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§ The Wiener-Hopf Technique as Applied to Mixed Boundary Value Problems

The Wiener-Hopf technique was invented by Wiener and Hopf in 1931 to solve a special type of integral equation. Later, it was noted by Schwinger and Copson that the Sommerfeld's half-plane problem can be formulated in terms of an integral equation solvable by the Wiener-Hopf technique. The technique is also known as the Fock-Wiener-Hopf technique to Russian readers in order not to ignore the important contribution of Fock.

The Wiener-Hopf technique provides a significant and natural extension of the range of problems solvable by the use of Fourier, Laplace, Hankel and Mellin transforms. The technique can be employed to solve many problems with semi-infinite geometry which can be reduced to a two dimensional problem, like diffraction from a set of semi-infinite parallel plates, radiation from an open-end parallel plate waveguide, bifurcated waveguide etc.

In the Wiener-Hopf technique, the aim is to obtain an equation of Wiener-Hopf type. This equation can be obtained from a set of dual integral equations, an integral equation or from the differential equation directly. For our case, we shall derive the dual integral equations first for better physical interpretation and derive the Wiener-Hopf equation from there.

Properties of Fourier Transform

Since the Wiener-Hopf technique makes use of Fourier transform, we shall review some of its properties here relevant to the Wiener-Hopf technique. Given the Fourier transform integrals

$$F(x) = \int_{-\infty}^{\infty} f(k_x)\, e^{ik_x x}\, dk_x \tag{1a}$$

$$f(k_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x)\, e^{-ik_x x}\, dx \tag{1b}$$

we can deduce some analyticity properties of $f(k_x)$ if we know $F(x)$.

(i) If $F(x)$ is a positive semi-infinite function, i.e. nonzero only for $a < x < \infty$, and $|F(x)| \sim O(e^{-xt_1})$ where $x \to \infty$, then $f(k_x)$ is analytic in the region where $\mathrm{Im}(k_x) < t_1$. This follows from the fact that the second Fourier integral (1b) will be well-defined only if $\mathrm{Im}(k_x) < t_1$.

(ii) If $F(x)$ is a negative semi-infinite function, i.e. nonzero only for $-\infty < x < b$, and $|F(x)| \sim O(e^{xt_2})$ when $x \to -\infty$, then $f(k_x)$ is analytic in the region where $\mathrm{Im}(k_x) > -t_2$. The reason being the same as the above.

(iii) If $F(x)$ is a positive semi-infinite function, (without loss of generality, we consider $F(x)$ to be nonzero for $0 < x < +\infty$), and $F(x) \sim A\, x^{\eta}$ when $x \to 0^{+}$, then $f(k_x) \sim O(k_x^{-\eta-1})$ when $k_x \to \infty$ in the region of analyticity implied in (i). This follows from the fact that

$$f(k_x) = \frac{1}{2\pi} \int_0^{\infty} F(x)\, e^{-ik_x x}\, dx \tag{2}$$

By the argument of stationary phase, when k_x tends to infinity in the region where the above integral is defined, most of the contribution to the above integral comes from around $x \simeq 0$. Hence, we can approximate

$$\begin{aligned} f(k_x) &\sim \frac{A}{2\pi} \int_0^{\infty} x^{\eta}\, e^{-ik_x x}\, dx = \frac{A}{2\pi}\, k_x^{-\eta-1} \int_0^{\infty} t^{\eta}\, e^{-it}\, dt \\ &\sim O(k_x^{-\eta-1}), \qquad k_x \to \infty. \end{aligned} \tag{3}$$

(iv) If $F(x)$ is a negative semi-infinite function, and $F(x) \sim A\, x^{\eta}$ when $x \to 0^{-}$, then $f(k_x) \sim O(k_x^{-\eta-1})$ when $k_x \to \infty$ in the region of analyticity implied in (ii).

Dual Integral Equations

Mixed boundary value problems associated with semi-infinite planes, e.g. the Sommerfeld's half-plane problem can always be formulated in terms of dual integral equations. The boundary condition on the plane that contains the half-plane is mixed. As we have shown in the previous lecture, without loss of generality, we need only to consider a two dimensional problem with the scalar wave equation. For a scalar potential satisfying the wave equation

$$(\nabla^2 + k^2)\ \Phi^s(x, y) = 0 \tag{4}$$

the mixed boundary conditions can be of the type:

(i) continuity of potential for all x, in particular,

$$\Phi^s(x, 0) = -\ \Phi^i(x, 0) \qquad x \geq 0 \tag{5a}$$

and

$$\frac{\partial}{\partial y}\ [\Phi^s(x, 0^+) - \Phi^s(x, 0^-)] = J(x) = 0 \qquad x < 0 \tag{5b}$$

as in the TM electromagnetic wave or

(ii) continuity of normal derivation of potential for all x, in particular,

$$\frac{\partial}{\partial y} \Phi^s(x, 0) = - \frac{\partial \Phi^i}{\partial y}(x, 0) \qquad x \geq 0 \qquad (6a)$$

and

$$\Phi^s(x, 0^+) - \Phi^s(x, 0^-) = D(x) = 0 \qquad x < 0 \qquad (6b)$$

as in the TE electromagnetic wave.

Taking case (i) as an illustrative example, it is known that the scattered potential can be expressed as the normal derivative of the potential on the surface of the screen (half-plane, etc).

$$\Phi^s(x, y) = \int_{-\infty}^{\infty} g(x - x', y) \frac{\partial}{\partial y} [\Phi^s(x', 0^+) - \Phi^s(x', 0^-)] dx'$$

$$= \int_{-\infty}^{\infty} g(x - x', y) \, J(x') dx' \qquad (7)$$

where $g(x, y)$ is the appropriate Green's function. Using Parseval's theorem, the above can be written as

$$\Phi^s(x, y) = \int_{-\infty}^{\infty} \tilde{g}(k_x, y) \, \tilde{J}(k_x) \, e^{ik_x x} \, dk_x. \qquad (7a)$$

Thus, the dual integral equation for the problem stated in (ii) can be written as

$$\phi^s(x, 0) = \int_{-\infty}^{\infty} \tilde{g}(k_x, 0)\, J(k_x)\, e^{ik_x x}\, dk_x = -\phi^i(x, 0)$$

$$x \geq 0 \qquad (8a)$$

$$J(x) = \int_{-\infty}^{\infty} \tilde{J}(k_x)\, e^{ik_x x}\, dk_x = 0 \qquad x < 0 \qquad (8b)$$

Denoting the Fourier transforms of $\phi^s(x, 0)$ as

$$\phi^s(k_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi^s(x, 0)\, e^{-ik_x x}\, dx = -\frac{1}{2\pi} \int_{0}^{\infty} \phi^i(x, 0)\, e^{-ik_x x}\, dx$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{0} \phi^s(x, 0)\, e^{-ik_x x}\, dx \qquad (9)$$

we can write (8a) as

$$\phi^s(k_x) = \tilde{g}(k_x, 0)\, \tilde{J}(k_x) = -\phi^i(k_x) + \phi^u(k_x), \qquad (10)$$

where $\phi^u(k_x)$ corresponds to the second integral in (9). In the above equation, $\tilde{J}(k_x)$ and $\phi^u(k_x)$ are unknowns to be determined. Even though they are unknowns, they are transforms of semi-infinite functions whose analyticity properties can be determined as discussed in the previous section. Thus, we deduce that $\tilde{J}(k_x)$ and $\phi^u(k_x)$ are both analytic functions in the lower half-plane and the upper half-plane of the complex k_x-plane respectively. Hence we denote them by $\tilde{J}_-(k_x)$ and $\phi_+^u(k_x)$. Furthermore, by assuming slight loss in the medium, or that $k = k' + ik''$, we deduce that

$$\Phi^s(x_i) \sim O(e^{k''x}) \qquad x \to -\infty \tag{11}$$

and that $\phi_+^u(k_x)$ is analytic for $Im(k_x) > -k''$. Since $\Phi^i(x, 0) = e_o\, e^{ik_x^i x}$ where we assume k_x real, then $\phi^i(k_x) = \phi_-^i(k_x)$ is analytic for $Im(k_x) < 0$. Since $J(x)$ is induced by $\phi^i(x, y)$, $|J(x)| \sim O(1)$ when $x \to \infty$, or $\tilde{J}(k_x)$ is analytic for $Im(k_x) < 0$. Consequently, (10) becomes

$$\tilde{g}(k_x, 0)\, J_-(k_x) = -\,\phi_-^i(k_x) + \phi_+^u(k_x) \tag{12}$$

which is the Wiener-Hopf equation. It is solvable, even though there are two unknowns involved, by using the Wiener-Hopf technique. Similar equations can be derived for case (ii) concerning the TE waves. It is to be noted in the subsequent section that even though $\tilde{g}(k_x, 0)$ is the Fourier transform of $g(-x, y)$, it is often simpler to derive $\tilde{g}(k_x, y)$ directly from the differential equation.

<u>The Wiener-Hopf Technique</u>

The ability to solve the Wiener-Hopf equation (12) relies on the factorization of $\tilde{g}(k_x, 0)$ symmetrically as

$$\tilde{g}(k_x, 0) = G_+(k_x)\, G_-(k_x) \tag{13}$$

where $G_+(k_x)$ is analytic and non-zero for $\mathrm{Im}[k_x] > -k''$ and $G_-(k_x)$ is analytic and non-zero for $\mathrm{Im}[k_x] < k''$. Substituting (13) into (12) and dividing the result by $G_+(k_x)$, we obtain

$$G_-(k_x)\, \tilde{J}_-(k_x) = -\frac{\phi_-^{\,i}(k_x)}{G_+(k_x)} + \frac{\phi_+^{\,u}(k_x)}{G_+(k_x)} . \tag{14}$$

The idea is to group functions analytic over the upper half-plane and the lower half-plane to each side of the equation. But, $\phi_-^{\,i}(k_x)$ is not analytic in the upper half-plane as wanted. However, $\phi_-^{\,i}(k_x)$ is known, and say if it has only a pole singularity at $k_x = k_x^{\,p}$, we can remove the singularity as follows by subtraction.

$$G_-(k_x)\, \tilde{J}_-(k_x) + \frac{\phi_-^{\,i}(k_x)}{G_+(k_x^{\,p})} = -\phi_-^{\,i}(k_x)\left[\frac{1}{G_+(k_x)} - \frac{1}{G_-(k_x^{\,p})}\right] + \frac{\phi_+^{\,u}(k_x)}{G_+(k_x)} . \tag{15}$$

The left-hand side of (15) is analytic for $Im[k_x] < 0$ while the right hand side of (15) is analytic for $Im[k_x] > -k''$. Since the function on the RHS of (15) and that on the LHS share a strip of analyticity, by complex variable theory, they represent the same function which is analytic over the entire complex plane, viz.,

$$LHS(15) = RHS(15) = P(k_x) \tag{16}$$

where $P(k_x)$ is an entire function for all k_x. We can further study the behavior of $P(k_x)$ at infinity, by observing how LHS (15) and RHS (15) behave when $k_x \to \infty$ for $Im(k_x) < 0$ and $k_x \to \infty$ for $Im[k_x] > 0$ respectively. Even though $\tilde{J}_-(k_x)$ and $\phi_+{}^u(k_x)$ are not known, their amplitude at infinity can be deduced from the edge condition, i.e. the asymptotic values of $J(x)$ and $\phi^u(x)$ when $x \to 0^{+-}$. Often times, it can be shown that $P(k_x) \to 0$ when $k_x \to \infty$ on the whole of the complex plane. In such a case, Louiville's theorem can be invoked implying that $P(k_x) = 0$ for all k_x. Therefore, we deduce that

$$\tilde{J}_-(k_x) = -\frac{\phi_-{}^i(k_x)}{G_+(k_x{}^p)\ G_-(k_x)} \tag{17}$$

and

$$\phi^s(k_x) = \tilde{g}(k_x, 0)\ \tilde{J}_-(k_x) = -\ \phi_-^i(k_x)\ \frac{G_+(k_x)}{G_+(k_x^p)}\ . \tag{18}$$

From the above, we can derive $\Phi^s(x, y)$, the scattered potential.

Factorization

The success of the Wiener-Hopf technique often depends on the factorization as in (13). Occasionally, the factorization can be performed by inspection. When the factorization is not obvious, we resort to Cauchy's integral formula [Noble, Wiener-Hopf, Tech., 1958]. Defining a function

$$K(k_x) = \ln\ g(k_x) \tag{19}$$

where $g(k_x)$ has a strip of analyticity around the real axis $-t_1 \leq \text{Im}(\xi) \leq t_2$. If $K(k_x)$ can be separated as $K(k_x) = K_+(k_x) + K_-(k_x)$, then $g(k_x)$ can be factorized as $g(k_x) = G_+(k_x)\ G_-(k_x)$ where

$$G_\pm(k_x) = e^{K_\pm(x)}\ . \tag{20}$$

Assuming that $g(k_x) \to 1$ when $k_x \to \pm\infty$, and that $g(k_x)$

is an even function of k_x, then $K(k_x) \to 0$ when $k_x \to \infty$. Using Cauchy's integral formula, we can write $K(k_x)$ as

$$K(k_x) = \frac{1}{2\pi i} \int_C \frac{K(\xi)}{\xi - k_x} d\xi$$

where C is defined in Fig. 1, since $K(\xi)$ is analytic for $-t_1 < \mathrm{Im}(\xi) < t_2$. $C = C_1 + C_2$ since the contribution from the two extremes of C is vanishingly small when they recede to infinity. As such, (20) can be written as

$$K(k_x) = \frac{1}{2\pi i} \int_{C_2} \frac{K(\xi)}{\xi - k_x} d\xi - \frac{1}{2\pi i} \int_{-C_1} \frac{K(\xi)}{\xi - k_x} d\xi . \tag{21}$$

The first integral is well-defined if $\mathrm{Im}(k_x) > -t_1$ and thus is analytic for $\mathrm{Im}(k_x) > -t_1$. The second integral, by similar argument, is analytic for $\mathrm{Im}(k_x) < t_2$. Thus,

$$K_+(k_x) = \frac{1}{2\pi i} \int_{C_2} \frac{K(\xi)}{\xi - k_x} d\xi \tag{22a}$$

$$K_-(k_x) = - \frac{1}{2\pi i} \int_{-C_1} \frac{K(\xi)}{\xi - k_x} d\xi. \tag{22b}$$

Therefore, we derive $G_\pm(k_x)$ from (20) where $G_+(k_x)$ is analytic

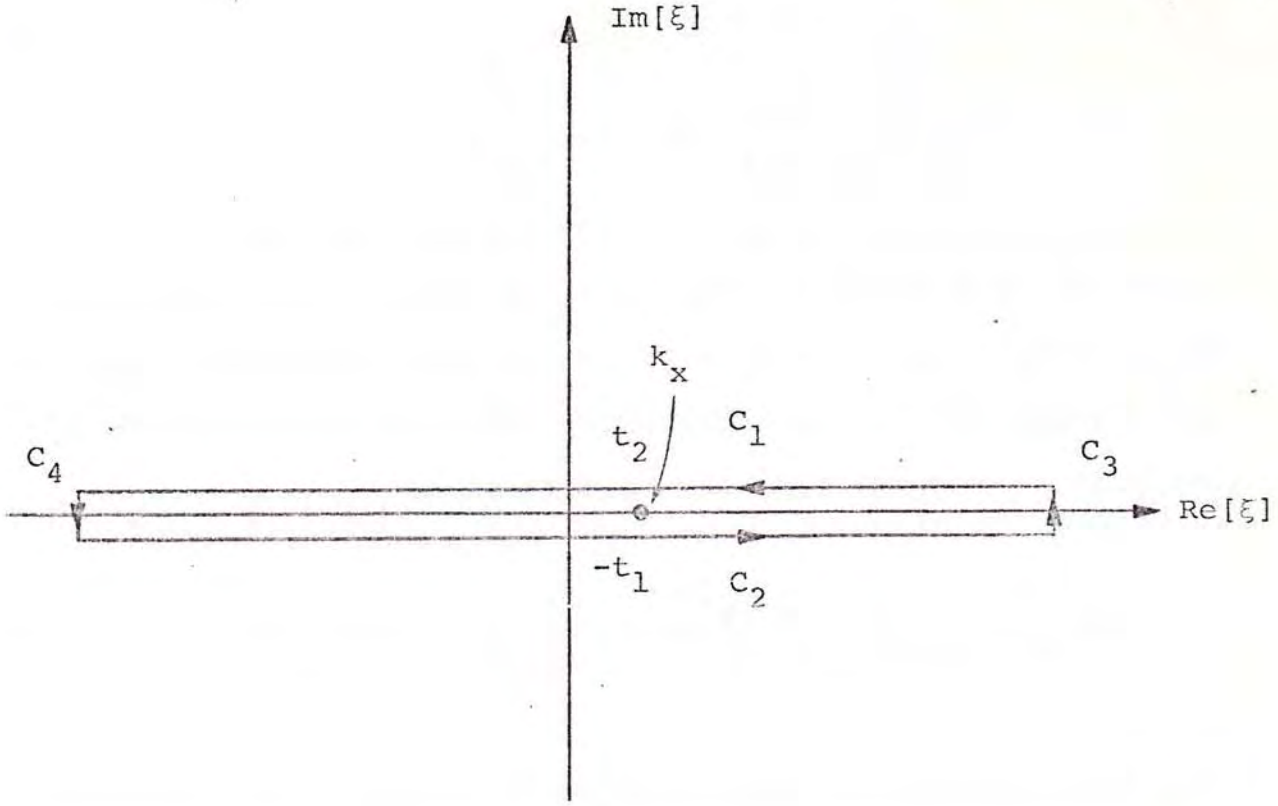


Figure 1

and nonzero for $\mathrm{Im}(k_x) > -t_1$ and $G_-(k_x)$ is analytic and nonzero for $\mathrm{Im}(k_x) < t_2$.

The above represents a systematic method of factorizing $g(k_x)$ though the integral in (22a) and (22b), as you will find out in your homework, can be difficult to perform.

Examples

(i) The Sommerfeld's Half-Plane Problem

In the Sommerfeld's half plane problem involving the TM wave, the potential $\Phi^s(x, y)$ is specified on the half-plane. By symmetry, $\Phi^s(x, -y) = \Phi^s(x, y)$ and we can write $\Phi^s(x, y)$ as

$$\Phi^s(x, y) = \int_{-\infty}^{\infty} \phi^s(k_x)\, e^{ik_y|y| + ik_x x}\, dk_x \tag{23}$$

where $k_y = \sqrt{k^2 - k_x^2}$ and $k = k' + ik''$. The expression (23) does not ensure the continuity of the normal derivative of $\Phi^s(x, y)$ for $x < 0$. Thus we require that

$$J(x) = \frac{\partial}{\partial y}\,[\Phi^s(x, 0^+) - \Phi^s(x, 0^-)] = 2i \int_{-\infty}^{\infty} k_y\, \phi^s(k_s)\, e^{ik_x x}\, dk_x = 0$$

$$\text{for} \quad x < 0. \tag{24}$$

We relate that the Fourier transform of $J(x)$ is

$$\tilde{J}(k_x) = 2ik_y\, \phi^s(k_x). \tag{25}$$

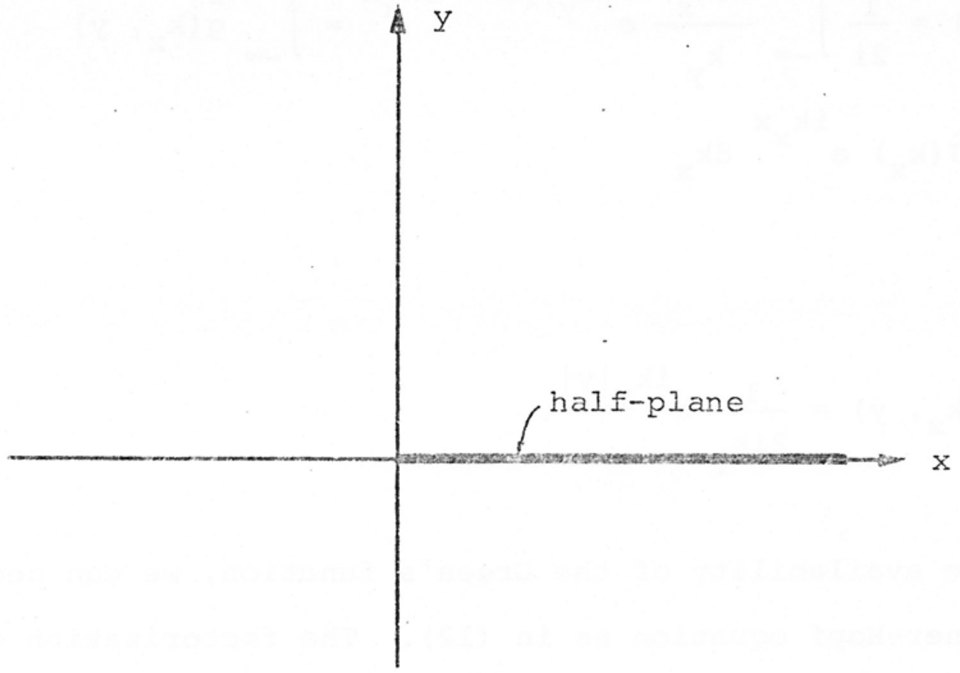


Figure 2

Substituting (25) into (23), we obtain

$$\phi^s(x, y) = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{\tilde{J}(k_x)}{k_y} e^{ik_y|y| + ik_x x} = \int_{-\infty}^{\infty} \tilde{g}(k_x, y)$$

$$\tilde{J}(k_x)\, e^{ik_x x}\, dk_x \tag{26}$$

where

$$\tilde{g}(k_x, y) = \frac{1}{2ik_y} e^{ik_y|y|}. \tag{26a}$$

With the availability of the Green's function, we can deduce the Wiener-Hopf equation as in (12). The factorization of $\tilde{g}(k_x, 0)$ can be performed by inspection

$$\tilde{g}(k_x, 0) = \frac{1}{2i\sqrt{k^2 - k_x{}^2}} = G_+(k_x)\, G_-(k_x) \tag{27}$$

where

$$G_\pm(k_x) = \frac{1}{\sqrt{2i}} (k \pm k_x)^{-1/2}. \tag{27a}$$

We can thus obtain an equation similar to (15). Given that $\phi^i(x, 0) = e_o\, e^{ik_x^i x}$, we deduce that

$$\phi_-^{\,i}(k_x) = \frac{e_o}{2\pi i(k_x - k_x^{\,i})} \tag{28}$$

with a pole location at $k_x = k_x^{\,i}$. We notice that $G_\pm(k_x) \sim O(k_x^{-1/2})$ when $k_x \to \infty$. Also, since $J(x)$ is proportional to $\partial\Phi^s/\partial y$, from the edge condition, $J(x) \sim O(x^{-1/2})$ at most when $x \to 0^+$. In other words, $\tilde{J}(k_x) \sim O(k_x^{-1/2})$ when $k_x \to \infty$. Since $\phi_-^{\,i}(k_x) \sim O(k_x^{-1})$ when $k_x \to \infty$, the LHS of (15) vanishes on the lower half-plane. Noticing that $\Phi^s(x, 0) = -\Phi^i(x, 0) \sim O(1)$ when $x \to 0^+$, by continuity of potential, $\Phi^s(x, 0) \sim O(1)$ too when $x \to 0^-$. Hence $\phi_+^{\,u}(k_x) \sim O(k_x^{-1})$ when $k_x \to \infty$. We observe also that the RHS of (15) vanishes when $k_x \to \infty$ on the upper half-plane. It can be shown [similar to (11) and the subsequent discussion] that the function on the RHS of (15) and the LHS of (15) share a strip of analyticity. Thus they represent an analytic function $P(k_x)$ which is identically zero everywhere. From (18) we conclude that

$$\phi^s(k_x) = -\phi_-^{\,i}(k_x)\,\frac{G_+(k_x)}{G_+(k_x^{\,i})} = -\frac{e_o}{2\pi i(k_x - k_x^{\,i})}\left(\frac{k + k_x^{\,i}}{k + k_x}\right)^{1/2}. \tag{29}$$

As such

$$\Phi^s(x, y) = -\frac{e_o(k + k_x^{\,i})^{1/2}}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{ik_x x + ik_y y}}{(k_x - k_x^{\,i})(k_x + k)^{1/2}}\, dk_x. \tag{30}$$

The inversion path in (30) is taken to be in the strip of analyticity shared by $\phi_-^{\,i}(k_x)$ and $G_+(k_x)$ as shown in Fig. 3a. Letting

$$k_x = k\cos\alpha, \quad k_x^{\,i} = -k\cos\phi^i, \quad x = \rho\cos\phi, \quad y = \rho\sin\phi \tag{31}$$

we have

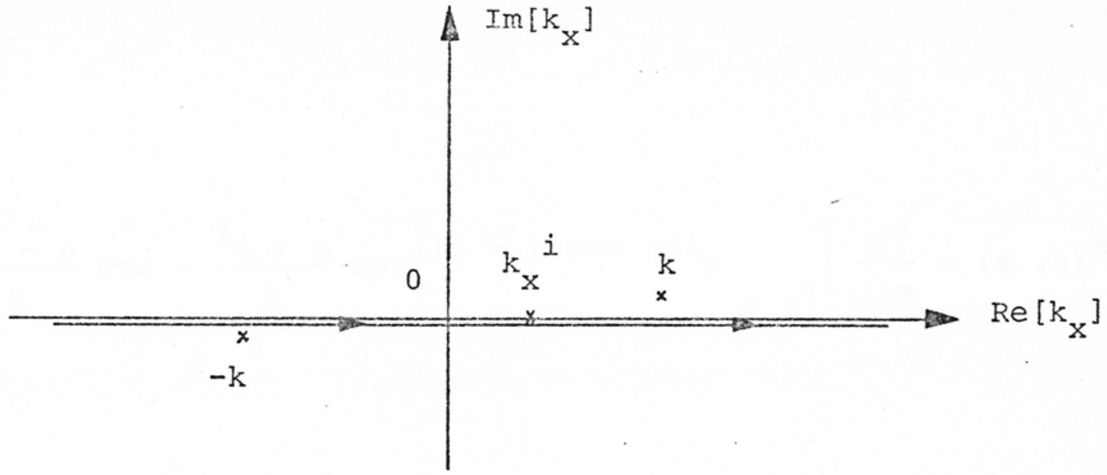
$$\phi^s(\rho,\phi) = \frac{e_0 \sin\frac{\phi^i}{2}}{2\pi i}\int_{\Gamma_{1,2}} \frac{\sin\frac{\alpha}{2}\, e^{ik\rho\cos(\alpha-\phi)}}{\cos\alpha + \cos\phi^i}\, d\alpha \tag{32}$$

where $\Gamma_{1,2}$ is shown in Fig. 3b. Γ_1 is chosen when $0 < \phi < \pi$ while Γ_2 is chosen when $\pi < \phi < 2\pi$. Letting

$$2\sin\frac{\phi^i}{2}\sin\frac{\alpha}{2} = \cos\frac{\phi^i - \alpha}{2} - \cos\frac{\phi^i + \alpha}{2} \tag{33a}$$

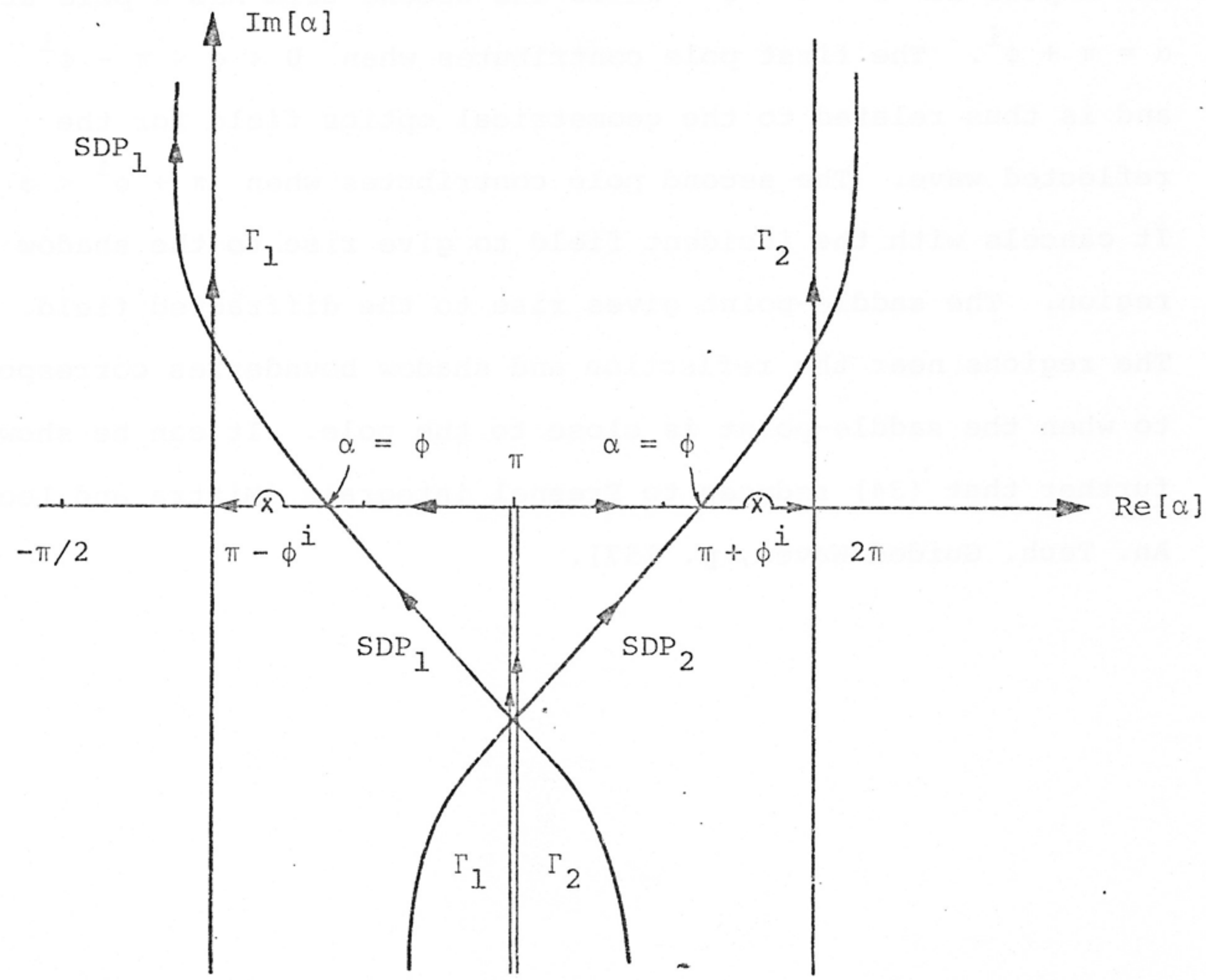
$$\cos\alpha + \cos\phi^i = 2\cos\left(\frac{\alpha + \phi^i}{2}\right)\cos\left(\frac{\alpha - \phi^i}{2}\right) \tag{33b}$$

(32) becomes


Im[kx]
0
kx i
k
Re[kx]
-k

(a)

Figure 3


Im[α]
SDP1
Γ1
Γ2
α = φ
π
α = φ
Re[α]
-π/2
π - φ i
π + φ i
2π
SDP1
SDP2
Γ1
Γ2

(b)

$$\Phi^{s}(\rho, \phi) = \frac{e_o}{8\pi i} \int_{\Gamma_{1,2}} e^{ik\rho \cos(\alpha - \phi)} \left[\sec \frac{\alpha + \phi^i}{2} - \sec \frac{\alpha - \phi^i}{2}\right] d\alpha. \tag{34}$$

The above integral does not converge along Γ_1 when $\pi < \phi < 2\pi$ and thus Γ_2 has to be taken to be the path of integration. There is a saddle-point at $\alpha = \phi$. The first term in the bracket has a pole at $\alpha = \pi - \phi^i$ while the second term has a pole at $\alpha = \pi + \phi^i$. The first pole contributes when $0 < \phi < \pi - \phi^i$ and is thus related to the geometrical optics field for the reflected wave. The second pole contributes when $\pi + \phi^i < \phi < 2\pi$. It cancels with the incident field to give rise to the shadow region. The saddle-point gives rise to the diffracted field. The regions near the reflection and shadow boundaries correspond to when the saddle-point is close to the pole. It can be shown further that (34) reduces to Fresnel integrals [Mittra and Lee, An. Tech. Guided Waves, p. 157].