The Casimir Effect

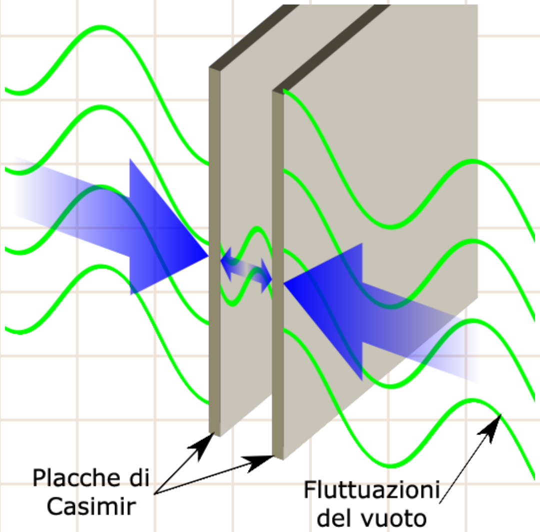
QI DAI
LRS FEB 27 2014

Force from Nowhere

Hendrik Casimir (1909-2000)
Dutch theoretical physicist,
Philips Research Lab

- Two mirrors face each other in empty space, what happens?
- Nothing at all?
- Hendrik Casimir's answer in 1948

They are mutually attracted to each other!



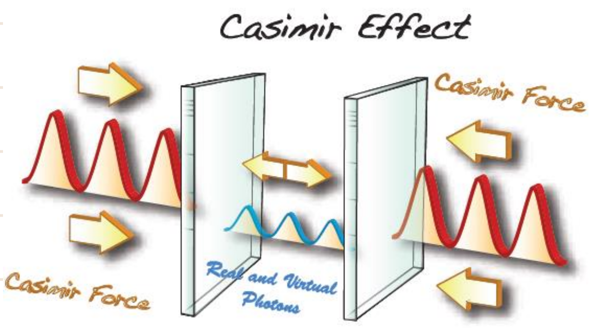
"On the attraction between two perfectly conducting plates"

'Summer or autumn 1947 (but I am not absolutely certain that it [was] not somewhat earlier or later) I mentioned my results to Niels Bohr, during a walk. "That is nice," he said, "That is something new." I told him that I was puzzled by the extremely simple form of the expressions for the interaction at very large distance and he mumbled something about zero-point energy. That was all, but it put me on a new track.

Zero-Point Energy

- Simple calculation via zero-point energy
- Energy sum-over-modes

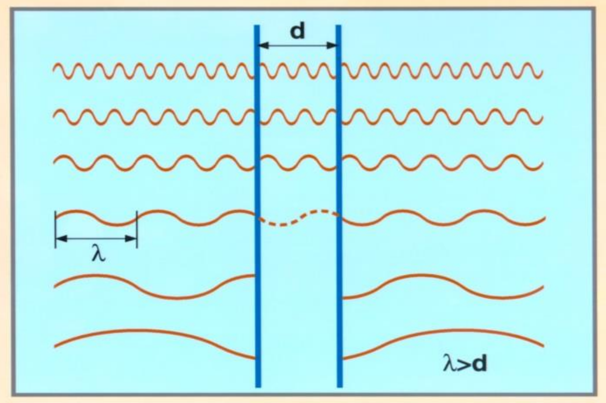
$$E = \underbrace{\sum_{p,\mathbf{k}} \frac{\hbar}{2} \left[\sum_n \omega_n^p \right]_L}_{\text{Infinite zero point energy}} - \underbrace{\sum_{p,\mathbf{k}} \frac{\hbar}{2} \left[\sum_n \omega_n^p \right]_{L \to \infty}}_{\text{Setting the zero}}$$



$$E \propto \frac{\hbar c}{L^\alpha}$$

- Force between neutral conducting plates

$$\mathcal{F} = -\frac{\hbar c \pi^2}{240 d^4}$$



Are Zero-Point Energies Real?

R. L. Jaffe, Jane and Otto Morningstar professor of physics, MIT. He was formerly director of MIT center for theoretical physics

- Casimir effect – Evidence that zero point energies are "real"
- In 2005, Jaffe points out, "Casimir effect gives no more (or less) support for the "reality" of the vacuum energy of fluctuating quantum fields than any other one-loop effect in quantum electrodynamics, like the vacuum polarization contribution to the Lamb shift, for example."

Are Zero-Point Energies Real?

Julian Schwinger (1918 – 1994), UCLA professor. Nobel prize in physics (1965) for his work on QED

$$\mathcal{F} = -\frac{\hbar c \pi^2}{240 d^4}$$

Seems universal

- Casimir force vanishes as the fine structure constant goes to zero
- Casimir energy can be expressed entirely in terms of Feynman diagrams with external legs – i.e. in terms of S-matrix elements which make no reference to the vacuum. (Schwinger, DeRaad, Milton)

"…one of the least intuitive consequences of quantum electrodynamics" – Julian Schwinger

" It is clear that this zero-point energy has no physical reality." – Pauli (Nobel lecture, 1945)

Peter Milonni, Professor of Physics, University of Rochester

A Small Force

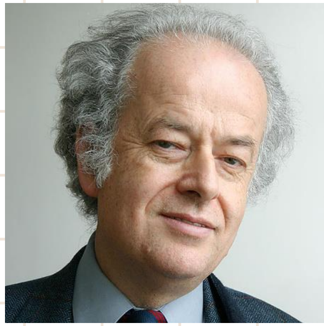
- Casimir force is small, only 0.013 dyne for two 1x1 cm plates separated by 1μm

- This is comparable to the Coulomb force on the electron in the hydrogen atom, or to the gravitational attraction between two one-pound weights separated by half an inch. Or about 1/1000 the weight of a housefly. – P. Milonni (Los Alamos National Lab)'s lecture at Institute for Quantum Computing in U of Waterloo

Experiments

Steven Lamoreaux

Umar Mohideen



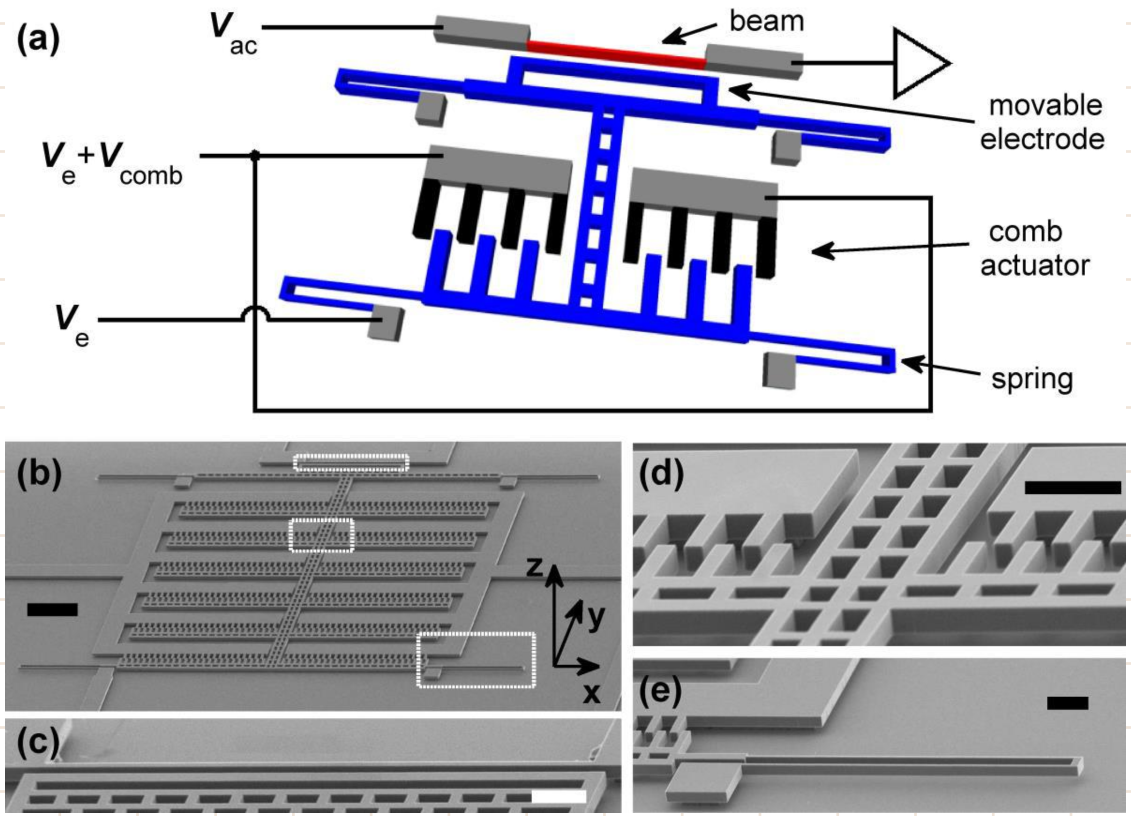
Federico Capasso

Experiments in 1950s-1970s (Derjaguin& Abrikosova, Sparnaay, Tabor & Winterton, Hunklinger, van Blokland & Overbeek). Casimir force has been measured to an accuracy of a few percent since mid-1990s

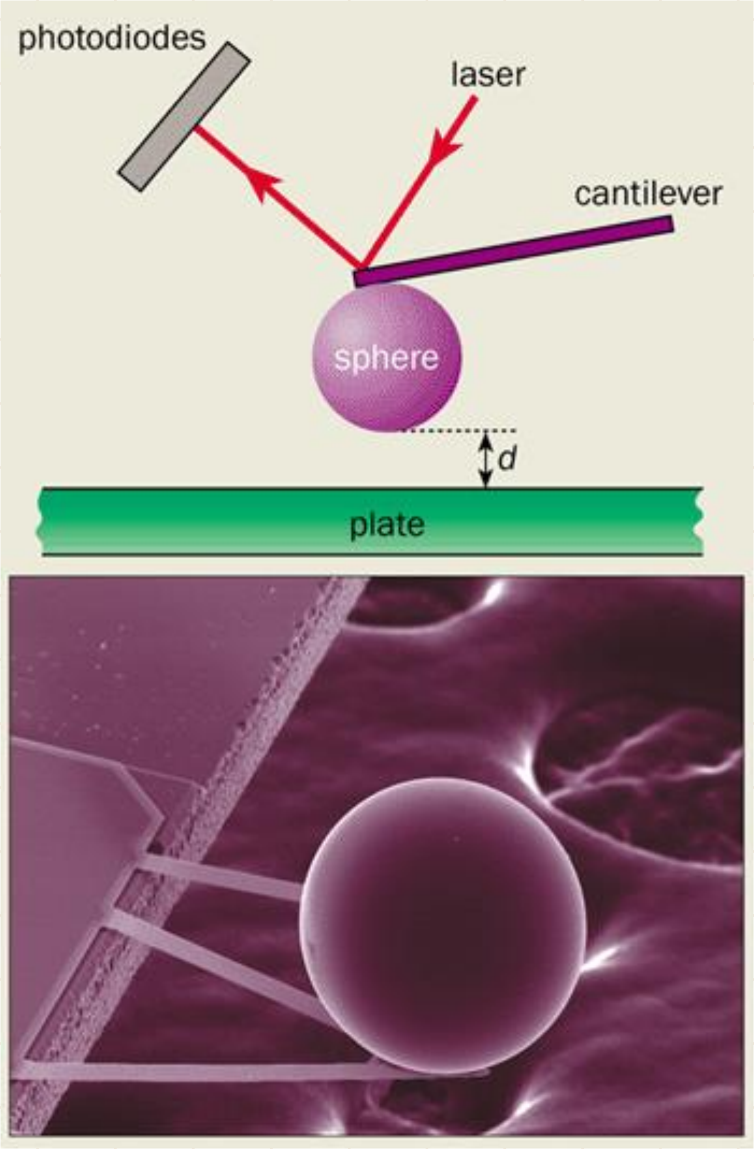
- Lamoreaux (U of Washington, later Los Alamos, now Yale), 1997
- Mohideen et al. (U of California at Riverside), 1998
- Chan, Capasso et al. (Harvard), 2001
- Bressl et al. (Dartmouth), 2002
- Decca et al. (Indianna), 2003
- Zou, Chan et al. (HKUST), 2013

Experiments

Chan et al., "Casimir forces on a silicon micromechanical chip", Nature Communications, 2013



Umar Mohideen, PRL, 1998.
Casimir force tips the balance.



Quantum Vacuum

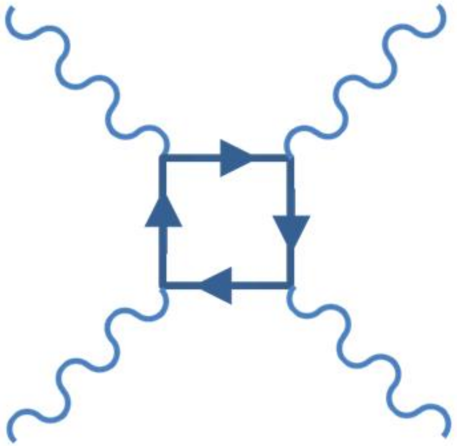
- QED vacuum – lowest energy state of EM field when fields are quantized
- "Field" – space filled with interconnected vibrating balls and springs. Field strength (ball's displacement from rest position) is quantized at each point in space.

> The quantum theory asserts that a vacuum, even the most perfect vacuum devoid of any matter, is not really empty. Rather the quantum vacuum can be depicted as a sea of continuously appearing [pairs of] particles that manifest themselves in the apparent jostling of particles that is quite distinct from their thermal motions. These particles are 'virtual', as opposed to real, particles… At any given instant, the vacuum is full of such virtual pairs, which leave their signature behind, by affecting the energy levels of atoms.
>
> -Joseph Silk on the shores of the unknown, p.62

Quantum Fluctuation

- Quantum fluctuation – temporary appearance of energetic particles out of empty space, as allowed by $\Delta E \Delta t \geq \frac{\hbar}{2}$
- Virtual particle – transient fluctuation exhibiting ordinary particle's characteristics, but exists for a limited time
- Interaction between actual particles are calculated in terms of exchanges of virtual particles (QFT's perturbation theory)

A Feynman diagram (one-loop diagram) for photo-photon scattering, one photon scatters from the transient vacuum charge fluctuations of the other. Solid lines correspond to a fermion propagator, wavy ones to bosons.



Second Quantization

Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{J}\right)$$

Source-free

$$\phi = \mathbf{J} = 0$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

Separation of variables

$$\mathbf{A}\left(\mathbf{r}, t\right) = \alpha(t)\mathbf{A}_0(\mathbf{r}) + \alpha^*(t)\mathbf{A}_0^*(\mathbf{r})$$
$$= \alpha(0)e^{-i\omega t}\mathbf{A}_0(\mathbf{r}) + \alpha^*(0)e^{i\omega t}\mathbf{A}_0^*(\mathbf{r})$$

$$\nabla^2 \mathbf{A}_0(\mathbf{r}) + k^2\mathbf{A}_0(\mathbf{r}) = 0$$

Vector and scalar potentials

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0$$

$$\mathbf{E}(\mathbf{r}, t) = -\frac{1}{c}\left[\dot{\alpha}(t)\mathbf{A}_0(\mathbf{r}) + \dot{\alpha}^*(t)\mathbf{A}_0^*(\mathbf{r})\right]$$

$$\mathbf{B}(\mathbf{r}, t) = \alpha(t)\nabla \times \mathbf{A}_0(\mathbf{r}) + \alpha^*(t)\nabla \times \mathbf{A}_0^*(\mathbf{r})$$

Second Quantization

Hamiltonian

$$H_F = \frac{1}{8\pi}\int_V d\mathbf{r}\,(\mathbf{E}^2+\mathbf{B}^2) = \frac{1}{8\pi c^2}\dot{\alpha}(t)^2\int_V d\mathbf{r}\mathbf{A}_0(\mathbf{r})^2 + \frac{1}{8\pi c^2}\dot{\alpha}^*(t)^2\int_V d\mathbf{r}\mathbf{A}_0^*(\mathbf{r})^2$$
$$+\frac{1}{4\pi c^2}|\dot{\alpha}(t)|^2\int_V d\mathbf{r}\,|\mathbf{A}_0(\mathbf{r})|^2 + \frac{1}{8\pi}\alpha(t)^2\int_V d\mathbf{r}\,[\nabla\times\mathbf{A}_0(\mathbf{r})]^2$$
$$+\frac{1}{8\pi}\alpha^*(t)^2\int_V d\mathbf{r}\,[\nabla\times\mathbf{A}_0^*(\mathbf{r})]^2 + \frac{1}{4\pi}|\alpha(t)|^2\int_V d\mathbf{r}\,|\nabla\times\mathbf{A}_0(\mathbf{r})|^2$$

$$\int_V d\mathbf{r}\,[\nabla\times\mathbf{A}_0(\mathbf{r})]^2 = k^2\int_V d\mathbf{r}\mathbf{A}_0(\mathbf{r})^2$$

normalization

$$\int_V d\mathbf{r}\,|\mathbf{A}_0(\mathbf{r})|^2 = 1$$

$$H_F = \frac{k^2}{2\pi}|\alpha(t)|^2$$

Classical real quantities

$$q(t) = \frac{i}{c^2\sqrt{4\pi}}[\alpha(t)+\alpha^*(t)]$$
$$p(t) = \frac{k}{c\sqrt{4\pi}}[\alpha(t)-\alpha^*(t)]$$

Simple harmonic oscillator

$$H_F = \frac{1}{2}(p^2+\omega^2 q^2)$$

Second Quantization

Poisson bracket

$$\dot{q} = \frac{\partial H}{\partial p} = \{q, H\}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = \{p, H\}$$

Canonical commutation, Heisenberg picture

$$\dot{\hat{q}} = \frac{1}{i\hbar}\left[\hat{q}, \hat{H}\right] = \frac{\hat{p}}{m}$$

$$\dot{\hat{p}} = \frac{1}{i\hbar}\left[\hat{p}, \hat{H}\right] = -m\omega^2 \hat{q}$$

Lowering
Raising

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{q})$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{q})$$

Vector potential operator

$$\mathbf{A}(\mathbf{r}, t) = \sqrt{\frac{2\pi\hbar c^2}{\omega}}\left[\hat{a}(t)\mathbf{A}_0(\mathbf{r}) + \hat{a}^\dagger(t)\mathbf{A}_0^*(\mathbf{r})\right]$$

Field operators

$$\mathbf{E}(\mathbf{r}, t) = i\sqrt{2\pi\hbar\omega}\left[\hat{a}(t)\mathbf{A}_0(\mathbf{r}) + \hat{a}^\dagger(t)\mathbf{A}_0^*(\mathbf{r})\right]$$

$$\mathbf{B}(\mathbf{r}, t) = \sqrt{\frac{2\pi\hbar c^2}{\omega}}\left[\hat{a}(t)\nabla \times \mathbf{A}_0(\mathbf{r}) + \hat{a}^\dagger(t)\nabla \times \mathbf{A}_0^*(\mathbf{r})\right]$$

$$\hat{\alpha}(t) = \sqrt{\frac{2\pi\hbar c^2}{\omega}}\hat{a}(t)$$

$$\hat{\alpha}^*(t) = \sqrt{\frac{2\pi\hbar c^2}{\omega}}\hat{a}^\dagger(t)$$

Hamiltonian

$$\hat{H}_F = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

Zero-photon state

$$\langle\mathbf{E}(\mathbf{r}, t)\rangle = \langle\mathbf{B}(\mathbf{r}, t)\rangle = 0$$

Regularization

- Summing over all possible oscillators at all points in space gives an infinite quantity
- Renormalization to remove infinity
- Removal of infinity presents a challenge in the search for a Theory of Everything
- Regulator in summation

Exponential regulator

$$\langle E(t)\rangle = \frac{1}{2}\sum_n \hbar|\omega_n| \exp(-t|\omega_n|)$$

Gaussian regulator

$$\langle E(t)\rangle = \frac{1}{2}\sum_n \hbar|\omega_n| \exp(-t^2|\omega_n|^2)$$

$t \to 0^+$

Zeta function regulator

$$\langle E(s)\rangle = \frac{1}{2}\sum_n \hbar|\omega_n||\omega_n|^{-s}$$

$s \to 0$

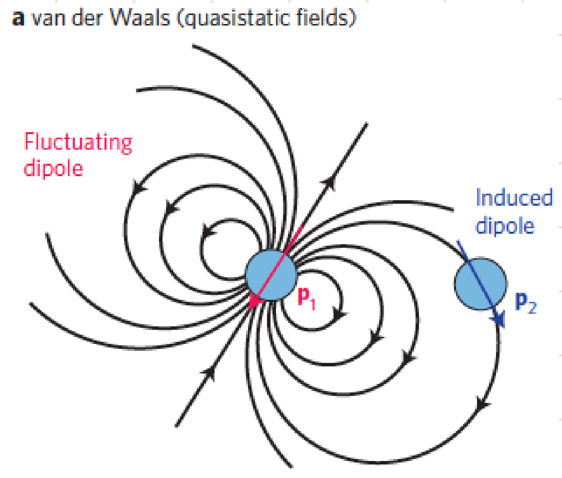
Van Der Waals Force

van der Waals (1837-1923)

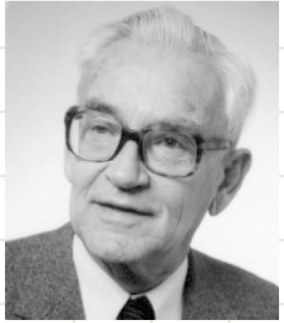
Fritz London (1900-1954)

Intermolecular attraction between molecules

- Orientation effect (W. H. Keesom, 1912)
  - between permanent dipoles
- Induction effect (Debye, 1920, Falkenhagen, 1922)
  - Permanent dipole and induced dipole
- Dispersion effect (London, 1937)
  - Fluctuating dipole and inductive effects
  - Potential scales as d^{-6}
  - Fails when molecules are at large separations



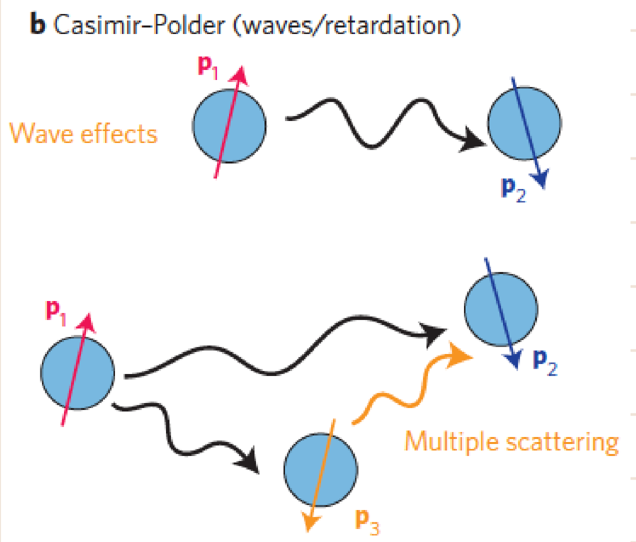
Casimir-Polder Force



Dirk Polder (1919-2001),
Dutch physicist

Evgeny Lifshitz (1915-1985),
Soviet physicist

- Finite speed of light makes different!
- At large separation, potential scales as d^{-7}
- Casimir force is a retarded van der Waals force, not additive
- Out of quantum and thermal fluctuation



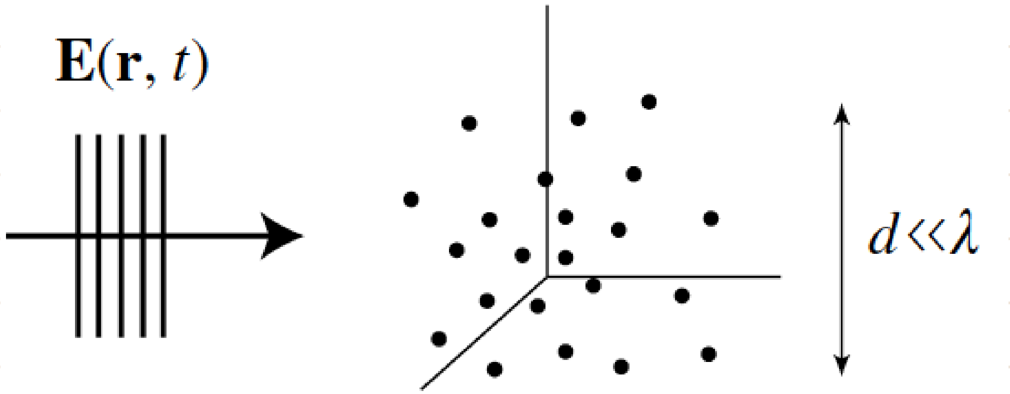
Fluctuation dissipation theorem

Analysis of arbitrary dielectric and realistic metal plates, multilayer planar geometries (Lifshitz and his students, 1961)

Fluctuation Dissipation Theorem

- Mostly derived using Fermi's golden rule
- Callen and Welton's derivation (1951)

A purely classical but intuitive derivation following Novoltny's book



$$f_{eq}(s) = f_0 e^{-H_0(s)/kT} \qquad s = [q_1 \dots q_N;\ p_1 \dots p_N]$$

$$\int f_{eq}\, ds = 1$$

Fluctuation Dissipation Theorem

System's ensemble average dipole moment

$$\langle \boldsymbol{\mu}(s,t) \rangle = \frac{\int f_{\mathrm{eq}}(s)\, \boldsymbol{\mu}(s,t)\, \mathrm{d}s}{\int f_{\mathrm{eq}}(s)\, \mathrm{d}s} = \langle \boldsymbol{\mu} \rangle$$

Perturbed Hamiltonian

$$H = H_0 + \delta H = H_0 - \boldsymbol{\mu}(s,t) \cdot \mathbf{E}(t) = H_0 - \sum_k \mu_k(s,t) E_k(t) \qquad k = x, y, z$$

Dipole moment deviation due to perturbation

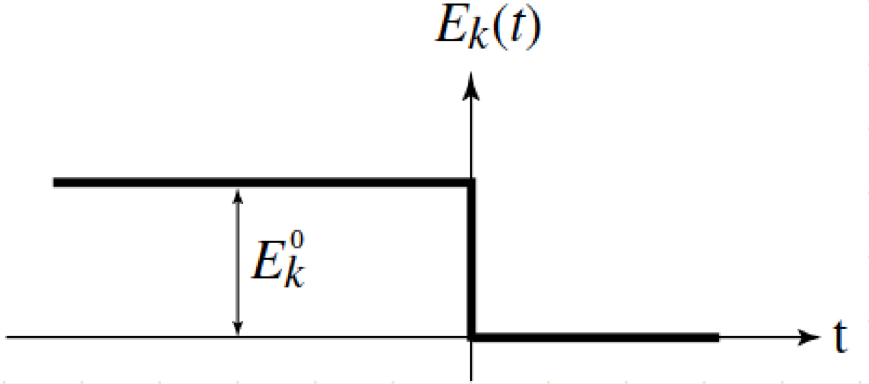
$$\delta \bar{\boldsymbol{\mu}}(t) = \bar{\boldsymbol{\mu}}(t) - \langle \boldsymbol{\mu} \rangle$$

$$\delta \bar{\mu}_j(t) = \frac{1}{2\pi} \sum_k \int_{-\infty}^{t} \tilde{\alpha}_{jk}(t-t')\, E_k(t')\, \mathrm{d}t' \qquad j, k = x, y, z$$

$$\tilde{\alpha}_{jk}(t,t') = \tilde{\alpha}_{jk}(t-t')$$

$$\tilde{\alpha}_{jk}(t-t') = 0 \text{ for } t' > t$$

Fluctuation Dissipation Theorem



Simple perturbation promotes system from a relaxed state to another

$$\delta\bar{\mu}_j(t) = \frac{E_k^0}{2\pi}\int_{-\infty}^{0}\tilde{\alpha}_{jk}(t-t')\,\mathrm{d}t' = \frac{E_k^0}{2\pi}\int_{t}^{\infty}\tilde{\alpha}_{jk}(\tau)\,\mathrm{d}\tau$$

$$\tilde{\alpha}_{jk}(t) = -\frac{2\pi}{E_k^0}\,\Theta(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\delta\bar{\mu}_j(t)$$

Fluctuation Dissipation Theorem

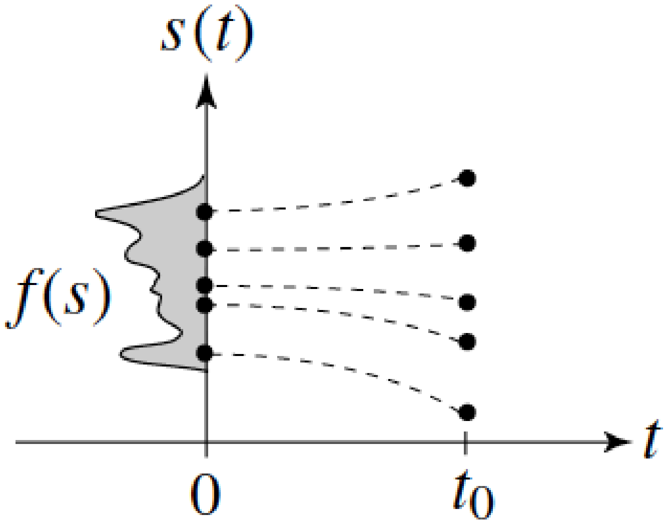
Average dipole moment at t

$$\bar{\boldsymbol{\mu}}(t) = \frac{\int f(s)\,\boldsymbol{\mu}(s,t)\,\mathrm{d}s}{\int f(s)\,\mathrm{d}s}$$

$$\bar{\boldsymbol{\mu}}(t) = \left\langle \boldsymbol{\mu} \right\rangle - \frac{1}{kT}\left[\left\langle \delta H(s)\,\boldsymbol{\mu}(s,t)\right\rangle - \left\langle \boldsymbol{\mu}(s,t)\right\rangle\left\langle \delta H(s)\right\rangle\right]$$

$$\delta\bar{\mu}_j(t) = \bar{\mu}_j(t) - \langle \mu_j \rangle = \frac{E_k^0}{kT}\left\langle \delta\mu_k(0)\,\delta\mu_j(t)\right\rangle$$

$$\tilde{\alpha}_{jk}(t) = -\frac{2\pi}{kT}\,\Theta(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \delta\mu_k(0)\,\delta\mu_j(t)\right\rangle \quad (classical)$$



Newton's motion equation maps each s at t=0 to a certain s at t=t0

$$f(s) \propto \mathrm{e}^{-[H_0+\delta H]/kT} = f_{\mathrm{eq}}(s)\,\mathrm{e}^{-\delta H(s)/kT} = f_{\mathrm{eq}}(s)\left[1 - \frac{1}{kT}\,\delta H(s) + \cdots\right]$$

$$[1 - \langle \delta H \rangle / kT]^{-1} \approx [1 + \langle \delta H \rangle / kT - \ldots]$$

Fluctuation Dissipation Theorem

Classical
$$\left[\alpha_{jk}(\omega) - \alpha^*_{kj}(\omega)\right]\delta(\omega-\omega') = \frac{2\pi i\omega}{kT}\left\langle \delta\hat{\mu}_j(\omega)\,\delta\hat{\mu}^*_k(\omega')\right\rangle$$

$$kT \to \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1} + \hbar\omega$$

Quantum
$$\left\langle \delta\hat{\mu}_j(\omega)\,\delta\hat{\mu}^*_k(\omega')\right\rangle = \frac{1}{2\pi i\omega}\left[\frac{\hbar\omega}{1 - e^{-\hbar\omega/kT}}\right]\left[\alpha_{jk}(\omega) - \alpha^*_{kj}(\omega)\right]\delta(\omega-\omega')$$

EM source
$$\left\langle \delta\hat{j}_j(\mathbf{r},\omega)\,\delta\hat{j}^*_k(\mathbf{r}',\omega')\right\rangle = \frac{\omega\varepsilon_0}{\pi}\,\varepsilon''(\omega)\left[\frac{\hbar\omega}{1 - e^{-\hbar\omega/kT}}\right]\delta(\omega-\omega')\,\delta(\mathbf{r}-\mathbf{r}')\,\delta_{jk}$$

EM field
$$\left\langle \delta\hat{E}_j(\mathbf{r},\omega)\,\delta\hat{E}^*_k(\mathbf{r}',\omega')\right\rangle = \frac{\omega}{\pi c^2\varepsilon_0}\left[\frac{\hbar\omega}{1 - e^{-\hbar\omega/kT}}\right]\mathrm{Im}\{G_{jk}(\mathbf{r},\mathbf{r}';\omega)\}\,\delta(\omega-\omega')$$

$$\delta\hat{\mathbf{E}}(\mathbf{r},\omega) = i\omega\mu_0\int_{V_0}\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}_0;\omega)\,\delta\hat{\mathbf{j}}(\mathbf{r}_0,\omega)\,d^3\mathbf{r}_0$$

$$\sum_n\int_{V_0} G_{jn}(\mathbf{r},\mathbf{r}_0;\omega)\,\varepsilon''(\omega)\,G_{kn}(\mathbf{r}',\mathbf{r}_0;\omega)\,d^3\mathbf{r}_0 = \mathrm{Im}\{G_{jk}(\mathbf{r},\mathbf{r}';\omega)\}$$

Force on a Polarizable Particle

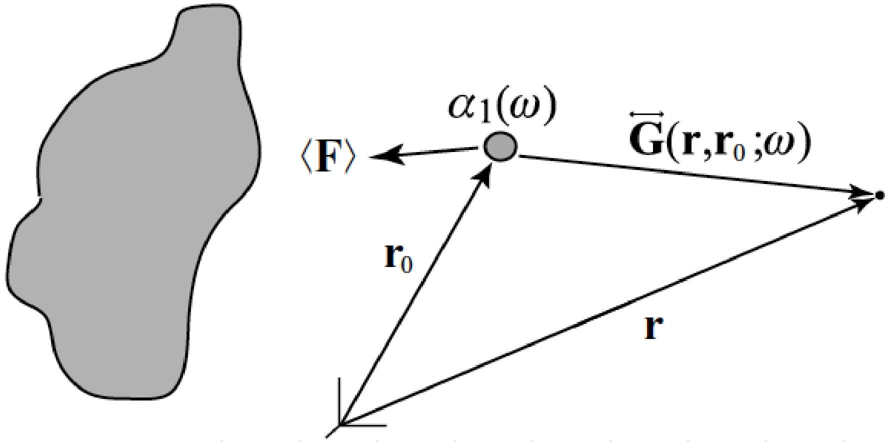
$$\langle \mathbf{F}(\mathbf{r}_0)\rangle = \sum_i \Big[\langle \mu_i^{(\mathrm{in})}(t)\,\nabla E_i^{(\mathrm{fl})}(\mathbf{r}_0,t)\rangle + \langle \mu_i^{(\mathrm{fl})}(t)\,\nabla E_i^{(\mathrm{in})}(\mathbf{r}_0,t)\rangle\Big]$$

$$= \sum_i \iint_{-\infty}^{\infty} \big\langle \hat{\mu}_i^{(\mathrm{in})}(\omega)\ \nabla \hat{E}_i^{*\,(\mathrm{fl})}(\mathbf{r}_0,\omega')\big\rangle\, \mathrm{e}^{\mathrm{i}(\omega'-\omega)t}\,\mathrm{d}\omega'\,\mathrm{d}\omega$$

$$+ \sum_i \iint_{-\infty}^{\infty} \big\langle \hat{\mu}_i^{(\mathrm{fl})}(\omega)\ \nabla \hat{E}_i^{*\,(\mathrm{in})}(\mathbf{r}_0,\omega')\big\rangle\, \mathrm{e}^{\mathrm{i}(\omega'-\omega)t}\,\mathrm{d}\omega'\,\mathrm{d}\omega$$

$$= \sum_i \iint_{-\infty}^{\infty} \alpha_1(\omega)\,\nabla_2 \big\langle \hat{E}_i^{*\,(\mathrm{fl})}(\mathbf{r}_0,\omega)\ \hat{E}_i^{*\,(\mathrm{fl})}(\mathbf{r}_0,\omega')\big\rangle\, \mathrm{e}^{\mathrm{i}(\omega'-\omega)t}\,\mathrm{d}\omega'\,\mathrm{d}\omega$$

$$+ \sum_{i,j} \iint_{-\infty}^{\infty} \frac{\omega'^2}{c^2}\frac{1}{\varepsilon_0}\,\nabla_1 G_{ij}^{*}(\mathbf{r}_0,\mathbf{r}_0;\omega')\,\big\langle \hat{\mu}_i^{(\mathrm{fl})}(\omega)\ \hat{\mu}_j^{*\,(\mathrm{fl})}(\omega')\big\rangle\, \mathrm{e}^{\mathrm{i}(\omega'-\omega)t}\,\mathrm{d}\omega'\,\mathrm{d}\omega$$



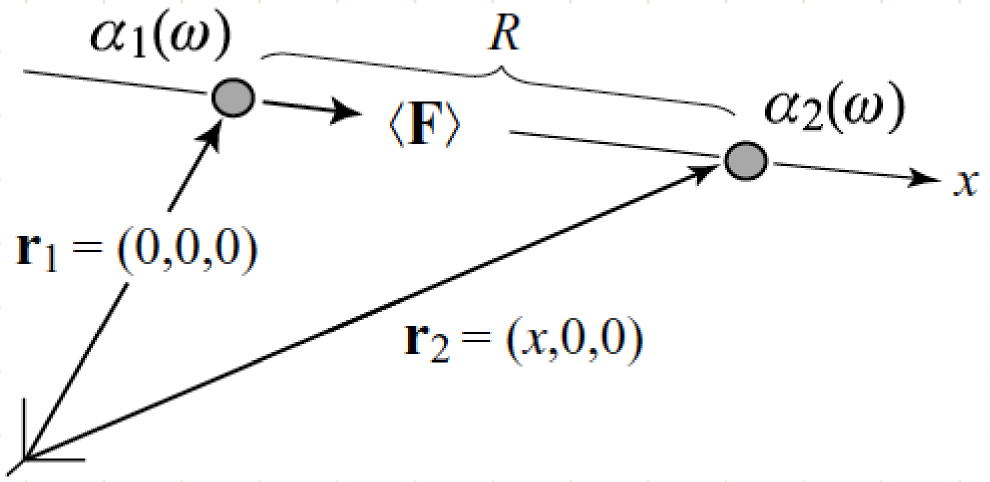
$$\hat{\boldsymbol{\mu}}^{(\mathrm{in})}(\omega) = \alpha_1(\omega)\,\hat{\mathbf{E}}^{(\mathrm{fl})}(\mathbf{r}_0,\omega)$$

$$\hat{\mathbf{E}}^{(\mathrm{in})}(\mathbf{r},\omega) = \frac{\omega^2}{c^2}\frac{1}{\varepsilon_0}\ \overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}_0;\omega)\cdot\hat{\boldsymbol{\mu}}^{(\mathrm{fl})}(\omega)$$

Force on a Polarizable Particle

$$\langle \mathbf{F}(\mathbf{r}_0)\rangle = \sum_i \int_{-\infty}^{\infty} \frac{\omega}{\pi c^2 \varepsilon_0}\left[\frac{\hbar\omega}{1-\mathrm{e}^{-\hbar\omega/kT}}\right] \mathrm{Im}\left\{\alpha_1(\omega)\, \nabla_1 G_{ii}(\mathbf{r}_0, \mathbf{r}_0; \omega)\right\}\, \mathrm{d}\omega$$

Casimir-Polder force



$$\overleftrightarrow{\mathbf{G}}(\mathbf{r}, \mathbf{r}_1; \omega) = \overleftrightarrow{\mathbf{G}}^0(\mathbf{r}, \mathbf{r}_1; \omega) + \frac{\omega^2}{c^2}\frac{1}{\varepsilon_0}\overleftrightarrow{\mathbf{G}}^0(\mathbf{r}, \mathbf{r}_2; \omega)\alpha_2(\omega)\overleftrightarrow{\mathbf{G}}^0(\mathbf{r}_2, \mathbf{r}_1; \omega)$$

Force on a Polarizable Particle

Casimir-Polder potential valid for any separation R

$$U = -\int \langle F(x) \rangle \, \mathrm{d}x = \frac{\hbar}{16\pi^3 \varepsilon_0^2 x^6} \, \mathrm{Im} \int_0^\infty \alpha_1(\omega)\, \alpha_2(\omega)\, \mathrm{e}^{2\mathrm{i}x\omega/c}$$

$$\times \left[-3 + 6\mathrm{i}\left(\frac{\omega}{c}x\right) + 5\left(\frac{\omega}{c}x\right)^2 - 2\mathrm{i}\left(\frac{\omega}{c}x\right)^3 - \left(\frac{\omega}{c}x\right)^4 \right] \mathrm{d}\omega$$

$$U(R \to 0) = -\frac{6\hbar}{32\pi^3 \varepsilon_0^2} \frac{1}{R^6} \int_0^\infty \alpha_1(\mathrm{i}\eta)\, \alpha_2(\mathrm{i}\eta)\, \mathrm{d}\eta$$

$$U(R \to \infty) = -\frac{23\hbar c}{64\pi^3 \varepsilon_0^2} \frac{\alpha_1(0)\, \alpha_2(0)}{R^7}$$

Stress Tensor Approach

- Rodriguez et al. (MIT) 2007 FDFD 2009 FDTD, Xiong et al. 2009 MoM
  - FEM may need local regularization
- Net force acting on a surface S, after Wick rotation

$$\mathbf{F} = \oint_S \langle \overline{\mathbf{T}}(\mathbf{r}') \rangle \cdot d\mathbf{s}'$$

Lifshitz & Pitaevskii

$$T_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\left(\epsilon_0 \sum_{k=1}^{3} E_k^2 + \frac{1}{\mu_0}\sum_{k=1}^{3} B_k^2\right)\delta_{ij}$$

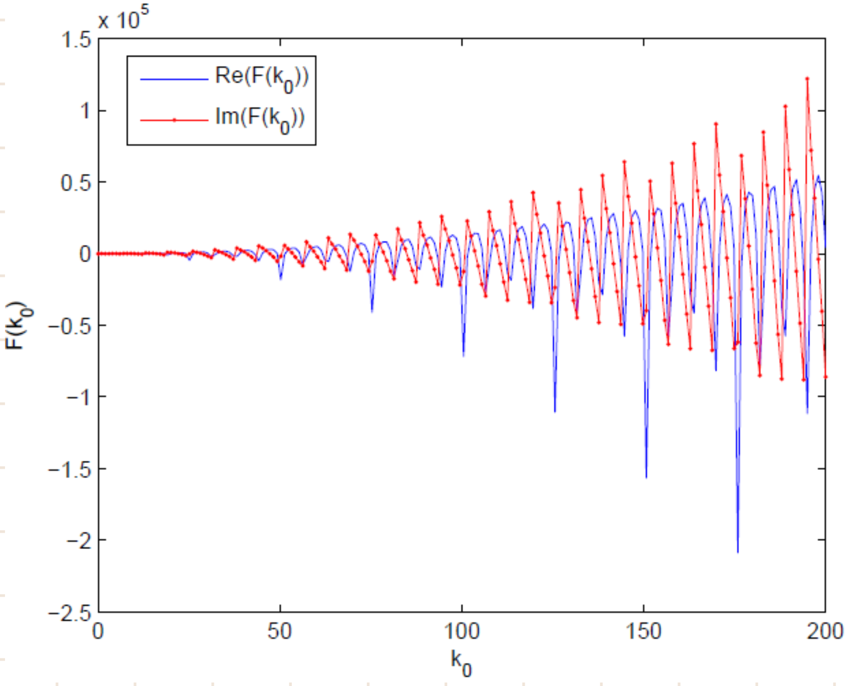
- Fluctuation dissipation theorem

$$\langle 0|\hat{E}_i(\mathbf{r},t)\hat{E}_j(\mathbf{r}',t)|0\rangle = \frac{\hbar}{\pi}\mathrm{Im}\int_0^\infty \omega^2 G_{ij}(\mathbf{r},\mathbf{r}',\omega)d\omega$$

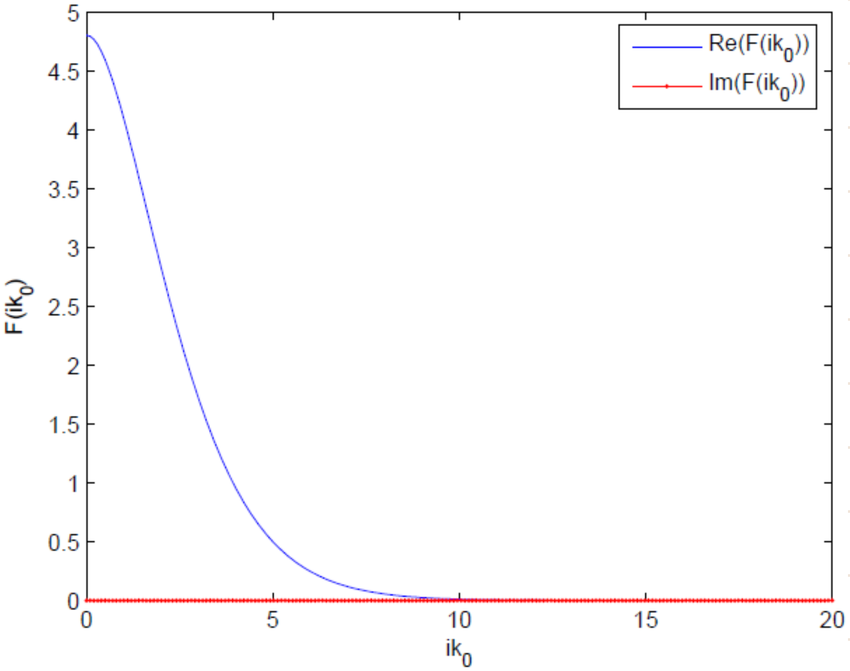
$$\langle 0|\hat{B}_i(\mathbf{r},t)\hat{B}_j(\mathbf{r}',t)|0\rangle = \frac{\hbar}{\pi}\mathrm{Im}\int_0^\infty (\nabla\times)_{il}(\nabla'\times)_{jm} G_{lm}(\mathbf{r},\mathbf{r}',\omega)d\omega$$

Stress Tensor Approach

- Wick rotation (parallel plates)



Integrand along real axis, highly oscillating

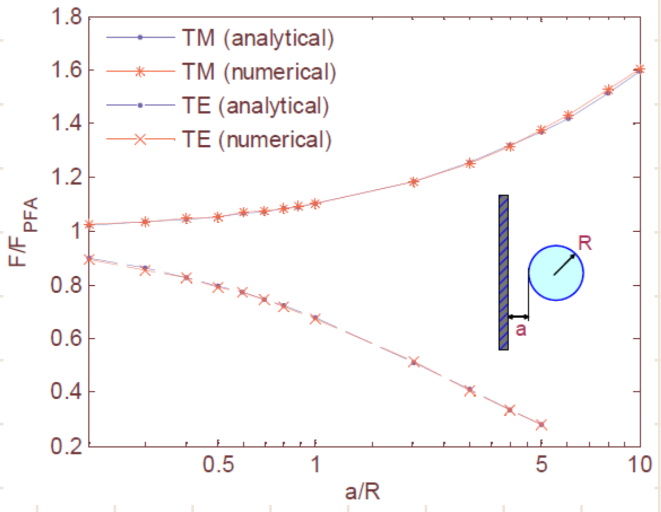


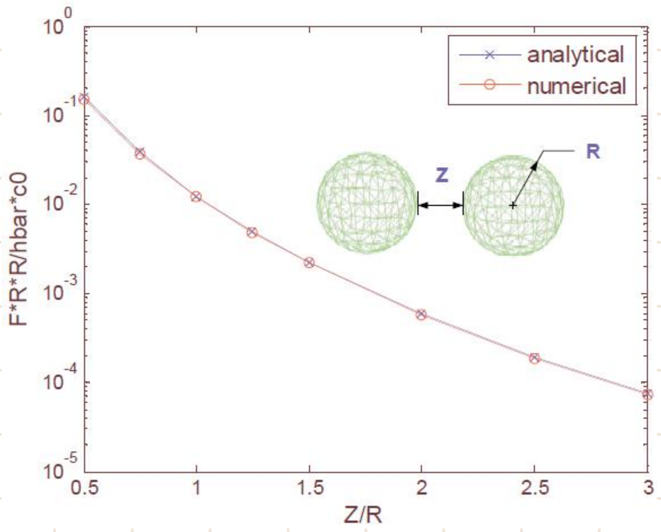
Integrand along imaginary axis, smooth and exponetially decaying, purely real

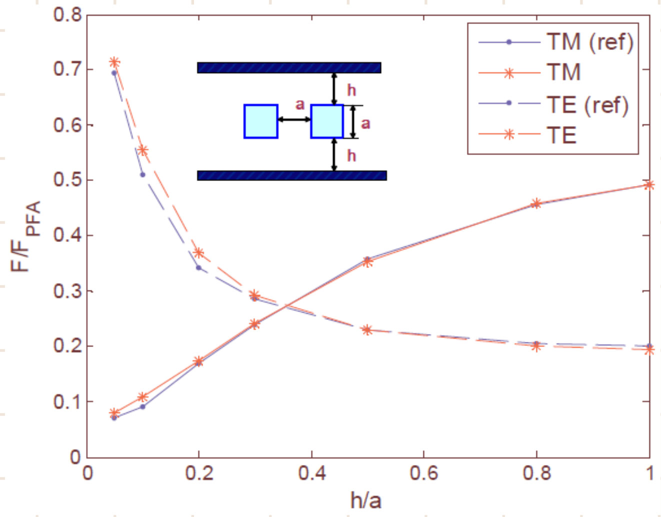
J. Xiong, COMPUTATIONAL ELECTROMAGNETICS FOR MICROSTRIP AND MEMS STRUCTURES, 2010

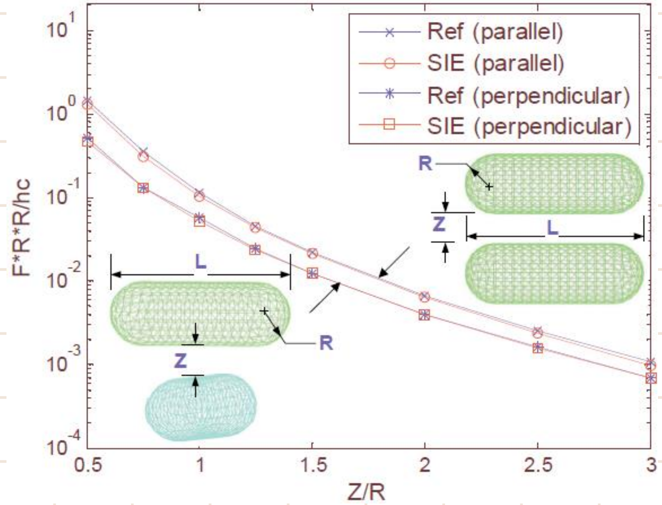
Stress Tensor Approach, EFIE

- 2-D results
- 3-D results









J. Xiong, COMPUTATIONAL ELECTROMAGNETICS FOR MICROSTRIP AND MEMS STRUCTURES, 2010

Path Integral Approach

- Bordag et al. 1985, Li and Kardar 1991, Golestanian, Kardar et al. 1997
- Emig & Buescher 2005 –scattering theory
- Reid, Rodriguez, White, & Johnson (RRWJ) –BEM

$$U = -\frac{\hbar c}{2\pi}\int_0^\infty \log\det \frac{Z(\xi)}{Z_\infty(\xi)} d\xi$$

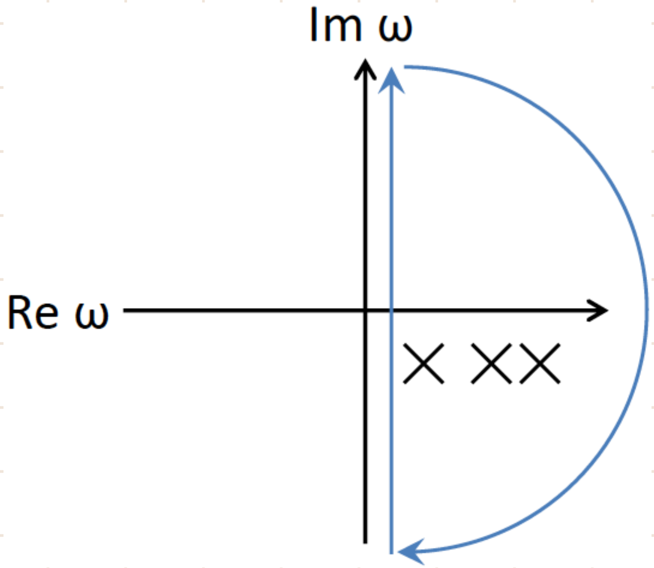
$$Z(\xi) = \int \mathcal{D}\mathbf{J} e^{-\frac{1}{2}\iint d^2\mathbf{x} \iint d^2\mathbf{x}' \mathbf{J}(\mathbf{x})\cdot\mathbf{G}_\xi(\mathbf{x}-\mathbf{x}')\cdot\mathbf{J}(\mathbf{x}')}$$

$$U = +\frac{\hbar c}{2\pi}\int_0^\infty \log\det\left[A_\infty(\xi)^{-1}A(\xi)\right] d\xi$$

$$\mathbf{J} \approx \sum c_k \mathbf{b}_k(\mathbf{x})$$

$$A_{jk} = \int \mathbf{b}_j \cdot \mathbf{G}_\xi \cdot \mathbf{b}_k$$

Mode Summation Approach



- Van Kampen et al. 1968, Schram 1973

Lambrecht & Marachevsky, 2008

- Argument principle

$$\frac{1}{2\pi i}\oint \phi(\omega)\frac{d}{d\omega}\ln f(\omega)d\omega = \sum_i \phi(\omega_{0,i}) - \sum_j \phi(\omega_{\infty,j})$$

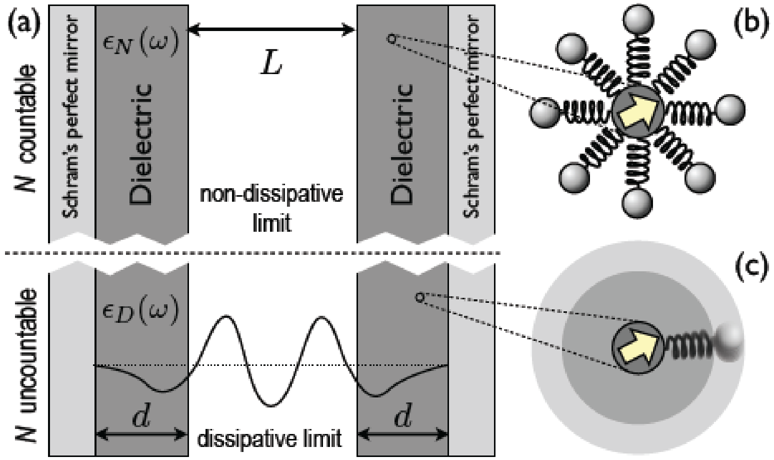
- Let $\phi(\omega) = \omega \quad f(ic\kappa) = \frac{\det \overline{\mathbf{M}}(\kappa)}{\det \overline{\mathbf{M}}_\infty(\kappa)}$

$$\int_{-\infty}^{\infty} \ln \frac{\det \overline{\mathbf{M}}(\kappa)}{\det \overline{\mathbf{M}}_\infty(\kappa)} d\kappa = \frac{2\pi}{c}\left[\sum_i \phi(\omega_{0,i}) + \sum_j \phi(\omega_{\infty,j})\right] = \frac{2\pi}{c}\left[\sum_i \omega_i + \sum_j \omega_{\infty,j}\right]$$

$$\mathcal{E} = \frac{\hbar c}{2\pi}\int_0^\infty d\kappa \ln \frac{\det \overline{\mathbf{M}}(\kappa)}{\det \overline{\mathbf{M}}_\infty(\kappa)}$$

Sum over Complex Modes

- Difficulty in dissipative system
- Barash & Ginzburg 1975, Sernelius 2006, Rosa & Milonni et al. 2010 2011, Intravaia & Behunin 2013

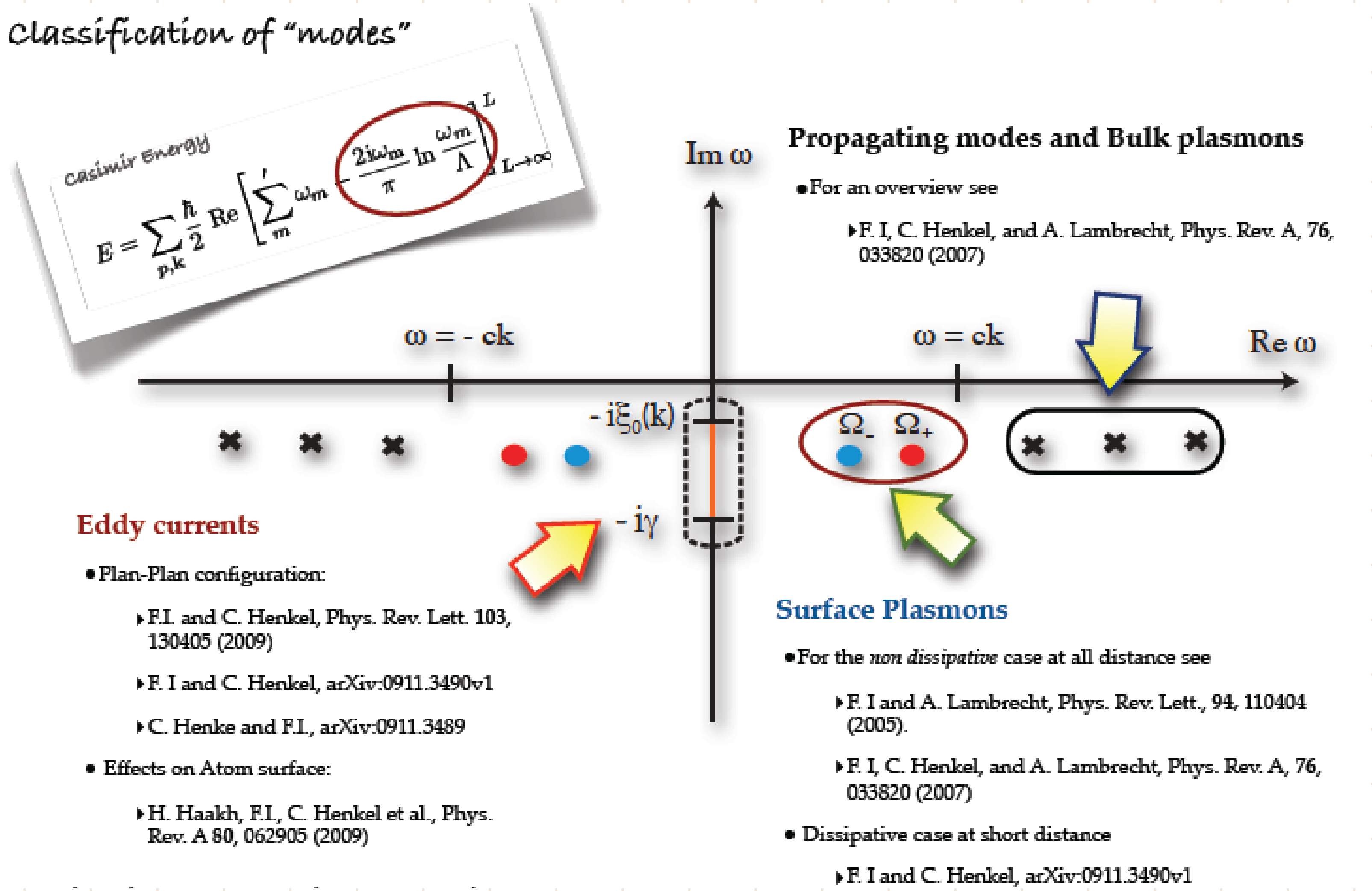


(a) A cavity formed by parallel planes where the optical properties of the mirrors are determined from the dynamics of the electric polarization field coupled to a bath of harmonic oscillators. **The system as a whole is closed and at equilibrium.** As long as the number of bath oscillators is countable the modes of the total system are **real** and the dielectric medium does not exhibit dissipation.

(b) and (c) show that the local electric polarization within the dielectric body couples to the **bath's "hidden" degrees of freedom** allowing the bath and polarization field to **exchange energy**. In the non-dissipative limit depicted in (b) the polarization field couples to a discrete countable number of oscillators. For this case energy lost to the bath from the matter will return in a finite time. Dissipation is introduced when the number of bath oscillators becomes uncountable. This is illustrated in subfigure (c) where the discrete oscillators shown in (b) are smeared into a continuum. In this case dissipation manifests as an **irreversible transfer of energy** from the polarization to the bath resulting ultimately in the thermalization of the matter.

$$\epsilon_N(\omega) \to D_N^p(\omega_n, \mathbf{k}) = 0 \quad E = \lim_{N \to \text{continuum}} -\frac{1}{2\pi}\int_0^\infty \frac{\hbar\omega}{2} \sum_{p,\mathbf{k}} \text{Im}\left[\partial_\omega \ln \frac{D_{N,L}^p(\omega, \mathbf{k})}{D_{N,L\to\infty}^p(\omega, \mathbf{k})}\right] d\omega$$

Sum over Complex Modes



A-EFIE for Low-Frequency Breakdown

- Atkins et al. 2012
- A-EFIE matrix system (Qian and Chew 2009)

$$\begin{bmatrix} \overline{\mathbf{V}} & \overline{\mathbf{D}}^T \cdot \overline{\mathbf{P}} \\ \overline{\mathbf{D}} & k_0^2 \overline{\mathbf{I}} \end{bmatrix} \cdot \begin{bmatrix} ik_0 \mathbf{J} \\ c_0 \boldsymbol{\rho} \end{bmatrix} = \begin{bmatrix} \eta_0^{-1} \mathbf{V} \\ \mathbf{0} \end{bmatrix}$$

- Casimir energy and force

$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^\infty d\kappa \ln \frac{\det \overline{\mathbf{Z}}_A(\kappa)}{\det \overline{\mathbf{Z}}_{A,\infty}(\kappa)}$$

$$\mathbf{F} = -\frac{\hbar c}{2\pi} \int_0^\infty d\kappa \nabla_i \ln \frac{\det \overline{\mathbf{Z}}_A(\kappa)}{\det \overline{\mathbf{Z}}_{A,\infty}(\kappa)}$$

A-EFIE for Low-Frequency Breakdown

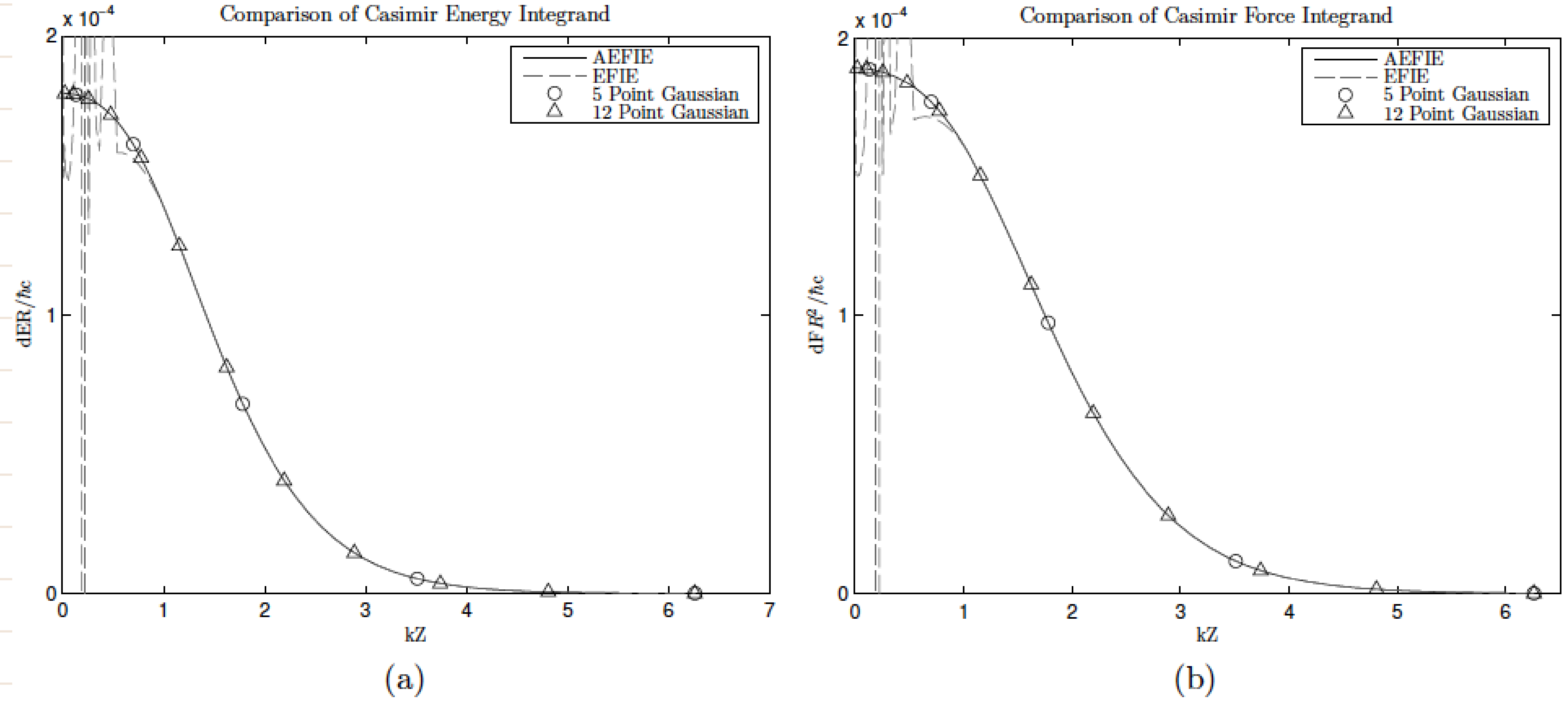


Figure 3.3: Comparison of the EFIE and A-EFIE results for (a) Casimir energy integrand and (b) Casimir force integrand for two PEC spheres.

A-EFIE for Low-Frequency Breakdown

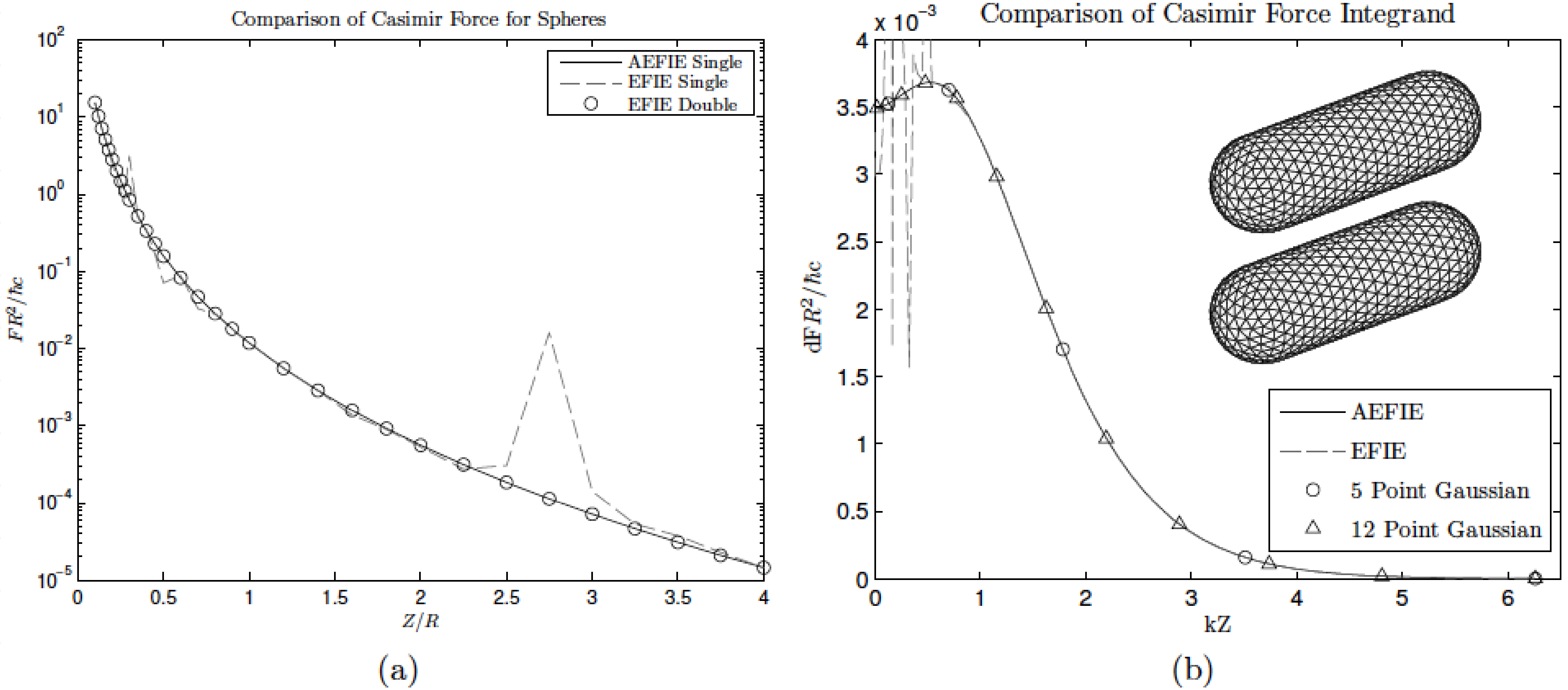


Figure 3.4: Comparison of the EFIE and A-EFIE results for (a) Casimir force between 2 PEC spheres and (b) Casimir force integrand for two rounded PEC capsules.

Domain Decomposition through EPA

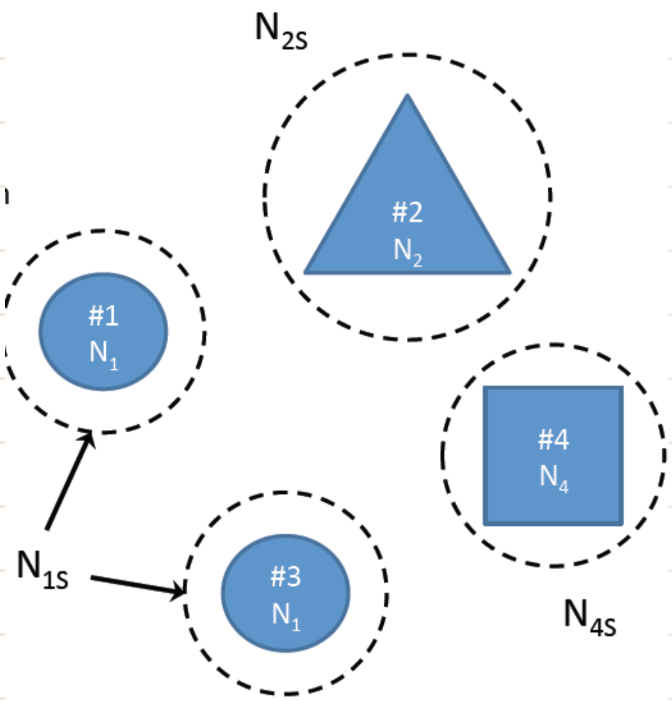
Limitation of EFIE based approach (Atkins 2013)

- EFIE uses dense matrix algorithms that have limitations in the size and complexity of object
  - CPU scales as O(N^3), memories scales as O(N^2)
  - Largest solvable system, unknowns # 10^4
- Interaction of objects over multiple distance and orientations
  - Each needs to generate and solve a new dense matrix

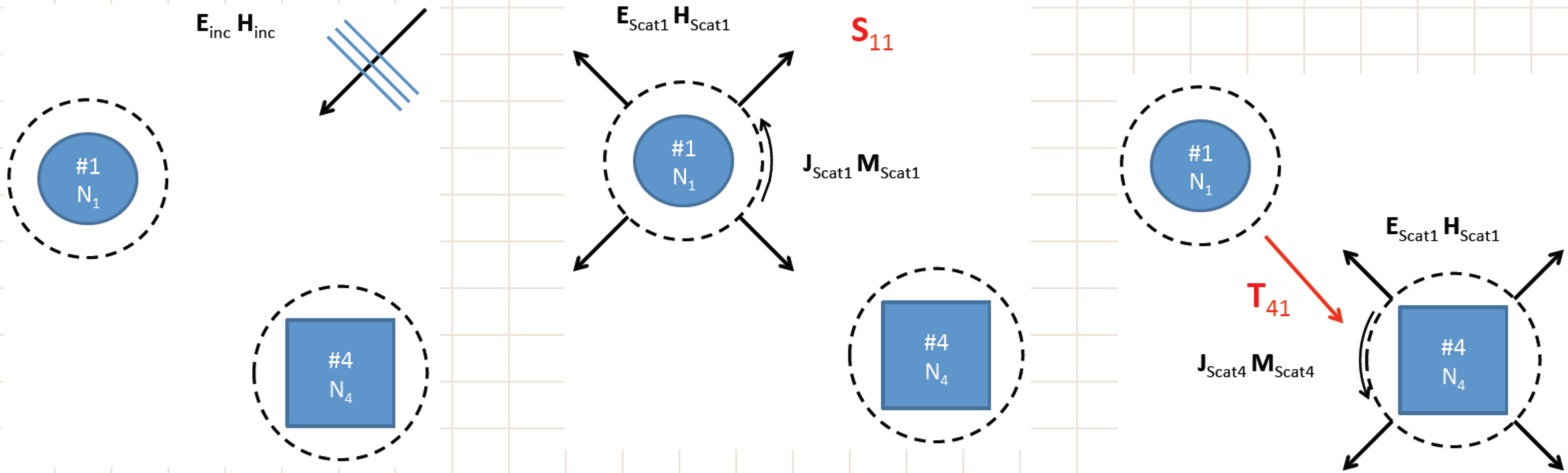
Domain Decomposition through EPA

EPA (Li and Chew 2006)

- Domain decomposition, decompose large problem into multiple subdomains enclosed by equivalent surface
- Solve each subdomain individually, and recombine results to solve original problem
- Information of subdomain can be reused
- Reduced number of unknowns



Domain Decomposition through EPA



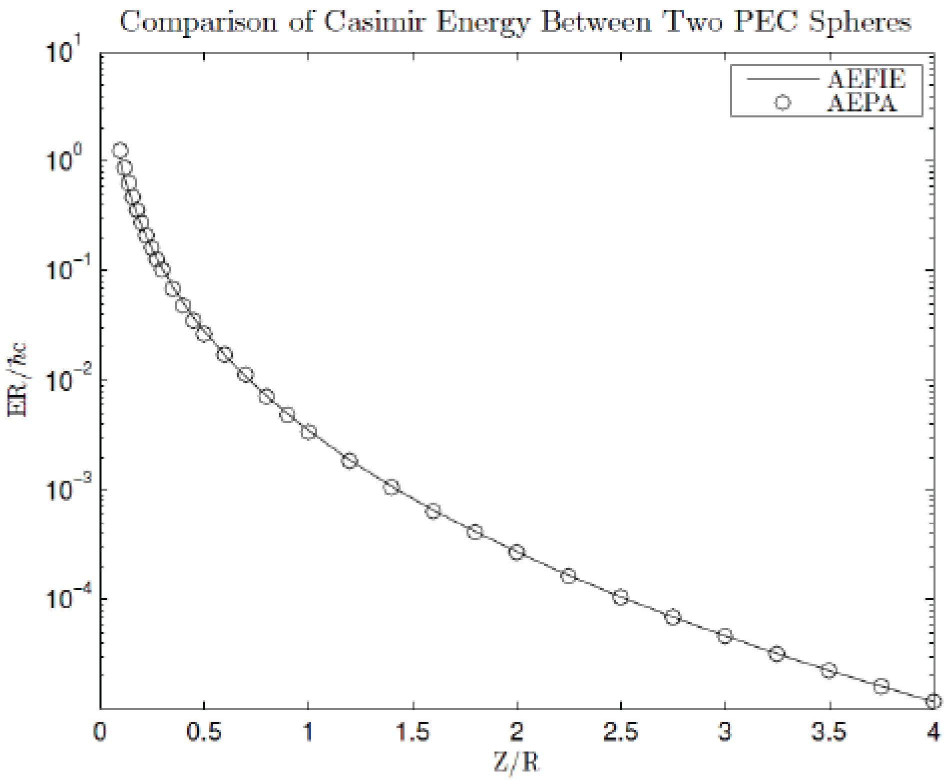
$$\overline{\mathbf{M}} = \begin{bmatrix} \overline{\mathbf{I}}_{11} & -\overline{\mathbf{S}}_{11} \cdot \overline{\mathbf{T}}_{12} & -\overline{\mathbf{S}}_{11} \cdot \overline{\mathbf{T}}_{13} \\ -\overline{\mathbf{S}}_{22} \cdot \overline{\mathbf{T}}_{21} & \overline{\mathbf{I}}_{22} & -\overline{\mathbf{S}}_{22} \cdot \overline{\mathbf{T}}_{23} \\ -\overline{\mathbf{S}}_{33} \cdot \overline{\mathbf{T}}_{31} & -\overline{\mathbf{S}}_{33} \cdot \overline{\mathbf{T}}_{32} & \overline{\mathbf{I}}_{33} \end{bmatrix}$$

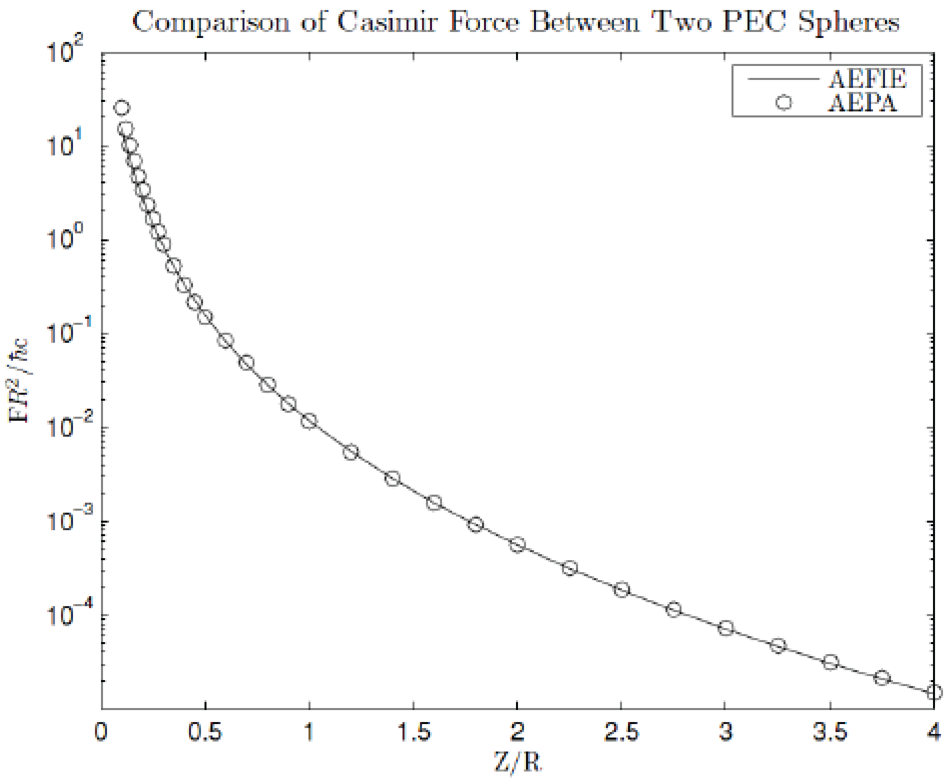
$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^\infty d\kappa \ln \det \overline{\mathbf{M}}$$

$$\mathbf{F}_i = -\frac{\hbar c}{2\pi} \int_0^\infty d\kappa \nabla_i \ln \det \overline{\mathbf{M}}$$

Domain Decomposition through EPA

P. Atkins 2013



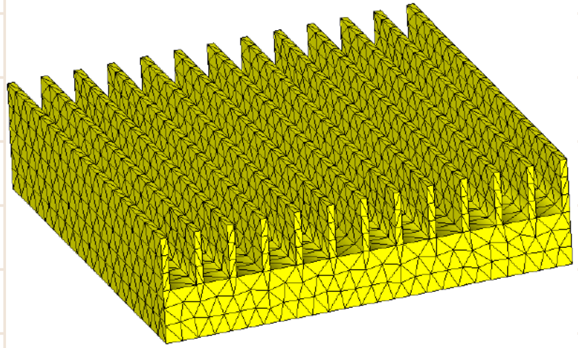


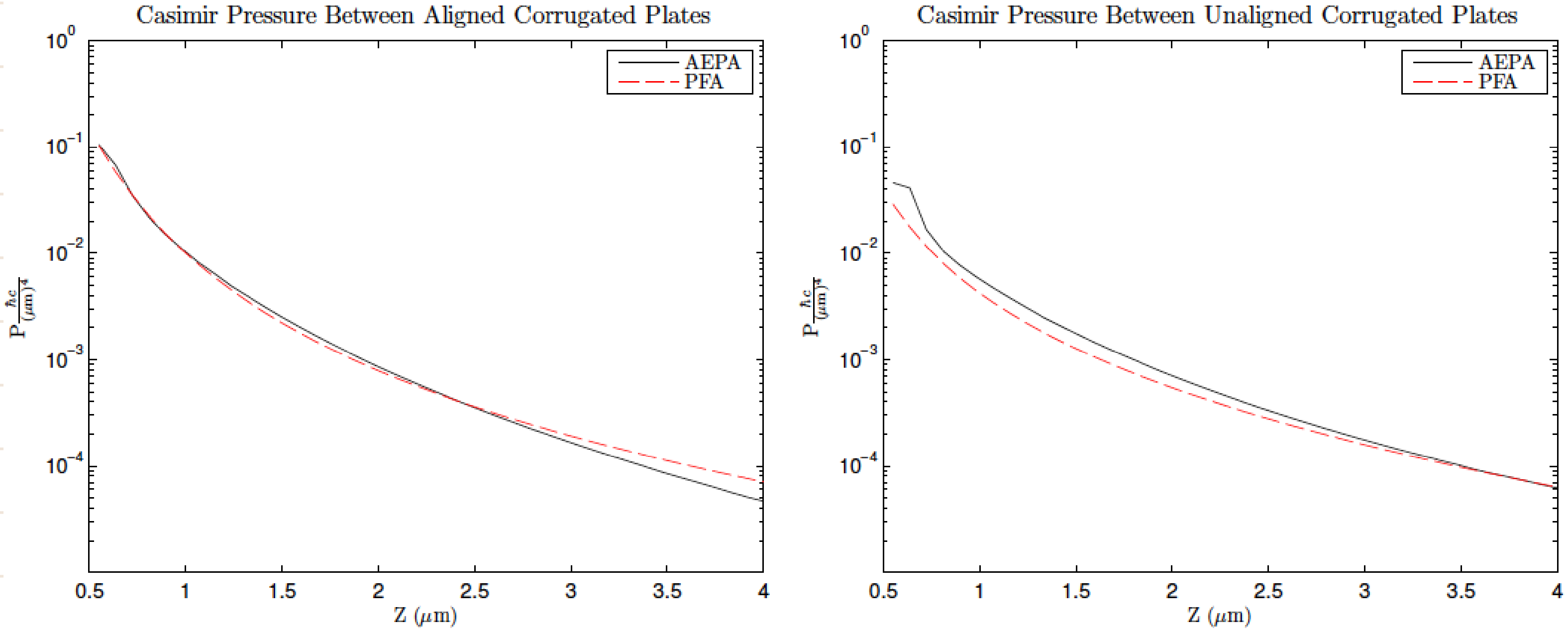
Z/R > 0.5	A-EFIE (h:m:s)	A-EFIE (MB)	A-EPA (h:m:s)	A-EPA (MB)
Energy	4:50:00	120	0:12:54	69 MB
Force	2:05:00	120	0:23:41	92 or 114 MB

Domain Decomposition through EPA

P. Atkins, 2013

Memory: A-EFIE 11.92 GB, A-EPA 620 MB





(a) (b)

Figure 4.7: The A-EPA results for the (a) aligned and (b) anti-aligned corrugated plates compared to the PFA results for μm scale dimensions.

THANK YOU!