# An FFT T-Matrix Method for 3D Electromagnetic Scattering Solutions from Random Discrete Scatterers †

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**Abstract**

A method to solve efficiently the scattering by a cluster of randomly located discrete scatterer is described. The spheres could have an arbitrary shape, and only their T matrices need be known. An iterative method is suggested to solve this problem. To expedite The matrix-vector multiply, we first aggregate the random scattering centers of the scatterers into centers which reside on a regular array. Then, FFT method is used to perform the matrix-vector multiply in $O(N \log N)$ operations. This method can solve a problem with a large number of unknowns, and involving thousands of discrete scatterers on a workstation.

## 1. Introduction

In this paper, we propose a fast and efficient method of solving the random discrete scatterer problems numerically. We assume that we have a cluster of discrete random scatterers whose T matrices [1-2] exist in closed form or can be computed by other means. A set of linear algebraic equations is easily derived to solve for the scattering amplitudes from the spheres. If instead, the discrete scatterers are randomly distributed, we aggregate the discrete scattering centers to scattering centers which reside on an array using the addition theorem. When the discrete scatterers reside on a regular array, no

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such aggregation is required. As a result, we have a block-Toeplitz matrix structure. An iterative solver such as BiCG (bi-conjugate gradient) method can be used to solve for the solution of the matrix equation. Exploiting the block-Toeplitz structure, we can perform the matrix-vector multiplication in $O(N \log N)$ operations by FFT, where $N$ is the total number of spheres involved. The method requires $O(N)$ memory storage. Hence, it can be used to solve fairly large three-dimensional (3D) problems with as many as 90,000 unknowns on a SPARC 10 workstation. For 3D random discrete scatterers, we have solved a problem with as many as 10,000 scatterers.

## 2. Formulation and Implementation

When a cluster of scatterers are illuminated by an incident wave, the total field can be written as [2]

$$\mathbf{E}(\mathbf{r}) = \boldsymbol{\psi}^t(k_0, \mathbf{r}_s) \cdot \mathbf{a}_s + \sum_{i=1}^{N} \boldsymbol{\psi}^t(k_0, \mathbf{r}_i) \cdot \mathbf{b}_i, \tag{1}$$

where $\mathbf{r}_i = \mathbf{r} - \mathbf{r}_i'$ and $\mathbf{r}_i'$ is the location of the scattering center of the $i$-th scatterer. Here, $\boldsymbol{\psi}^t(k_0, \mathbf{r}_i)$ is a row vector containing the vector spherical harmonics from each scatterer. The vectors $\mathbf{a}_s$ and $\mathbf{b}_i$ contain the amplitudes of the incident field harmonics and the unknown scattered field harmonics from the $i$-th scatterer respectively. We can use the addition theorem or the translation matrix [2] to obtain the equation

$$\mathbf{b}_j = \overline{\mathbf{T}}_{j(1)} \cdot \left[ \overline{\boldsymbol{\alpha}}_{js} \cdot \mathbf{a}_s + \sum_{\substack{i=1 \\ i \neq j}}^{N} \overline{\boldsymbol{\alpha}}_{ji} \cdot \mathbf{b}_i \right], \quad j = 1, \cdots, N, \tag{2}$$

where $\overline{\mathbf{T}}_{j(1)}$ is the T matrix of the $j$-th scatterer. The $\overline{\boldsymbol{\alpha}}_{ji}$ matrices are translation matrices defined in [1,2]. The unknowns $\mathbf{b}_i$ above can be solved iteratively. But the major computational cost will be in the summation above.

If the scatterers are located on a regular array, $\overline{\boldsymbol{\alpha}}_{ji}$ only depends on $\mathbf{r}_j' - \mathbf{r}_i'$ and hence the above summation, which is a matrix-vector multiply, can be cast into a matrix-vector multiply involving a Toeplitz matrix, which can then be expedited using FFT.

If the scatterers are randomly distributed, for every scatterer, we separate the surrounding scatterers into near-neighbor scatterers and far scatterers. To this end, we divide the 3D space into regular boxes, or cells. A scatterer is then assigned to the cell which contains the center of the scatterer. Consequently, we can rewrite Equation (2) as

$$\overline{\mathbf{T}}_{j(1)} \cdot \overline{\boldsymbol{\alpha}}_{js} \cdot \mathbf{a}_s = \mathbf{b}_j - \overline{\mathbf{T}}_{j(1)} \cdot \left[ \sum_{\substack{i \in \mathcal{N}_j \\ i \neq j}} \overline{\boldsymbol{\alpha}}_j \cdot \mathbf{b}_i + \sum_{i \notin \mathcal{N}_j} \overline{\boldsymbol{\alpha}}_{ji} \cdot \mathbf{b}_i \right], \qquad j = 1, \cdots, N, \quad (3)$$

where $\mathcal{N}_j$ is the set of indices of the near-neighbor scatterers of the $j$-th scatterer. These near-neighbor scatterers are either in the same cell to which the $j$-th scatterer belongs, or in its near-neighbor cells.

The first term in the bracket above is the near-neighbor interaction, which can be computed with $O(N)$ operations. The second term accounts for "far" interactions between the scatterers. For a large number of scatterers, this is the most time consuming part of the computation, requiring $O(N^2)$ operations. However, we can expedite the process by rewriting $\overline{\boldsymbol{\alpha}}_{ij}$ using the translational addition theorem [2] as

$$\overline{\boldsymbol{\alpha}}_{ji} = \overline{\boldsymbol{\beta}}_{jm} \cdot \overline{\boldsymbol{\alpha}}_{mn} \cdot \overline{\boldsymbol{\beta}}_{ni} \quad (4)$$

where $m$ is the center of the cell to which scatterer $j$ belongs, and $n$ is the center of the cell to which scatterer $i$ belongs . In this manner, $\overline{\boldsymbol{\alpha}}_{mn}$ depends only on $\mathbf{r}'_m - \mathbf{r}'_n$ where $\mathbf{r}'_m$ and $\mathbf{r}'_n$ are points on a regular grid. Using (4) in (3), we can express the second term in the square bracket in (3) as

$$\sum_{i \notin \mathcal{N}_j} \overline{\boldsymbol{\alpha}}_{ji} \cdot \mathbf{b}_i = \overline{\boldsymbol{\beta}}_{jm} \cdot \sum_{i \notin \mathcal{N}_j} \overline{\boldsymbol{\alpha}}_{mn} \cdot \overline{\boldsymbol{\beta}}_{mi} \cdot b_i, \qquad j = 1, \cdots, N. \quad (5)$$

However, the summation over scatterers not in the near neighbor could be written as a summation over all scatterers $i$ in cell $n$ followed by a summation over different cell $n$ not in the near neighbor of cell $m$ which contains the $j$-th scatterer. In ther words,

$$\sum_{i \notin \mathcal{N}_j} = \sum_{n \notin \mathcal{N}_m} \sum_{i \in \mathcal{C}_n} \quad (6)$$

where $\mathcal{C}_n$ is the set of indices of the scatterer in the $n$-th cell. Consequently, we can rewrite (5) as

$$\sum_{i \notin \mathcal{N}_j} \overline{\boldsymbol{\alpha}}_{ji} \cdot \mathbf{b}_i = \overline{\boldsymbol{\beta}}_{jm} \cdot \sum_{n \notin \mathcal{N}_m} \overline{\boldsymbol{\alpha}}_{mn} \cdot \sum_{i \in \mathcal{C}_n} \overline{\boldsymbol{\beta}}_{ni} \cdot b_i, \qquad j = 1, \cdots, N. \tag{7}$$

The computation of the inner summation for all cells is an $O(N)$ operation. The computation of the second summation, by exploiting the Toeplitz structure of $\overline{\boldsymbol{\alpha}}_{mn}$, can be performed with FFT in $O(N \log N)$ operations. The last operation is also an $O(N)$ operation. Hence, the summation in Equation (3), which constitutes a matrix-vector multiply, can be effected with $O(N \log N)$ operation. This method is similar but not identical to that expressed in [3].

## 3. Results and Conclusions

We have used the method to calculate for the scattering from both inhomogeneous scatterer as well as random distribution of scatterers. An inhomogeneous scatterer is model by small cubes which are in turned modeled by equivolume spheres. The algorithm had been used to calculate the scattering from a dielectric cube involving 90,000 unknowns, but the result was not validated. Figure 1 shows the result of scattering from an inhomogeneously layered sphere involving 42,000 unknowns, which can be validated with analytic result [2] showing good agreement. The calculation converged in 53 iterations and took about 1.5 hours on a SPARC 10 workstation with double precision.

Next, we used this method to calculate the scattering from a random cluster of 10,000 spheres with $ka = 0.1$ and $\epsilon_r = 3.2$. For random clusters, the code was validated with RATMA [1] using several hundred spheres. The scattered field, which takes about 15 min to calculate for each realization, is averaged over 20 different realizations of the random cluster for Monte-Carlo simulation [4]. The simulation validates the QCA-CP theory for small scatterers.

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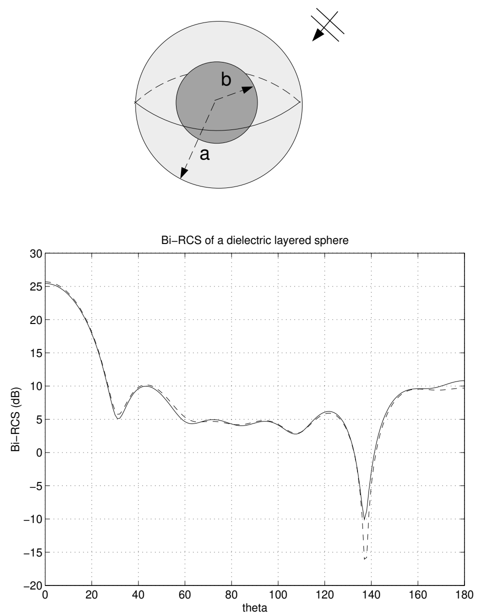


**Figure 1.** The Bi-RCS of a spherically layered scatterer. The solid line is from the Mie series solution; the dash line is the numerical solution. Here, $b = 0.5\lambda_0, \epsilon_{r1} = 1.2, a = 1.0\lambda_0$ and $\epsilon_{r2} = 2.4$. A $30 \times 30 \times 30$ grid is used.

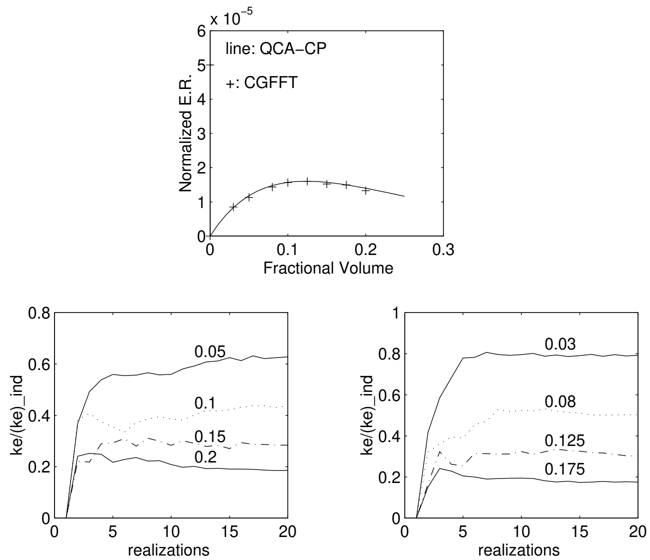


**Figure 2.** The extinction coefficient obtained by using Monte-Carlo simulation of 20 different realizations of 10,000 random spheres, with $ka = 0.1$ and $\epsilon_r = 3.2$.