A Ray-Propagation Fast Multipole Algorithm†

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ABSTRACT

A new technique is presented for accelerating the fast multipole method, allowing rapid solution of surface integral equations for wave scattering problems. A non-nested, ray propagation approach is used to compute a matrix-vector multiply in $O(N^{4/3})$ operations, where N is the number of unknowns in the discretized integral equation.

I. INTRODUCTION

Recently, many researchers have been investigating the use of radiation beam ideas to extend classical integral equation techniques for solution of wave scattering problems into the high frequency regime [1-3]. Here, we use a "ray" propagation approach to accelerate the fast multipole method (FMM) of Rokhlin [4-8]. This new technique is discussed briefly in [9]. A more complete treatment is given in this letter. The FMM allows the rapid solution of surface integral equations arising in electromagnetics. In this approach, the integral equation is discretized using, for

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example, the method of moments (MOM), then solved using an iterative method, such as conjugate gradient. The main bottleneck in the computation is the cost of computing a matrix-vector multiply. A standard matrix-vector multiply for a problem with N unknowns has a cost of $O(N^2)$. A straight-forward application of the FMM reduces this cost to $O(N^{3/2})$. A two-level nested application further reduces the cost to $O(N^{4/3})$ [4]. Here we present a *non-nested* method, using a ray propagation approach, to reduce the cost of a FMM matrix-vector multiply to $O(N^{4/3})$. We call this the ray-propagation fast multipole algorithm (RPFMA).

II. SCATTERING COMPUTATIONS USING THE FMM

Here, we consider the problem of computing the scattering of a TM (E_z) polarized electromagnetic wave from a two-dimensional conducting cylinder with boundary contour C. The incident wave E_z^{inc} excites a surface current J_z on the surface of the cylinder. Enforcing the boundary condition that the total field E_z must vanish on C yields the electric field integral equation (EFIE):

$$E_z^{inc}(\mathbf{r}) = -i\omega\mu \int_C dl' J_z(\mathbf{r}')g_0(\mathbf{r},\mathbf{r}'), \qquad \mathbf{r} \in C, \tag{1}$$

where g_0 is the two-dimensional Green's function $[g_0(\mathbf{r},\mathbf{r}') = (i/4)H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|)]$, and k is the wavenumber.

Because the EFIE is ill conditioned at the resonant frequencies of the cylinder C, we make use of the combined field integral equation (CFIE), which is free of resonance problems. The CFIE may be derived by applying the differential operator $P(\mathbf{r}) = 1 + \alpha\hat{n}\cdot\nabla$ to both sides of Equation (1), yielding

$$E_z^{inc}(\mathbf{r}) + \alpha\frac{\partial E_z^{inc}(\mathbf{r})}{\partial n} = -i\omega\mu \int_C dl' J_z(\mathbf{r}')[g_0(\mathbf{r},\mathbf{r}') + \alpha\frac{\partial g_0(\mathbf{r},\mathbf{r}')}{\partial n}], \qquad \mathbf{r} \in C. \tag{2}$$

The constant α is the CFIE combination parameter.

Discretizing Equation (2) using a pulse basis for J_z and point matching yields the MOM matrix formulation

$$\mathbf{e} = \mathbf{Z}\cdot\mathbf{j}, \tag{3}$$

where $\mathbf{Z}$ is a full non-symmetric matrix with elements which can be approximated as

$$Z_{ij} = \frac{\omega\mu}{4}\begin{cases} \triangle_j\left[1 + i\frac{2}{\pi}\ln\left(\frac{e^{\gamma}k\triangle_j}{4e}\right)\right] - 2i\alpha, & i = j \\ \triangle_j\left[H_0^{(1)}(k|\mathbf{r}_i-\mathbf{r}_j|) - \alpha k(\hat{n}_i\cdot\frac{\mathbf{r}_i-\mathbf{r}_j}{|\mathbf{r}_i-\mathbf{r}_j|})H_1^{(1)}(k|\mathbf{r}_i-\mathbf{r}_j|)\right], & i \neq j, \end{cases} \tag{4}$$

where $\gamma = 0.5772$ is Euler's constant and $\triangle_i$, $\mathbf{r}_i$, and $\hat{n}_i$ are the width, center position, and outgoing normal to C, respectively, for the ith pulse.

Because the matrix $\mathbf{Z}$ is dense, Equation (3) is expensive to solve. If a direct solution (such as LU decomposition or Gaussian elimination) is used, the cost will be $O(N^3)$, where N is the number of unknowns in the expansion of the surface current J_z. If an iterative method such as conjugate gradient is used, the cost will be $O(N^2)$ per iteration, which is the cost of a matrix-vector multiply. The goal of the FMM is to reduce the cost of a matrix-vector multiply, thus speeding the iterative solution of Equation (3).

As discussed in [6], the FMM provides a sparse decomposition of the dense MOM impedance matrix $\mathbf{Z}$:

$$\mathbf{Z} = \mathbf{Z}' + \mathbf{UTV}, \tag{5}$$

where $\mathbf{Z}'$, $\mathbf{U}$, $\mathbf{T}$, and $\mathbf{V}$ are all sparse matrices. The matrix $\mathbf{Z}'$ is the part of $\mathbf{Z}$ that represents the interaction between near neighbors, as given by Equation (4). The matrix $\mathbf{UTV}$ is the FMM decomposition of the remainder of $\mathbf{Z}$. The physical interpretation of this decomposition is discussed below. Detailed derivation of this decomposition is presented in [4] and [8].

The matrix $\mathbf{V}$ is a block diagonal matrix which transforms the currents in a group of M adjacent current elements (MOM basis functions) into partial field components radiated in a set of Q directions. In the two-dimensional case, these directions are

$$\phi_q = \frac{2\pi(q-1)}{Q}, \qquad q = 1, \ldots, Q. \tag{6}$$

This representation of the scattered field converges in $Q \sim M$ terms [4] (this also follows from the Nyquist theorem).

The matrix $\mathbf{T}$ translates the Q terms described above from the transmitting group to a receiving group elsewhere on the surface of the scatterer. For each pair of groups, the translation operator is diagonal. That is, there is no interaction between partial field components in different directions ϕ_q. For this reason, $\mathbf{T}$ is sparse.

Finally, the block diagonal matrix $\mathbf{U}$ transforms the total field received by a group from the partial fields in Q directions into the spatial domain field on the surface of the M elements of the group. This is illustrated in Figure 1.

The cost of a matrix-vector multiply using this decomposition of $\mathbf{Z}$ is easily computed as follows. The current pulses are grouped together in groups of M adjacent elements. The radiation from each group is expressed in terms of a set of Q directions, where $Q \sim M$. Let $L = N/M$ denote the total number of groups. For each group, the block diagonal matrix $\mathbf{V}$ transforms M currents into Q directions, so $\mathbf{V}$ consists of L submatrices of size $Q \times M$. The cost of multiplying $\mathbf{V}$ with a vector is thus LQM. Similarly, $\mathbf{U}$ is a block diagonal matrix which essentially reverses this process, transforming the Q partial received fields into spatial domain fields at M observation points within each group, at a total cost of LQM. For each pair of groups, $\mathbf{T}$ propagates the Q partial fields from the transmitting to the receiving group. As mentioned above, for each pair of groups $\mathbf{T}$ is diagonal, with no coupling between partial field components in different directions, so the cost per group is Q, and the total cost of applying $\mathbf{T}$ is L^2Q. Finally, $\mathbf{Z}'$ transforms the $O(M)$ currents in a group and its near neighbors into the received field at M points within that group. Carrying this out for all L groups has a cost of $O(LM^2)$. Thus, the total cost of a matrix-vector multiply using the FMM decomposition is

$$\text{Cost} = C_1 \times LQM + C_2 \times L^2Q + C_3 \times LM^2. \tag{7}$$

Using the fact that $Q \sim M$ and $L = N/M$, this is

$$\text{Cost} = C_4 \times NM + C_5 \times N^2/M. \tag{8}$$

The above cost is minimized by the choice $M \sim N^{1/2}$, yielding a total cost of $O(N^{3/2})$ for a FMM matrix-vector multiply. As discussed in [4,5], a two-level nested application of the FMM can reduce this cost to $O(N^{4/3})$. However, we have found that this cost can also be reduced to $O(N^{4/3})$ using a simple, non-nested, ray propagation approach.

III. RAY PROPAGATION FAST MULTIPOLE ALGORITHM (RPFMA)

Here we focus on the matrix $\mathbf{T}$ which translates the field radiated from a transmitting group in some direction ϕ_q into the received field component in the same direction at a receiving group. Intuitively, one expects the interaction to be strongest for fields radiated along the line joining the transmitting and receiving groups. This is indeed the case, and this forms the basis for our approach to accelerating the FMM.

Taking the transmitting group to be l and the receiving group to be l', the element of $\mathbf{T}$ which translates the partial field in direction ϕ_q from l to l' is given by

$$(9) \qquad T_{l',l}(\phi_q) = \sum_{p=-Q/2}^{Q/2} H_p^{(1)}(kr_{l,l'})e^{-ip(\phi_{l,l'}-\phi_q-\pi/2)},$$

where k is the wavenumber, $r_{l,l'}$ is the distance between groups l and l', and $\phi_{l,l'}$ is the angle between the groups, with respect to the x axis. The geometry is illustrated in Figure 1. Using the far-field approximation for the Hankel functions above yields a simpler form which offers more insight into $\mathbf{T}$:

$$(10) \qquad T_{l',l}(\phi_q) \approx f(r_{l,l'}) \cdot \frac{\sin[(Q+1)(\phi_q - \phi_{l,l'})/2]}{\sin[(\phi_q - \phi_{l,l'})/2]}.$$

Thus, the interaction between the two groups as a function of angle ϕ_q takes a form similar to a linear antenna array with $Q+1$ elements and a main beam centered at $\phi_q = \phi_{l,l'}$. The angular dependence of Equation (10) is illustrated in Figure 2. The key observation is that most of the interaction between the two groups occurs in the main beam of this array factor. For a certain accuracy of computation, many side lobes can be ignored, reducing the cost of applying the matrix $\mathbf{T}$. To apply this idea computationally a relative threshold ϵ is selected, and for each pair of groups the elements of the associated array factor which are less than ϵ times the maximum value are set to zero.

To take full advantage of this idea, it is also important to note that Equation (9) is the truncation of an infinite summation [8]. From Equation (9), it can be seen that $T_{l',l}(\phi_q)$ is actually equivalent to a truncated Fourier series or a Fourier series computed with a square window in the variable p. It is this square window that gives rise to the sinc function in Equation (10). However, it is equally valid to use another window function which makes a smooth transition from zero to one and back to zero. The advantage of such a window is that its Fourier transform will have lower sidelobes than the square window used in Equation (9). These lower sidelobes will allow the discarding of more elements of the array factor for a given threshold level, producing greater sparsity in $\mathbf{T}$. This is similar to the use of a windowed addition theorem for cylindrical harmonics discussed in [10]. The window used has a broad flat middle section centered around $p = 0$ and has a cosine taper on both ends [10].

Thus, denoting the window function by w_p, the elements of $\mathbf{T}$ are calculated as

$$T_{l',l}(\phi_q) = \sum_{p=-Q/2}^{Q/2} w_p H_p^{(1)}(kr_{l,l'}) e^{-ip(\phi_{l,l'}-\phi_q-\pi/2)}. \tag{11}$$

This expression is not based on the far-field approximation used to obtain Equation (10), and so is valid in both the far- and near-field regions.

As the problem size grows, the width of the main beam joining two groups decreases as $1/M$, while the angular sampling density Q grows as M. Thus, the number of directions which lie in the main beam (and significant sidelobes) remains approximately constant as a function of problem size. Therefore, the number of partial field components which need to be translated between two groups is independent of problem size, rather than growing as M in the original FMM. These field components form a "ray" which propagates the field from the transmitting to the receiving group.

Using this ray propagation idea, the cost of applying the matrix $\mathbf{T}$ is reduced from L^2Q to $O(L^2)$. Since the cost of applying $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{Z}'$ is unchanged, the total cost of a FMM matrix-vector multiply (Equation (8)) becomes

$$\text{Cost} = C_4 \times NM + C_6 \times N^2/M^2. \tag{12}$$

This cost is minimized when $M \sim N^{1/3}$, and the resulting cost of a matrix-vector multiply is reduced to $N^{4/3}$.

IV. RESULTS

Figure 3 shows the result of applying this technique to compute the scattering of an E_z polarized plane wave from a conducting ogive of length 18λ and half-height 3λ. In this case, about half the elements in the translation matrix $\mathbf{T}$ were discarded. The result shows good agreement with an independent MOM calculation of the scattering.

The computational complexity of the method is demonstrated in Figure 4, which shows the cost of a matrix-vector multiply for computing the scattering from a circular cylinder, as the size of the cylinder is increased. For both the FMM and RPFMA calculations, the accuracy of the final solution was held near the 1% level. The data clearly indicates the reduced computational complexity of the new approach.

Additional computer results indicate that the accuracy of this method can be controlled by adjusting the threshold level at which side lobes are discarded from the computation. Figure 5 shows relative residual error as a function of threshold parameter ϵ for E_z scattering from a circular cylinder of radius 12λ. The relative residual error is defined as $||\mathbf{Z} \cdot \mathbf{j} - \mathbf{e}||/||\mathbf{e}||$, where $\mathbf{j}$ is the solution to Equation (3) obtained using the technique described here, and $\mathbf{Z}$ is the original MOM matrix defined in Equation (4). The data shows that the error decreases smoothly as the threshold ϵ is decreased. As ϵ approaches zero, the error is reduced to that of the original FMM and conjugate gradient solver.

IV. CONCLUSIONS

We have presented the RPFMA, a modification of the FMM for 2-D scattering which reduces the cost of a matrix-vector multiply from $O(N^{3/2})$ to $O(N^{4/3})$ using a non-nested, ray propagation approach. The accuracy and efficiency of the new algorithm have been demonstrated through computer simulations of E_z scattering from metallic cylinders. We hope this simple approach may be useful in itself, and may also lead to further development of fast scattering techniques.

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FIGURE CAPTIONS

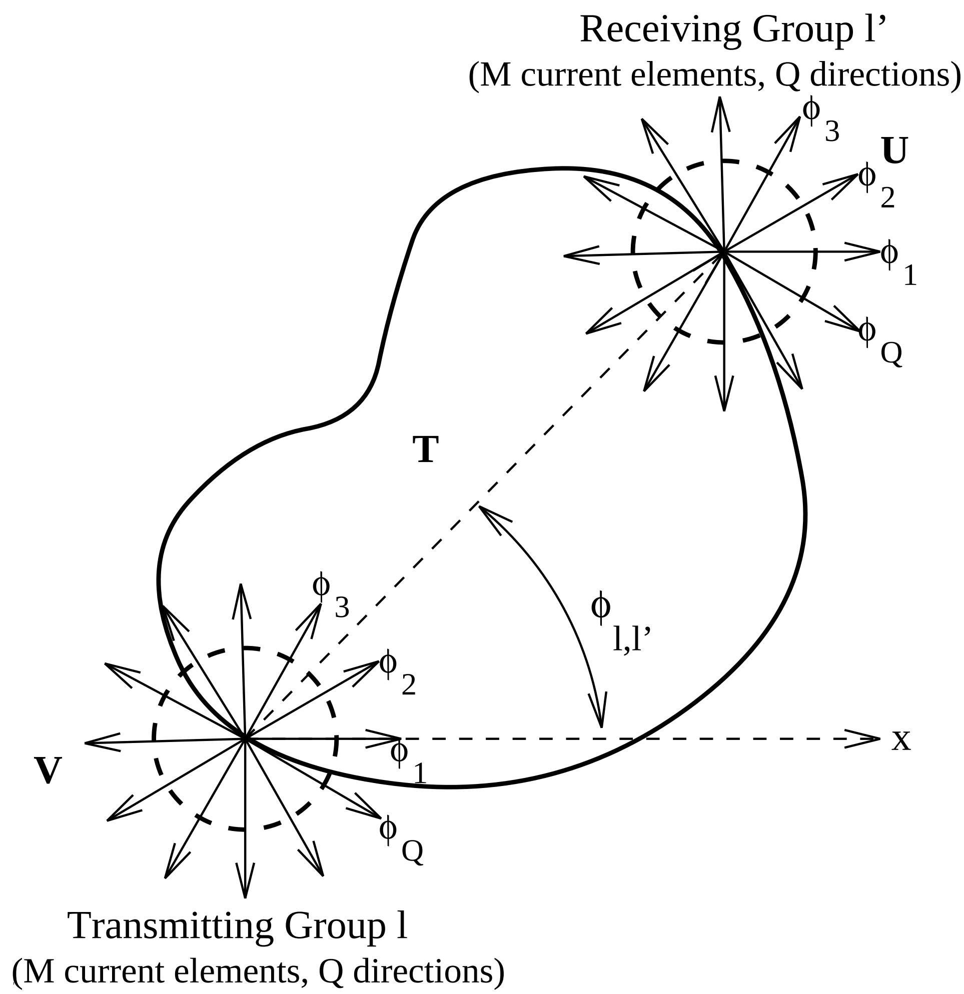
Figure 1. Physical interpretation of the FMM matrix decomposition. The matrix $\mathbf{V}$ transforms the currents in transmitting group l into the radiated field components in directions ϕ_q, $\mathbf{T}$ propagates the field components to the receiving group l', and $\mathbf{U}$ transforms the received field components into the spatial domain field on the surface of the scatterer inside group l'.

Figure 2. Angular dependence of the array factor of Equation (10), for the case $Q = 60$. The angle in the plot is $(\phi_q - \phi_{l,l'})$.

Figure 3. Bistatic RCS of a metallic ogive of length 18λ and half-height 3λ for E_z polarization. The incident wave is at $90°$, normal to the long axis of the ogive. The scatterer is discretized with 10 current pulses per wavelength.

Figure 4. Comparison of CPU time required for a matrix-vector multiply, for varying problem size. The matrix here is for computing the scattering of an E_z polarized wave from a metallic circular cylinder. The cylinder is discretized with 10 current pulses per wavelength. Times are for a SPARC 10 workstation.

Figure 5. Relative residual error versus threshold ϵ for E_z scattering from a metallic circular cylinder of radius 12λ.



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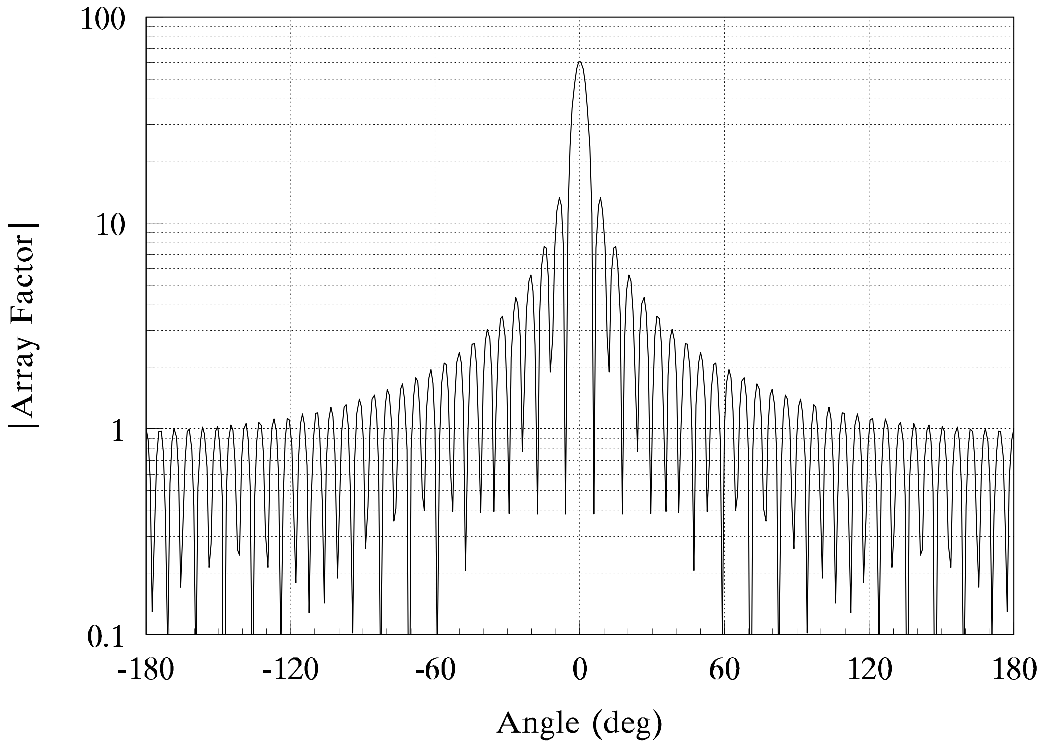

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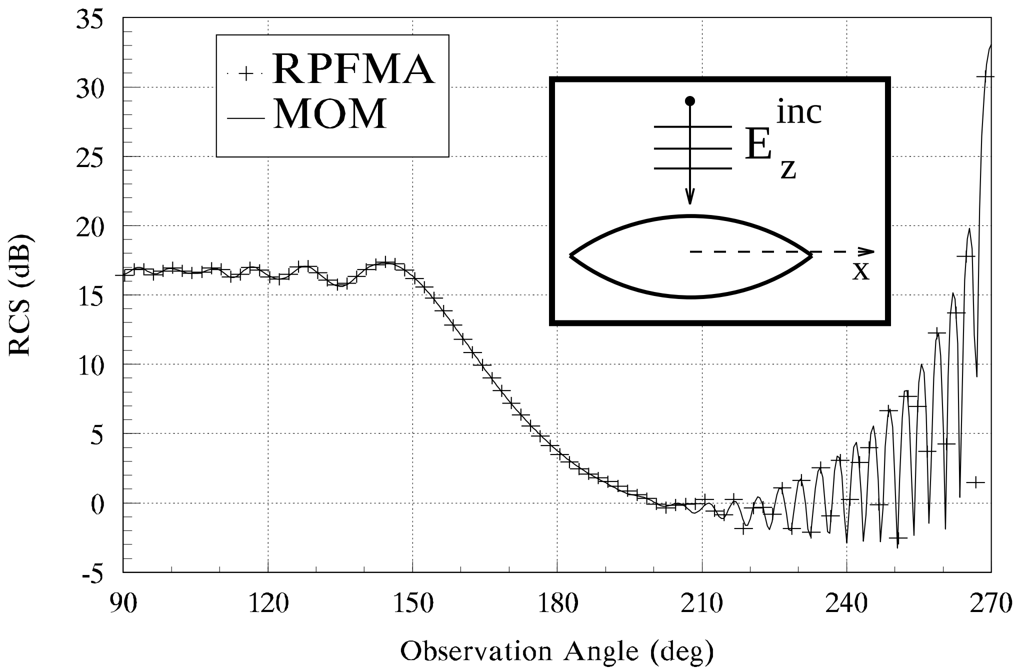

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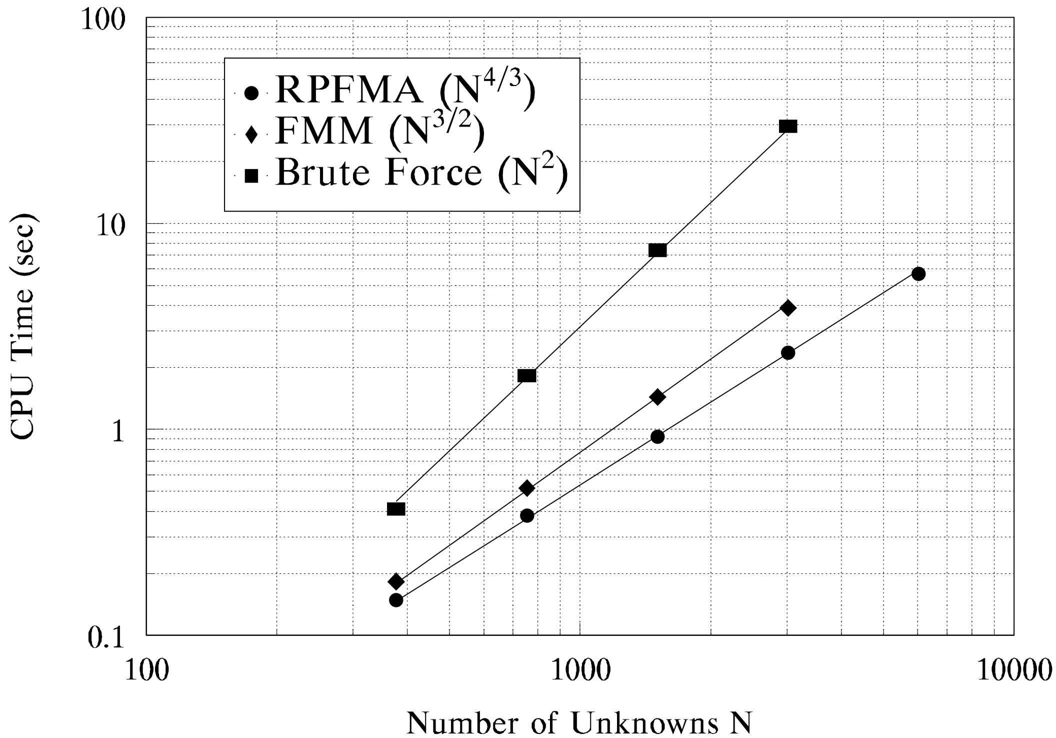

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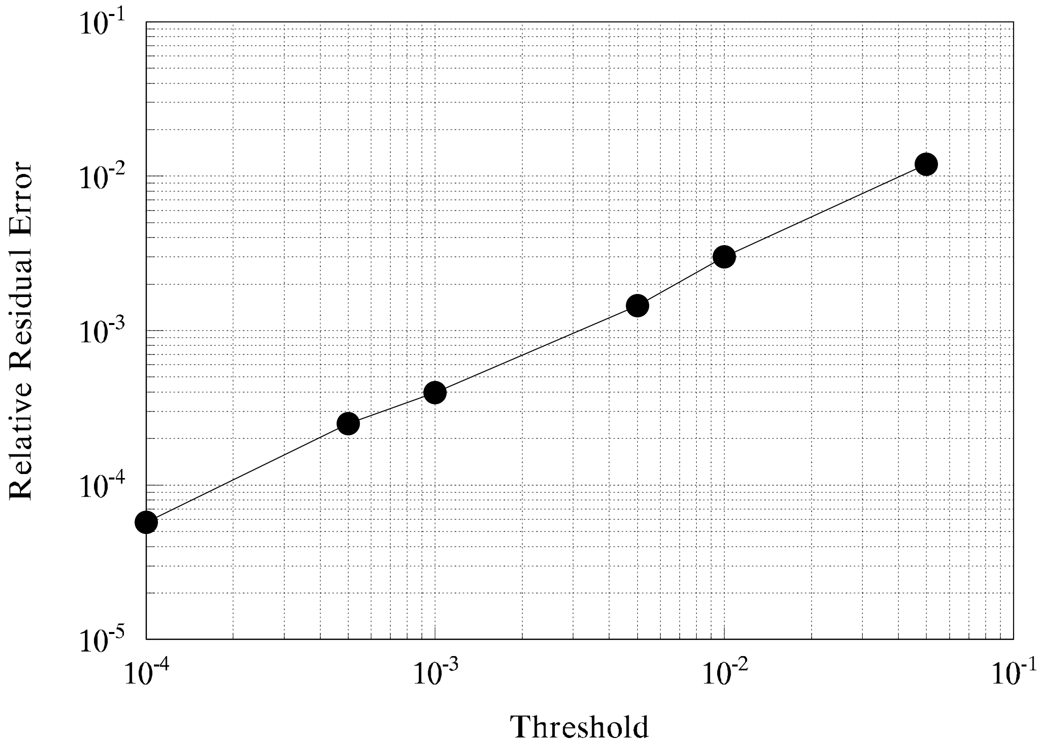

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