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Efficient Ways to Compute the Vector Addition Theorem

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Abstract– Two efficient ways of calculating the vector addition theorem are presented. One is obtained by relating the coefficients of the vector addition theorem to that of the scalar addition theorem for which an efficient recurrence relation exists. The second way is to derive recurrence relations directly for the coefficients of the vector addition theorem. These new ways of calculating the coefficients are of reduced computational complexity. Hence, when the number of coefficients required is large, the present methods are many times faster than the traditional method using Gaunt coefficients and Wigner 3j symbols.

1. INTRODUCTION

The translational addition theorem is of vital importance in the scattering theory of waves by multiple scatterers [1–10]. Even though the addition theorem is rather simple in two dimensions, the coefficients for the addition theorem are unusually complex in three dimensions [10–16]. The coefficients for the addition theorem are usually expressed in terms of summations over Gaunt coefficients. The Gaunt coefficients are in turn expressed in terms of Wigner 3j symbols involving a large number of factorials. Consequently, the complexity of calculating the addition theorem coefficients becomes a bottle neck in many scattering calculations. Recurrence relations have been derived for the Gaunt coefficients but they do not reduce the complexity of the calculation. Elegant formulas have been derived for these coefficients in terms of differential operators in [16], but the author just stops short of deriving recurrence relations.

Recently, recurrence relations for scalar addition theorem coefficients have been derived [17]. The recurrence relations reduce the computational complexity of calculating for the coefficients. Hence, when the number of coefficients is large, the recurrence relation method is a lot faster than the conventional means of calculating the coefficients. However, in many wave scattering theory, e.g., involving electromagnetic waves and elastic waves, the use of the vector addition theorem involving the vector wave functions is indispensable.

The vector addition theorem is [10, 12, 13, 15]

$$\mathbf{M}_{nm}(\mathbf{r}) = \sum_{\nu=1}^{\infty} \sum_{\mu=-\nu}^{\nu} [\mathbf{M}_{\nu\mu}(\mathbf{r}') A_{\nu\mu,nm} + \mathbf{N}_{\nu\mu}(\mathbf{r}') B_{\nu\mu,nm}] \tag{1}$$

where $\mathbf{r} = \mathbf{r}' + \mathbf{r}''$, and the vector wave functions are defined to be [10, 18]

$$\mathbf{M}_{nm}(\mathbf{r}) = \nabla \times \mathbf{r}\psi_{nm}(\mathbf{r}), \tag{2a}$$

$$\mathbf{N}_{nm}(\mathbf{r}) = \frac{1}{k}\nabla \times \mathbf{M}_{nm}(\mathbf{r}), \tag{2b}$$

which are divergence free, and

$$\psi_{nm}(\mathbf{r}) = j_n(kr) Y_{nm}(\theta, \phi), \tag{2c}$$

where

$$Y_{nm}(\theta, \phi) = (-1)^m \sqrt{\frac{(n-m)!}{(n+m)!} \frac{2n+1}{4\pi}} P_n^m(\cos\theta) e^{im\phi} \tag{3}$$

and $Y_{n,-m}(\theta, \phi) = (-1)^m Y_{nm}^*(\theta, \phi)$ [19]. The scalar wave function, $\psi_{nm}(\mathbf{r})$ satisfies the following addition theorem

$$\psi_{nm}(\mathbf{r}) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \psi_{\nu\mu}(\mathbf{r}') \beta_{\nu\mu,nm}. \tag{4}$$

Efficient recurrence relations have been derived for $\beta_{\nu\mu,nm}$ such that it can be derived from $\beta_{\nu\mu,00}$ [17].

The traditional method of calculating $A_{\nu\mu,nm}$ and $B_{\nu\mu,nm}$ is to express them in term of Gaunt coefficients, but this manner of computing them is extremely inefficient. However, if $A_{\nu\mu,nm}$ and $B_{\nu\mu,nm}$ can be related to $\beta_{\nu\mu,nm}$ (which can now be efficiently calculated), then, they too can be efficiently calculated. We shall seek to establish this relationship in the next section.

2. RELATIONSHIP TO THE SCALAR ADDITION THEOREM

The vector addition theorem can be efficiently computed if it can be related to the scalar addition theorem for which an efficient recurrence relation exists. To this end, we make use of (4), and the definition (2a) to establish that

$$\begin{aligned} \mathbf{M}_{nm}(\mathbf{r}) &= \sum_{\nu\mu} \nabla\psi_{\nu\mu}(\mathbf{r}') \times (\mathbf{r}' + \mathbf{r}'')\beta_{\nu\mu,nm} \\ &= \sum_{\nu\mu} \mathbf{M}_{\nu\mu}(\mathbf{r}')\beta_{\nu\mu,nm} + \sum_{\nu\mu} \nabla\psi_{\nu\mu}(\mathbf{r}') \times \mathbf{r}''\beta_{\nu\mu,nm}. \end{aligned} \tag{5}$$

The operator ∇, written in Cartesian coordinates, is invariant under coordinate translation. It remains to express the second term in (5) in terms of $\mathbf{M}_{\nu\mu}(\mathbf{r}')$ and $\mathbf{N}_{\nu\mu}(\mathbf{r}')$.

More explicitly, the second term in (5) is related to

$$\nabla\psi_{\nu\mu}(\mathbf{r}')\times\mathbf{r}''=x''\nabla'\times\hat{x}\psi_{\nu\mu}(\mathbf{r}')+y''\nabla'\times\hat{y}\psi_{\nu\mu}(\mathbf{r}')+z''\nabla'\times\hat{z}\psi_{\nu\mu}(\mathbf{r}'). \quad (6)$$

It can be shown that [Appendices C and D] for $k=1$,

$$\begin{aligned}\nabla'\times\hat{x}\psi_{\nu\mu}(\mathbf{r}')=&x_{\nu\mu}^{-+}\mathbf{M}_{\nu-1,\mu+1}(\mathbf{r}')+x_{\nu\mu}^{++}\mathbf{M}_{\nu+1,\mu+1}(\mathbf{r}')+x_{\nu\mu}^{0+}\mathbf{N}_{\nu,\mu+1}(\mathbf{r}')\\&+x_{\nu\mu}^{--}\mathbf{M}_{\nu-1,\mu-1}(\mathbf{r}')+x_{\nu\mu}^{+-}\mathbf{M}_{\nu+1,\mu-1}(\mathbf{r}')+x_{\nu\mu}^{0-}\mathbf{N}_{\nu,\mu-1}(\mathbf{r}')\end{aligned} \quad (7a)$$

$$\begin{aligned}\nabla'\times\hat{y}\psi_{\nu\mu}(\mathbf{r}')=&y_{\nu\mu}^{-+}\mathbf{M}_{\nu-1,\mu+1}(\mathbf{r}')+y_{\nu\mu}^{++}\mathbf{M}_{\nu+1,\mu+1}(\mathbf{r}')+y_{\nu\mu}^{0+}\mathbf{N}_{\nu,\mu+1}(\mathbf{r}')\\&+y_{\nu\mu}^{--}\mathbf{M}_{\nu-1,\mu-1}(\mathbf{r}')+y_{\nu\mu}^{+-}\mathbf{M}_{\nu+1,\mu-1}(\mathbf{r}')+y_{\nu\mu}^{0-}\mathbf{N}_{\nu,\mu-1}(\mathbf{r}')\end{aligned} \quad (7b)$$

$$\nabla'\times\hat{z}\psi_{\nu\mu}(\mathbf{r}')=z_{\nu\mu}^{-}\mathbf{M}_{\nu-1,\mu}(\mathbf{r}')+z_{\nu\mu}^{+}\mathbf{M}_{\nu+1,\mu}(\mathbf{r}')+z_{\nu\mu}^{0}\mathbf{N}_{\nu\mu}(\mathbf{r}'). \quad (7c)$$

Substituting (7) into (6) and hence (5), and gathering terms of the same kind, we have

$$\begin{aligned}\mathbf{M}_{nm}(\mathbf{r})=&\sum_{\nu\mu}\mathbf{M}_{\nu\mu}(\mathbf{r}')\beta_{\nu\mu,nm}+\sum_{\nu\mu}\Big\{m_{\nu\mu}^{-+}\beta_{\nu\mu,nm}\mathbf{M}_{\nu-1,\mu+1}(\mathbf{r}')\\&+m_{\nu\mu}^{++}\beta_{\nu\mu,nm}\mathbf{M}_{\nu+1,\mu+1}(\mathbf{r}')+m_{\nu\mu}^{--}\beta_{\nu\mu,nm}\mathbf{M}_{\nu-1,\mu-1}(\mathbf{r}')\\&+m_{\nu\mu}^{+-}\beta_{\nu\mu,nm}\mathbf{M}_{\nu+1,\mu-1}(\mathbf{r}')+m_{\nu\mu}^{-0}\beta_{\nu\mu,nm}\mathbf{M}_{\nu-1,\mu}(\mathbf{r}')\\&+m_{\nu\mu}^{+0}\beta_{\nu\mu,nm}\mathbf{M}_{\nu+1,\mu}(\mathbf{r}')+n_{\nu\mu}^{0+}\beta_{\nu\mu,nm}\mathbf{N}_{\nu,\mu+1}(\mathbf{r}')\\&+n_{\nu\mu}^{0-}\beta_{\nu\mu,nm}\mathbf{N}_{\nu,\mu-1}(\mathbf{r}')+n_{\nu\mu}^{00}\beta_{\nu\mu,nm}\mathbf{N}_{\nu\mu}(\mathbf{r}')\Big\}\end{aligned} \quad (8)$$

Rearranging the indices, we obtain

$$\begin{aligned}\mathbf{M}_{nm}(\mathbf{r})=&\sum_{\nu\mu}\mathbf{M}_{\nu\mu}(\mathbf{r}')\Big[\beta_{\nu\mu,nm}+m_{\nu+1,\mu-1}^{-+}\beta_{\nu+1,\mu-1,nm}\\&+m_{\nu-1,\mu-1}^{++}\beta_{\nu-1,\mu-1,nm}+m_{\nu+1,\mu+1}^{--}\beta_{\nu+1,\mu+1,nm}+m_{\nu-1,\mu+1}^{+-}\beta_{\nu-1,\mu+1,nm}\\&+m_{\nu+1,\mu}^{-0}\beta_{\nu+1,\mu,nm}+m_{\nu-1,\mu}^{+0}\beta_{\nu-1,\mu,nm}\Big]+\sum_{\nu\mu}\mathbf{N}_{\nu\mu}(\mathbf{r}')\\&\Big[n_{\nu,\mu-1}^{0+}\beta_{\nu,\mu-1,nm}+n_{\nu,\mu+1}^{0-}\beta_{\nu,\mu+1,nm}+n_{\nu\mu}^{00}\beta_{\nu\mu,nm}\Big].\end{aligned} \quad (9)$$

The above coefficients could be further simplified as follows:

$$m^{-+}_{\nu+1,\mu-1} = x'' x^{-+}_{\nu+1,\mu-1} + y'' y^{-+}_{\nu+1,\mu-1} = \frac{x'' - iy''}{2} \eta^{-}_{\nu+1,\mu-1(+)}$$
$$= r'' \sin\theta'' \frac{e^{-i\phi''}}{2} \frac{1}{\nu+1} \sqrt{\frac{(\nu-\mu+2)(\nu-\mu+1)}{(2\nu+1)(2\nu+3)}}, \quad (10a)$$

$$m^{++}_{\nu-1,\mu-1} = x'' x^{++}_{\nu-1,\mu-1} + y'' y^{++}_{\nu-1,\mu-1} = \frac{x'' - iy''}{2} \eta^{+}_{\nu-1,\mu-1(+)}$$
$$= r'' \sin\theta'' \frac{e^{-i\phi''}}{2} \frac{-1}{\nu} \sqrt{\frac{(\nu+\mu-1)(\nu+\mu)}{(2\nu-1)(2\nu+1)}}, \quad (10b)$$

$$m^{--}_{\nu+1,\mu+1} = x'' x^{--}_{\nu+1,\mu+1} + y'' y^{--}_{\nu+1,\mu+1} = \frac{x'' + iy''}{2} \eta^{-}_{\nu+1,\mu+1(-)}$$
$$= r'' \sin\theta'' \frac{e^{i\phi''}}{2} \frac{-1}{\nu+1} \sqrt{\frac{(\nu+\mu+2)(\nu+\mu+1)}{(2\nu+1)(2\nu+3)}}, \quad (10c)$$

$$m^{+-}_{\nu-1,\mu+1} = x'' x^{+-}_{\nu-1,\mu+1} + y'' y^{+-}_{\nu-1,\mu+1} = \frac{x'' + iy''}{2} \eta^{+}_{\nu-1,\mu+1(-)}$$
$$= r'' \sin\theta'' \frac{e^{i\phi''}}{2} \frac{1}{\nu} \sqrt{\frac{(\nu-\mu-1)(\nu-\mu)}{(2\nu-1)(2\nu+1)}}, \quad (10d)$$

$$m^{-0}_{\nu+1,\mu} = z'' z^{-}_{\nu+1,\mu} = r'' \cos\theta'' \frac{1}{\nu+1} \sqrt{\frac{(\nu+\mu+1)(\nu-\mu+1)}{(2\nu+1)(2\nu+3)}}, \quad (10e)$$

$$m^{+0}_{\nu-1,\mu} = z'' z^{+}_{\nu-1,\mu} = r'' \cos\theta'' \frac{1}{\nu} \sqrt{\frac{(\nu+\mu)(\nu-\mu)}{(2\nu-1)(2\nu+1)}}, \quad (10f)$$

$$n^{00}_{\nu\mu} = z'' z^{0}_{\nu\mu} = r \cos\theta'' \frac{i\mu}{\nu(\nu+1)}, \quad (11a)$$

$$n^{0+}_{\nu,\mu-1} = x'' x^{0+}_{\nu,\mu-1} + y'' y^{0+}_{\nu,\mu-1} = r'' \sin\theta'' \frac{e^{-i\phi''}}{2} \frac{i\sqrt{(\nu-\mu+1)(\nu+\mu)}}{\nu(\nu+1)}, \quad (11b)$$

$$n^{0-}_{\nu,\mu+1} = x'' x^{0-}_{\nu,\mu+1} + y'' y^{0-}_{\nu,\mu+1} = r'' \sin\theta'' \frac{e^{i\phi''}}{2} \frac{i\sqrt{(\nu+\mu+1)(\nu-\mu)}}{\nu(\nu+1)}. \quad (11c)$$

Consequently, we reduce the $A_{\nu\mu,nm}$ and $B_{\nu\mu,nm}$ in (1) to be

$$A_{\nu\mu,nm} = \beta_{\nu\mu,nm} + r'' \sin\theta'' \frac{e^{-i\phi''}}{2(\nu+1)} \sqrt{\frac{(\nu-\mu+2)(\nu-\mu+1)}{(2\nu+1)(2\nu+3)}} \beta_{\nu+1,\mu-1,nm}$$
$$- r'' \sin\theta'' \frac{e^{-i\phi''}}{2\nu} \sqrt{\frac{(\nu+\mu-1)(\nu+\mu)}{(2\nu-1)(2\nu+1)}} \beta_{\nu-1,\mu-1,nm}$$

$$
\begin{aligned}
&- r'' \sin\theta'' \frac{e^{i\phi''}}{2(\nu+1)} \sqrt{\frac{(\nu+\mu+2)(\nu+\mu+1)}{(2\nu+1)(2\nu+3)}} \beta_{\nu+1,\mu+1,nm} \\
&+ r'' \sin\theta'' \frac{e^{i\phi''}}{2\nu} \sqrt{\frac{(\nu-\mu)(\nu-\mu-1)}{(2\nu-1)(2\nu+1)}} \beta_{\nu-1,\mu+1,nm} \\
&+ r'' \cos\theta'' \frac{1}{\nu+1} \sqrt{\frac{(\nu+\mu+1)(\nu-\mu+1)}{(2\nu+1)(2\nu+3)}} \beta_{\nu+1,\mu,nm} \\
&+ r'' \cos\theta'' \frac{1}{\nu} \sqrt{\frac{(\nu+\mu)(\nu-\mu)}{(2\nu-1)(2\nu+1)}} \beta_{\nu-1,\mu,nm},
\end{aligned}
\tag{12a}
$$

$$
\begin{aligned}
B_{\nu\mu,nm} =& r'' \cos\theta'' \frac{i\mu}{\nu(\nu+1)} \beta_{\nu\mu,nm} + \frac{i r'' \sin\theta''}{2\nu(\nu+1)} \Big[\sqrt{(\nu-\mu)(\nu+\mu+1)} \\
& \cdot e^{i\phi''} \beta_{\nu,\mu+1,nm} + \sqrt{(\nu+\mu)(\nu-\mu+1)} e^{-i\phi''} \beta_{\nu,\mu-1,nm} \Big] .
\end{aligned}
\tag{12b}
$$

The results for $k \neq 1$ are obtained by replacing r'' by kr''. The above formulas relate the coefficients for the vector addition theorem, $A_{\nu\mu,nm}$ and $B_{\nu\mu,nm}$, to the coefficients for the scalaraddition theorem, $\beta_{\nu\mu,nm}$, for which an efficient method of calculation exists [17]. Equations (12) are similar to the results of Stein [12] and Wittmann [16] but they are derived quite differently here.

3. THE RECURRENCE RELATIONS

Alternatively, recurrence relations can be derived directly for the coefficients of the vector addition theorem, similar to that of the scalar addition theorem. To do so, we operate (1) with $\partial/\partial z = \partial/\partial z'$. First, similar to [17], it can be shown that

$$
\frac{\partial}{\partial z} \mathbf{M}_{nm}(\mathbf{r}) = a^{+}_{nm} \mathbf{M}_{n+1,m}(\mathbf{r}) + a^{-}_{nm} \mathbf{M}_{n-1,m}(\mathbf{r}) + \nabla \times \hat{z} \psi_{nm}(\mathbf{r}) \tag{13}
$$

where

$$
a^{+}_{nm} = -\left[\frac{(n+m+1)(n-m+1)}{(2n+1)(2n+3)}\right]^{\frac{1}{2}}, \quad a^{-}_{nm} = \left[\frac{(n+m)(n-m)}{(2n+1)(2n-1)}\right]^{\frac{1}{2}}. \tag{14}
$$

Using the result (C.9) for the expansion of $\nabla \times \hat{z}\psi_{nm}(\mathbf{r})$, we arrive at

$$
\frac{\partial}{\partial z} \mathbf{M}_{nm}(\mathbf{r}) = \lambda^{+}_{nm} \mathbf{M}_{n+1,m}(\mathbf{r}) + \lambda^{-}_{nm} \mathbf{M}_{n-1,m}(\mathbf{r}) + \lambda^{0}_{nm} \mathbf{N}_{nm}(\mathbf{r}). \tag{15}
$$

where

$$
\lambda^{+}_{nm} = a^{+}_{nm} + z^{+}_{nm} = \frac{n}{n+1}\, a^{+}_{nm}, \tag{16a}
$$

$$
\lambda^{-}_{nm} = a^{-}_{nm} + z^{-}_{nm} = \frac{n+1}{n}\, a^{-}_{nm}, \tag{16b}
$$

$$
\lambda^{0}_{nm} = \frac{im}{n(n+1)}. \tag{16c}
$$

Consequently, equation (1), after operating on by $\partial/\partial z = \partial/\partial z'$, becomes

$$\lambda^+_{nm}\mathbf{M}_{n+1,m}(\mathbf{r}) + \lambda^-_{nm}\mathbf{M}_{n-1,m}(\mathbf{r}) + \lambda^0_{nm}\mathbf{N}_{nm}(\mathbf{r})$$
$$= \sum_{\nu=1}^{\infty}\sum_{\mu=-\nu}^{\nu}\Big\{\mathbf{M}_{\nu\mu}(\mathbf{r}')\left[\lambda^+_{\nu-1,\mu}A_{\nu-1,\mu,nm} + \lambda^-_{\nu+1,\mu}A_{\nu+1,\mu,nm} + \lambda^0_{\nu\mu}B_{\nu\mu,nm}\right]$$
$$+\mathbf{N}_{\nu\mu}(\mathbf{r}')\left[\lambda^+_{\nu-1,\mu}B_{\nu-1,\mu,nm} + \lambda^-_{\nu+1,\mu}B_{\nu+1,\mu,nm} + \lambda^0_{\nu\mu}A_{\nu\mu,nm}\right]\Big\}. \qquad (17)$$

Expanding the left-hand side of (17) using addition theorem again [17], and equating like terms, we have

$$\lambda^+_{nm}A_{\nu\mu,n+1,m} + \lambda^-_{nm}A_{\nu\mu,n-1,m} + \lambda^0_{nm}B_{\nu\mu,nm}$$
$$= \lambda^+_{\nu-1,\mu}A_{\nu-1,\mu,nm} + \lambda^-_{\nu+1,\mu}A_{\nu+1,\mu,nm} + \lambda^0_{\nu\mu}B_{\nu\mu,nm}\,, \qquad (18a)$$

$$\lambda^+_{nm}B_{\nu\mu,n+1,m}+\lambda^-_{nm}B_{\nu\mu,n-1,m} + \lambda^0_{nm}A_{\nu\mu,nm}$$
$$= \lambda^+_{\nu-1,\mu}B_{\nu-1,\mu,nm} + \lambda^-_{\nu+1,\mu}B_{\nu+1,\mu,nm} + \lambda^0_{\nu\mu}A_{\nu\mu,nm}\,. \qquad (18b)$$

Next, we need to operate on equation (1) by $C_\pm = \frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}$. It can be shown similar to [17] that

$$C_\pm\mathbf{M}_{nm}(\mathbf{r}) = b^-_{nm(\pm)}\mathbf{M}_{n-1,m\pm1}(\mathbf{r}) + b^+_{nm(\pm)}\mathbf{M}_{n+1,m\pm1}(\mathbf{r}) + \nabla\times(\hat{x}\pm i\hat{y})\psi_{nm}(\mathbf{r}) \qquad (19)$$

where

$$b^-_{nm(\pm)} = \pm\left[\frac{(n\mp m)(n\mp m-1)}{(2n+1)(2n-1)}\right]^{\frac{1}{2}},$$
$$b^+_{nm(\pm)} = \pm\left[\frac{(n\pm m+2)(n\pm m+1)}{(2n+1)(2n+3)}\right]^{\frac{1}{2}}. \qquad (20)$$

Using the result of Appendix D for $\nabla\times(\hat{x}+i\hat{y})\ \psi_{nm}(\mathbf{r})$, we obtain

$$C_+\mathbf{M}_{nm}(\mathbf{r}) = \gamma^-_{nm}\mathbf{M}_{n-1,m+1}(\mathbf{r}) + \gamma^+_{nm}\mathbf{M}_{n+1,m+1}(\mathbf{r}) + \gamma^0_{nm}\mathbf{N}_{n,m+1}(\mathbf{r}) \qquad (21)$$

where

$$\gamma^-_{nm} = \frac{n+1}{n}b^-_{nm(+)}\,,\quad \gamma^+_{nm} = \frac{n}{n+1}b^+_{nm(+)}\,,\quad \gamma^0_{nm} = \frac{i\sqrt{(n-m)(n+m+1)}}{n(n+1)}\,. \qquad (22)$$

Consequently, operating on (1) by C_+, we arrive at

$$\gamma_{nm}^{-}\mathbf{M}_{n-1,m+1}(\mathbf{r}) + \gamma_{nm}^{+}\mathbf{M}_{n+1,m+1}(\mathbf{r}) + \gamma_{nm}^{0}\mathbf{N}_{n,m+1}(\mathbf{r}) = \sum_{\nu=1}^{\infty}\sum_{\mu=-\nu}^{\nu}$$

$$\Big\{\mathbf{M}_{\nu\mu}(\mathbf{r}')\left[\gamma_{\nu+1,\mu-1}^{-}A_{\nu+1,\mu-1,nm} + \gamma_{\nu-1,\mu-1}^{+}A_{\nu-1,\mu-1,nm} + \gamma_{\nu,\mu-1}^{0}B_{\nu,\mu-1,nm}\right]$$
$$+\mathbf{N}_{\nu\mu}(\mathbf{r}')\left[\gamma_{\nu+1,\mu-1}^{-}B_{\nu+1,\mu-1,nm} + \gamma_{\nu-1,\mu-1}^{+}B_{\nu-1,\mu-1,nm} + \gamma_{\nu,\mu-1}^{0}A_{\nu,\mu-1,nm}\right]\Big\} \tag{23}$$

Expanding the left hand side of (23) by the addition theorem again, we obtain

$$\begin{aligned}&\gamma_{nm}^{-}A_{\nu\mu,n-1,m+1} + \gamma_{nm}^{+}A_{\nu\mu,n+1,m+1} + \gamma_{nm}^{0}B_{\nu\mu,n,m+1}\\ &\quad = \gamma_{\nu+1,\mu-1}^{-}A_{\nu+1,\mu-1,nm} + \gamma_{\nu-1,\mu-1}^{+}A_{\nu-1,\mu-1,nm} + \gamma_{\nu,\mu-1}^{0}B_{\nu,\mu-1,nm}\end{aligned} \tag{24a}$$

$$\begin{aligned}&\gamma_{nm}^{-}B_{\nu\mu,n-1,m+1} + \gamma_{nm}^{+}B_{\nu\mu,n+1,m+1} + \gamma_{nm}^{0}A_{\nu\mu,n,m+1}\\ &\quad = \gamma_{\nu+1,\mu-1}^{-}B_{\nu+1,\mu-1,nm} + \gamma_{\nu-1,\mu-1}^{+}B_{\nu-1,\mu-1,nm} + \gamma_{\nu,\mu-1}^{0}A_{\nu,\mu-1,nm}\end{aligned} \tag{24b}$$

Another recurrence relation can be obtained by using C_-.

By letting $m = n$ in (24a) and (24b), we get

$$\gamma_{nn}^{+}A_{\nu\mu,n+1,n+1} = \gamma_{\nu+1,\mu-1}^{-}A_{\nu+1,\mu-1,nn} + \gamma_{\nu-1,\mu-1}^{+}A_{\nu-1,\mu-1,nn} + \gamma_{\nu,\mu-1}^{0}B_{\nu,\mu-1,nn} \tag{25a}$$

$$\gamma_{nn}^{+}B_{\nu\mu,n+1,n+1} = \gamma_{\nu+1,\mu-1}^{-}B_{\nu+1,\mu-1,nn} + \gamma_{\nu-1,\mu-1}^{+}B_{\nu-1,\mu-1,nn} + \gamma_{\nu,\mu-1}^{0}A_{\nu,\mu-1,nn} \tag{25b}$$

Equations (18) and (25) can be used as in reference [17] to find the coefficients of the addition theorem from $A_{\nu\mu,10}$, $B_{\nu\mu,10}$. In other words, (18) and (25) are difference equations with which the values of $A_{\nu\mu,nm}$ and $B_{\nu\mu,nm}$ can be found from their boundary values $A_{\nu\mu,10}$ and $B_{\nu\mu,10}$. A detailed description of such a procedure is described in reference [17].

4. THE VECTOR ADDITION THEOREMS FOR $\mathbf{M}_{10}(\mathbf{r})$ and $\mathbf{M}_{11}(\mathbf{r})$

The recurrence relations need $A_{\nu\mu,10}$ and $B_{\nu\mu,10}$ as initial values. Hence, we need a simple addition theorem for $\mathbf{M}_{10}(\mathbf{r})$. To begin, it can be shown that [Appendix C or (13)].

$$\nabla \times \hat{z}\psi_{00}(\mathbf{r}) = \frac{1}{\sqrt{3}}\mathbf{M}_{10}(\mathbf{r}). \tag{26}$$

Therefore, using the scalar addition theorem,

$$\mathbf{M}_{10}(\mathbf{r}) = \sqrt{3}\sum_{\nu\mu}(\nabla' \times \hat{z})\psi_{\nu\mu}(\mathbf{r}')\beta_{\nu\mu,00}. \tag{27}$$

However, from Appendix C, Equation (C9),

$$\nabla' \times \hat{z}\psi_{\nu\mu}(\mathbf{r}') = z^{+}_{\nu\mu}\mathbf{M}_{\nu+1,\mu}(\mathbf{r}') + z^{-}_{\nu\mu}\mathbf{M}_{\nu-1,\mu}(\mathbf{r}') + z^{0}_{\nu\mu}\mathbf{N}_{\nu\mu}(\mathbf{r}'). \tag{28}$$

Then, using (28) in (27), we have

$$\mathbf{M}_{10}(\mathbf{r}) = \sqrt{3}\sum_{\nu=1}^{\infty}\sum_{\mu=-\nu}^{\nu}\Big[\mathbf{M}_{\nu\mu}(\mathbf{r}')\left(z^{+}_{\nu-1,\mu}\beta_{\nu-1,\mu,00} + z^{-}_{\nu+1,\mu}\beta_{\nu+1,\mu,00}\right) + \mathbf{N}_{\nu\mu}(\mathbf{r}')z^{0}_{\nu\mu}\beta_{\nu\mu,00}\Big] \tag{29}$$

Notice that since $\mathbf{M}_{00}(\mathbf{r}) = \mathbf{N}_{00}(\mathbf{r}') = 0$, the above summation starts with $\nu = 1$. Consequently,

$$A_{\nu\mu,10} = \sqrt{3}\left(z^{+}_{\nu-1,\mu}\beta_{\nu-1,\mu,00} + z^{-}_{\nu+1,\mu}\beta_{\nu+1,\mu,00}\right), \tag{30a}$$

$$B_{\nu\mu,10} = \sqrt{3}\; z^{0}_{\nu\mu}\beta_{\nu\mu,00}. \tag{30b}$$

As $A_{\nu\mu,11}$ and $B_{\nu\mu,11}$ are needed as initial values as well, we need a simple addition theorem for $\mathbf{M}_{11}(\mathbf{r})$. It can be shown that [Appendix D or (19)]

$$\nabla \times (\hat{x} + i\hat{y})\psi_{00}(\mathbf{r}) = -\sqrt{\frac{2}{3}}\mathbf{M}_{11}(\mathbf{r}), \tag{31}$$

or similar to (27),

$$\mathbf{M}_{11}(\mathbf{r}) = -\sqrt{\frac{3}{2}}\sum_{\nu\mu}\nabla' \times (\hat{x} + i\hat{y})\psi_{\nu\mu}(\mathbf{r}')\beta_{\nu\mu,00}. \tag{32}$$

But from Appendix D, Equation (D.8)

$$\nabla' \times (\hat{x} + i\hat{y})\psi_{\nu\mu}(\mathbf{r}') = \eta^{-}_{\nu\mu(+)}\mathbf{M}_{\nu-1,\mu+1}(\mathbf{r}') + \eta^{+}_{\nu\mu(+)}\mathbf{M}_{\nu+1,\mu+1}(\mathbf{r}') + \eta^{0}_{\nu\mu(+)}\mathbf{N}_{\nu,\mu+1}(\mathbf{r}') \tag{33}$$

Consequently, using (33) in (32), we have

$$\mathbf{M}_{11}(\mathbf{r}) = -\sqrt{\frac{3}{2}}\sum_{\nu=1}^{\infty}\sum_{\mu=-\nu}^{\nu}\Big[\mathbf{M}_{\nu\mu}(\mathbf{r}')\left(\eta^{-}_{\nu+1,\mu-1(+)}\beta_{\nu+1,\mu-1,00} + \eta^{+}_{\nu-1,\mu-1(+)}\beta_{\nu-1,\mu-1,00}\right) + \mathbf{N}_{\nu\mu}(\mathbf{r}')\eta^{0}_{\nu,\mu-1,(+)}\beta_{\nu,\mu-1,00}\Big] \tag{34}$$

Therefore, we conclude that

$$A_{\nu\mu,11} = -\sqrt{\frac{3}{2}}\left(\eta^{-}_{\nu+1,\mu-1(+)}\beta_{\nu+1,\mu-1,00} + \eta^{+}_{\nu-1,\mu-1(+)}\beta_{\nu-1,\mu-1,00}\right), \tag{35a}$$

$$B_{\nu\mu,11} = -\sqrt{\frac{3}{2}}\eta^{0}_{\nu,\mu-1(+)}\beta_{\nu,\mu-1,00}. \tag{35b}$$

The above yields the initial values for the recurrence relations, and $\beta_{\nu\mu,00}$ is [17]

$$\beta_{\nu\mu,00} = (-1)^{\mu+\nu}\sqrt{4\pi}Y_{\nu,-\mu}(\theta'', \phi'')j_\nu(kr''). \tag{36}$$

Alternatively, $A_{\nu\mu,10}$, $B_{\nu\mu,10}$, $A_{\nu\mu,11}$ and $B_{\nu\mu,11}$ can be obtained from Equations (12a) and (12b).

5. RESULTS AND CONCLUSION

The above formulas have been tested against the conventional ways of calculating the vector translational addition theorem. For most applications, $A_{\nu\mu,mn}$ and $B_{\nu\mu,mn}$ are needed for $n = 1, \ldots, n_{max}$, and $\nu = 1, \ldots, \nu_{max}$ with m and μ assuming their proper values. In other words, arrays of $A_{\nu\mu,mn}$ and $B_{\nu\mu,mn}$ are needed. Due to the reduced computational complexity [17], the speed is about 100 times faster in finding these arrays when $n_{max} = \nu_{max} = 5$, and about 400 times faster when $n_{max} = \nu_{max} = 10$, compared to the method using Gaunt coefficient and Wigner 3j symbols. The difference is even larger when n_{max} is larger, because of the reduced complexity of this method. Even if an efficient method is available to calculate the Gaunt coefficients, the method described in this paper will eventually be more efficient when n_{max} and ν_{max} become large.

In conclusion, an efficient way of solving for the coefficients of the vector addition theorem is derived. This method will have a significant impact on methods of calculating vector wave scattering solutions where the addition theorem has to be invoked.

APPENDIX

In this Appendix, we derive Equations (7a) to (7c). Before that is done, we develop some identities in Appendices A and B.

Appendix A: What is $\mathbf{r} \cdot \nabla \times \nabla \times \hat{z}\psi_{nm}(\mathbf{r})$?

Letting

$$I = \mathbf{r} \cdot \nabla \times \nabla \times \hat{z}\phi_{nm}(\mathbf{r}) = \mathbf{r} \cdot [\nabla\nabla \cdot \hat{z}\phi_{nm}(\mathbf{r}) - \hat{z}\nabla^2\phi_{nm}(\mathbf{r})] \tag{A.1}$$

where

$$\phi_{nm}(\mathbf{r}) = P_n^m(\cos\theta)e^{im\phi}j_n(\mathbf{r}) \tag{A.2}$$

and assuming that $k = 1$. Then, (A.1) becomes

$$I = r\frac{\partial}{\partial r}\frac{\partial}{\partial z}\phi_{nm}(\mathbf{r}) + r\cos\theta\phi_{nm}(\mathbf{r}) \tag{A.3}$$

But from previous work [17],

$$\frac{\partial}{\partial z}\phi_{nm}(\mathbf{r}) = -\frac{n-m+1}{2n+1}\phi_{n+1,m}(\mathbf{r}) + \frac{n+m}{2n+1}\phi_{n-1,m}(\mathbf{r}) \tag{A.4}$$

Therefore,

$$I = -\frac{n-m+1}{2n+1}r\frac{\partial}{\partial r}\phi_{n+1,m}(\mathbf{r}) + \frac{n+m}{2n+1}r\frac{\partial}{\partial r}\phi_{n-1,m}(\mathbf{r}) + r\cos\theta\phi_{nm}(\mathbf{r}). \tag{A.5}$$

It can be shown, using the appropriate identity, that

$$r\frac{\partial}{\partial r}\phi_{n+1,m}(\mathbf{r}) = P_{n+1}^m(\cos\theta)e^{im\phi} r j_n(r) - (n+2)\phi_{n+1,m}(\mathbf{r}) \ , \quad \text{(A.6a)}$$

$$r\frac{\partial}{\partial r}\phi_{n-1,m}(\mathbf{r}) = -P_{n+1}^m(\cos\theta)e^{im\phi} r j_n(r) + (n-1)\phi_{n-1,m}(\mathbf{r}) \ . \quad \text{(A.6b)}$$

Therefore.

$$I = \frac{(n+2)(n-m+1)}{2n+1}\phi_{n+1,m}(\mathbf{r}) + \frac{(n-1)(n+m)}{2n+1}\phi_{n-1,m}(\mathbf{r}) + r j_n(r) e^{im\phi}$$
$$\left[\cos\theta P_n^m(\cos\theta) - \frac{n-m+1}{2n+1}P_{n+1}^m(\cos\theta) - \frac{n+m}{2n+1}P_{n-1}^m(\cos\theta)\right] \ . \quad \text{(A.7)}$$

The last term in (A.7) can be shown to be zero. Therefore,

$$\mathbf{r}\cdot\nabla\times\nabla\times\hat{z}\phi_{nm}(\mathbf{r}) = \frac{(n+2)(n-m+1)}{2n+1}\phi_{n+1,m}(\mathbf{r}) + \frac{(n-1)(n+m)}{2n+1}\phi_{n-1,m}(\mathbf{r}). \quad \text{(A.8)}$$

Using the normalized wave function defined in (2), then,

$$\mathbf{r}\cdot\nabla\times\nabla\times\hat{z}\psi_{nm}(\mathbf{r}) = \zeta_{nm}^+\psi_{n+1,m}(\mathbf{r}) + \zeta_{nm}^-\psi_{n-1,m}(\mathbf{r}) \quad \text{(A.9)}$$

where

$$\zeta_{nm}^+ = -(n+2)a_{nm}^+ = (n+2)\sqrt{\frac{(n+m+1)(n-m+1)}{2n+1)(2n+3)}} \ ,$$

$$\zeta_{nm}^- = (n-1)a_{nm}^- = (n-1)\sqrt{\frac{(n+m)(n-m)}{(2n+1)(2n-1)}} \ , \quad \text{(A.10)}$$

where $a_{nm}^\pm$ are defined in Equation (14).

Appendix B: What is $\mathbf{r}\cdot\nabla\times\nabla\times(\hat{x}\pm i\hat{y})\psi_{nm}(\mathbf{r})$?

First, let

$$I_+ = \mathbf{r}\cdot\nabla\times\nabla\times(\hat{x}+i\hat{y})\phi_{nm}(\mathbf{r}) = \mathbf{r}\cdot[\nabla\nabla\cdot(\hat{x}+i\hat{y})\phi_{nm}(\mathbf{r}) - (\hat{x}-i\hat{y})\nabla^2\phi_{nm}(\mathbf{r})], \quad \text{(B.1)}$$

where $\phi_{nm}(\mathbf{r})$ is as defined in Appendix A. Then, (B.1) becomes

$$I_+ = r\frac{\partial}{\partial r}C_+\phi_{nm}(\mathbf{r}) + r\sin\theta e^{i\phi}\phi_{nm}(\mathbf{r}), \quad \text{(B.2)}$$

where

$$C_+ = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \quad \text{(B.3)}$$

Using previously derived result for $C_+\phi_{nm}(\mathbf{r})$ [17], we have

$$I_+ = -\frac{1}{2n+1}r\frac{\partial}{\partial r}\phi_{n-1,m+1}(\mathbf{r}) - \frac{1}{2n+1}r\frac{\partial}{\partial r}\phi_{n+1,m+1}(\mathbf{r}) + r\sin\theta e^{i\phi}\phi_{nm}(\mathbf{r}). \quad \text{(B.4)}$$

Using identity (A.6), we have

$$I_+ = -\frac{n-1}{2n+1}\phi_{n-1,m+1}(\mathbf{r}) + \frac{n+2}{2n+1}\phi_{n+1,m+1}(\mathbf{r}) + rj_n(r)e^{i(m+1)\phi}\left[\frac{P_{n-1}^{m+1}(\cos\theta)}{2n+1} - \frac{P_{n+1}^{m+1}(\cos\theta)}{2n+1} + \sin\theta P_n^m(\cos\theta)\right] \quad \text{(B.5)}$$

The last term in (B.5) can be shown to be zero. Hence,

$$\mathbf{r}\cdot\nabla\times\nabla\times(\hat{x}+i\hat{y})\phi_{nm}(\mathbf{r}) = \frac{n+2}{2n+1}\phi_{n+1,m+1}(\mathbf{r}) - \frac{n-1}{2n+1}\phi_{n-1,m+1}(\mathbf{r}). \quad \text{(B.6)}$$

By using the normalized wave function, it can be shown that

$$\mathbf{r}\cdot\nabla\times\nabla\times(\hat{x}\pm i\hat{y})\psi_{nm}(\mathbf{r}) = \rho^+_{nm(\pm)}\psi_{n+1,m\pm1}(\mathbf{r}) + \rho^-_{nm(\pm)}\psi_{n-1,m\pm1}(\mathbf{r}) \quad \text{(B.7)}$$

where

$$\rho^+_{nm(\pm)} = -(n+2)b^+_{nm(\pm)} = \mp(n+2)\sqrt{\frac{(n\pm m+1)(n\pm m+2)}{(2n+1)(2n+3)}},$$

$$\rho^-_{nm(\pm)} = (n-1)b^-_{nm(\pm)} = \pm(n-1)\sqrt{\frac{(n\mp m)(n\mp m-1)}{(2n+1)(2n-1)}}, \quad \text{(B.8)}$$

where $b^\pm_{nm(\pm)}$ are defined in (19).

Appendix C: What is $\nabla\times\hat{z}\psi_{nm}(\mathbf{r})$?

First, let us expand

$$\nabla\times\hat{z}\psi_{nm}(\mathbf{r}) = \sum_{\nu\mu}[m_{nm,\nu\mu}\mathbf{M}_{\nu\mu}(\mathbf{r}) + n_{nm,\nu\mu}\mathbf{N}_{\nu\mu}(\mathbf{r})]. \quad \text{(C.1)}$$

It can be shown that (assuming $k=1$),

$$\mathbf{r}\cdot\mathbf{N}_{\nu\mu}(\mathbf{r}) = \nu(\nu+1)\psi_{\nu\mu}(\mathbf{r}). \quad \text{(C.2)}$$

Hence, dotting (C.1) with $\mathbf{r}$, we have

$$\mathbf{r}\cdot\nabla\times\hat{z}\psi_{nm}(\mathbf{r}) = \sum_{\nu\mu} n_{nm,\nu\mu}\nu(\nu+1)\psi_{\nu\mu}(\mathbf{r}), \quad \text{(C.3)}$$

since $\mathbf{r}\cdot\mathbf{M}_{\nu\mu}(\mathbf{r}) = 0$. But

$$\mathbf{r}\cdot\nabla\times\hat{z}\psi_{nm}(\mathbf{r}) = \frac{\partial}{\partial\phi}\psi_{nm}(\mathbf{r}) = im\psi_{nm}(\mathbf{r}). \quad \text{(C.4)}$$

Comparing (C.3) and (C.4), we have

$$n_{nm,\nu\mu} = \delta_{n\nu}\delta_{m\mu}\frac{im}{n(n+1)}. \quad \text{(C.5)}$$

Similarly, taking the curl of (C.1) and dotting with $\mathbf{r}$, we have

$$\mathbf{r}\cdot\nabla\times\nabla\times\hat{z}\psi_{nm}(\mathbf{r}) = \sum_{\nu\mu} m_{nm,\nu\mu}\nu(\nu+1)\psi_{\nu\mu}(\mathbf{r}) \quad \text{(C.6)}$$

after making use of the definition (2) and (C.2). But from Appendix A,

$$\mathbf{r}\cdot\nabla\times\nabla\times\hat{z}\psi_{nm}(\mathbf{r})=\zeta_{nm}^{+}\psi_{n+1,m}(\mathbf{r})+\zeta_{nm}^{-}\psi_{n-1,m}(\mathbf{r}). \tag{C.7}$$

Comparing (C.6) and (C.7), we have

$$m_{nm,\nu\mu}=\delta_{\nu,n+1}\delta_{\mu m}\frac{\zeta_{nm}^{+}}{(n+1)(n+2)}+\delta_{\nu,n-1}\delta_{\mu m}\frac{\zeta_{nm}^{-}}{n(n-1)}. \tag{C.8}$$

Consequently,

$$\nabla\times\hat{z}\psi_{nm}(\mathbf{r})=z_{nm}^{+}\mathbf{M}_{n+1,m}(\mathbf{r})+z_{nm}^{-}\mathbf{M}_{n-1,m}(\mathbf{r})+z_{nm}^{0}\mathbf{N}_{nm}(\mathbf{r}) \tag{C.9}$$

where

$$\begin{aligned}
z_{nm}^{+}&=\frac{\zeta_{nm}^{+}}{(n+1)(n+2)}=-\frac{a_{nm}^{+}}{n+1},\\
z_{nm}^{-}&=\frac{\zeta_{nm}^{-}}{n(n-1)}=\frac{a_{nm}^{-}}{n},\\
z_{nm}^{0}&=\frac{im}{n(n+1)}.
\end{aligned} \tag{C.10}$$

Appendix D: What is $\nabla\times(\hat{x}\pm i\hat{y})\psi_{nm}(\mathbf{r})$?

First, we let

$$\nabla\times(\hat{x}+i\hat{y})\psi_{nm}(\mathbf{r})=\sum_{\nu\mu}[p_{\nu\mu}\mathbf{M}_{\nu\mu}(\mathbf{r})+q_{\nu\mu}\mathbf{N}_{\nu\mu}(\mathbf{r})]. \tag{D.1}$$

Then, assuming $k=1$,

$$\mathbf{r}\cdot\nabla\times(\hat{x}+i\hat{y})\psi_{nm}(\mathbf{r})=\sum_{\nu\mu}q_{\nu\mu}\nu(\nu+1)\psi_{\nu\mu}(\mathbf{r}). \tag{D.2}$$

But

$$\begin{aligned}
\mathbf{r}\cdot\nabla\times(\hat{x}+i\hat{y})\psi_{nm}(\mathbf{r})&=(\hat{x}+i\hat{y})\cdot\mathbf{r}\times\nabla\psi_{nm}(\mathbf{r})\\
&=(\hat{x}+i\hat{y})\cdot\left[\hat{\phi}\frac{\partial}{\partial\theta}\psi_{nm}(\mathbf{r})-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\psi_{nm}(\mathbf{r})\right]\\
&=ie^{i\phi}\left[\frac{\partial}{\partial\theta}\psi_{nm}(\mathbf{r})-\frac{m\cos\theta}{\sin\theta}\psi_{nm}(\mathbf{r})\right]\\
&=i\sqrt{(n-m)(n+m+1)}\psi_{n,m+1}(\mathbf{r}).
\end{aligned} \tag{D.3}$$

Comparing (D.2) and (D.3), we have

$$q_{\nu\mu}=i\delta_{n\nu}\delta_{\mu,m+1}\sqrt{(n-m)(n+m+1)}/n(n+1). \tag{D.4}$$

Next, taking the curl of (D.1) and dotting with $\mathbf{r}$, we have

$$\mathbf{r}\cdot\nabla\times\nabla\times(\hat{x}+i\hat{y})\psi_{nm}(\mathbf{r})=\sum_{\nu\mu}p_{\nu\mu}\nu(\nu+1)\psi_{\nu\mu}(\mathbf{r}). \tag{D.5}$$

But from Appendix B, we have

$$\mathbf{r}\cdot\nabla\times\nabla\times(\hat{x}+i\hat{y})\psi_{nm}(\mathbf{r}) = \rho^{+}_{nm(+)}\psi_{n+1,m+1}(\mathbf{r}) + \rho^{-}_{nm(+)}\psi_{n-1,m+1}(\mathbf{r}). \quad (D.6)$$

Comparing (D.5) and (D.6), we have

$$p_{\nu\mu} = \delta_{\nu,n+1}\delta_{\mu,m+1}\frac{\rho^{+}_{nm(+)}}{(n+1)(n+2)} + \delta_{\nu,n-1}\delta_{\mu,m+1}\frac{\rho^{-}_{nm(+)}}{n(n-1)}\,. \quad (D.7)$$

Finally, we have

$$\begin{aligned}\nabla\times(\hat{x}\pm i\hat{y})\psi_{nm}(\mathbf{r})) =& \eta^{-}_{nm(\pm)}\mathbf{M}_{n-1,m\pm1}(\mathbf{r})\\ &+ \eta^{+}_{nm(\pm)}\mathbf{M}_{n+1,m\pm1}(\mathbf{r}) + \eta^{0}_{nm(\pm)}\mathbf{N}_{n,m\pm1}(\mathbf{r}) \quad (D.8)\end{aligned}$$

where

$$\begin{aligned}\eta^{-}_{nm(\pm)} &= \frac{\rho^{-}_{nm(\pm)}}{n(n-1)} = \frac{b^{-}_{nm(\pm)}}{n}\,,\\ \eta^{+}_{nm(\pm)} &= \frac{\rho^{+}_{nm(\pm)}}{(n+1)(n+2)} = -\frac{b^{+}_{nm(\pm)}}{n+1}\,,\\ \eta^{0}_{nm(\pm)} &= \frac{i\sqrt{(n\mp m)(n\pm m+1)}}{n(n+1)} \end{aligned} \quad (D.9)$$

With the above identities known, it is easy to show that

$$\begin{aligned}\nabla\times\hat{x}\psi_{nm}(\mathbf{r}) =& x^{-+}_{nm}\mathbf{M}_{n-1,m+1}(\mathbf{r}) + x^{++}_{nm}\mathbf{M}_{n+1,m+1}(\mathbf{r}) + x^{0+}_{nm}\mathbf{N}_{n,m-1}(\mathbf{r})\\ &+ x^{--}_{nm}\mathbf{M}_{n-1,m-1}(\mathbf{r}) + x^{+-}_{nm}\mathbf{M}_{n+1,m-1}(\mathbf{r}) + x^{0-}_{nm}\mathbf{N}_{n,m-1}(\mathbf{r})\end{aligned} \quad (D.10)$$

where

$$x^{-,\pm}_{nm} = \frac{1}{2}\eta^{-}_{nm(\pm)}, \quad x^{+,\pm}_{nm} = \frac{1}{2}\eta^{+}_{nm(\pm)}, \quad x^{0,\pm}_{nm} = \frac{1}{2}\eta^{0}_{nm(\pm)}. \quad (D.11)$$

Similarly,

$$\begin{aligned}\nabla\times\hat{y}\psi_{nm}(\mathbf{r}) =& y^{-+}_{nm}\mathbf{M}_{n-1,m+1}(\mathbf{r}) + y^{++}_{nm}\mathbf{M}_{n+1,m+1}(\mathbf{r}) + y^{0+}_{nm}\mathbf{N}_{n,m+1}(\mathbf{r})\\ &+ y^{--}_{nm}\mathbf{M}_{n-1,m-1}(\mathbf{r}) + y^{+-}_{nm}\mathbf{M}_{n+1,m-1}(\mathbf{r}) + y^{0-}_{nm}\mathbf{N}_{n,m-1}(\mathbf{r})\end{aligned} \quad (D.12)$$

where

$$y^{-,\pm}_{nm} = \frac{\pm1}{2i}\eta^{-}_{nm(\pm)}, \quad y^{+,\pm}_{nm} = \frac{\pm1}{2i}\eta^{+}_{nm(\pm)}, \quad y^{0,\pm}_{nm} = \frac{\pm1}{2i}\eta^{0}_{nm(\pm)} \quad (D.13)$$

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