of the system. Also, one notes that the relaxation happens when the capacitor is connected in series to an resistor. At the relaxation frequency, the impedance of the series connection loop is zero, namely, $1/(sC) + R = 0$. This is also the equation for the pole location.

2.2.4 Selecting the Bypass and Coupling Capacitors

The relaxation time of the circuit is determined by the RC time constants. Since usually, C_S sees the smallest resistor which is $(1/g_m \parallel R_S)$, and hence it has the smallest time constant or the highest frequency pole, it usually determines f_L in the discrete circuit. Then the coupling capacitors C_{C1} and C_{C2} are picked to make the other poles far enough so that the simple analysis previously discussed can be used.

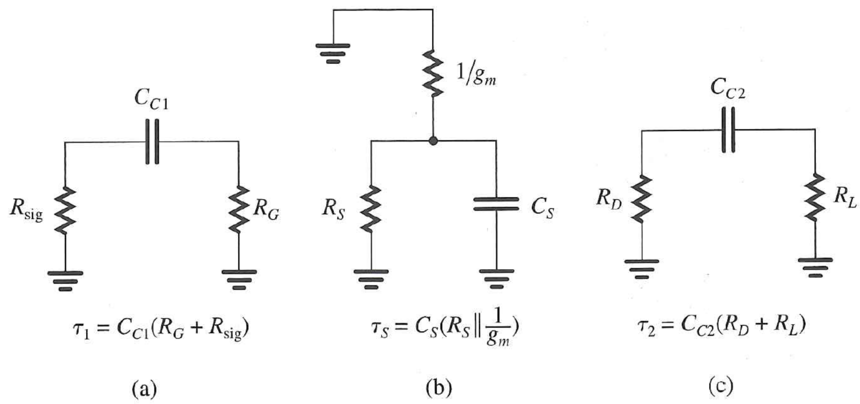

Figure 7: The three circuits that determine the relaxation time associated with the three capacitors (Courtesy of Sedra and Smith).

2.3 The Method of Short-Circuit Time Constants

In the previous example, the locations of the poles are simple because the three relaxation circuits are independent of each other. In other words, there is no coupling between these relaxation circuits. In general, finding the relaxation time of each capacitor can be rather complicated, because each of them may see a highly complex circuit and there could be inter-capacitor coupling in their relaxation times.

The poles of the circuit correspond to solution that exists when $V_{\text{sig}} = 0$. Then focusing on one capacitor, one looks at the impedance seen connected in series to this one capacitor. And the nonzero current solution that can exists in this loop is when

$$1/(sC) + Z(s) = 0 \tag{2.19}$$

where $Z(s)$ is the impedance seen by the capacitor. In other words, a current can flow in this loop without a voltage source since the impedance is zero. Usually, $Z(s)$ is a complicated function of s, and hence, solving the above equation is non-trivial. Here, $Z(s)$ can be found by the test current method, for example.

But one is usually interested in determining f_L, which corresponds to the highest-frequency pole of the system. Then one can make the assumption that at this highest frequency f_L, all the capacitors are short-circuited except for the capacitor of interest. This approximation definitely removes the inter-capacitor coupling. It is seen that at the highest frequency pole, the approximation is a good one. One then steps through the capacitors and find their relaxation frequencies that are associated with them with this approximation. Even though at the other lower frequency poles, the approximation is not a good one, it is in practice to calculate f_L using

$$\omega_L \approx \sum_{i=1}^{n} \frac{1}{C_i R_i} \tag{2.20}$$

The above sum will be dominated by the smallest relaxation time or the highest frequency pole, which is best approximated by the short-circuit time-constant calculation.

2.4 The CE Amplifier

The way to analyze MOSFET discrete-circuit at the low frequency end can be used to analyze a BJT discrete-circuit amplifier. This circuit is shown in Figure 8.

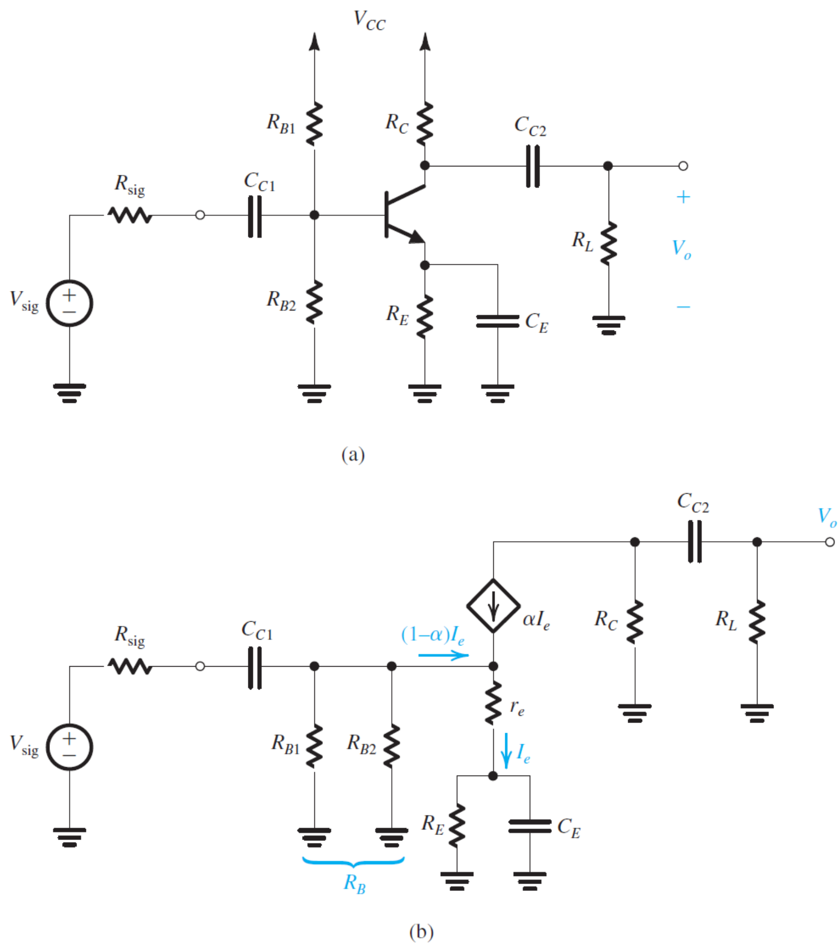

Figure 8: (a) A discrete-circuit common-emitter amplifier. (b) The corresponding small-signal equivalent circuit (Courtesy of Sedra and Smith).

The analysis is very similar to before, except that there is a difference when one finds the relaxation times of the capacitors. Here, the relaxation circuits cannot be easily divided into three independent ones, because of the inter-coupling effect between the capacitors C_{C1} and C_E. To remove the coupling, the short-circuit method is used to find their time constants. These relaxation circuits are shown in Figure 9.

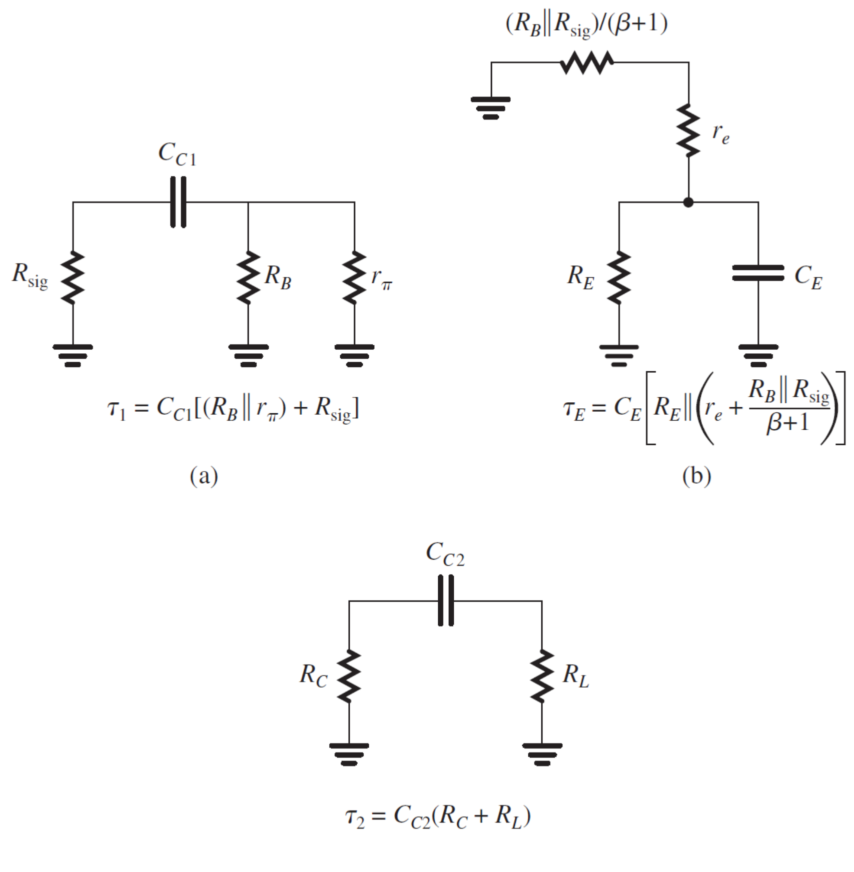

Figure 9: Circuits for determining the short-circuit time constants for the CE amplifier in this case (Courtesy of Sedra and Smith).