On Volume Integral Equations

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Abstract

- The focus of this paper is on the volume integral representations to be used in constructing integral equations for composite volume media.
- The major thrust of the paper is to identify where derivatives of a discontinuous function arise in the derivation of the volume representation.
- This paper identifies the sources of error in the incorrect representations and its major contribution is the rigorously correct derivation of the representations to be used in volume integral equations.

Sections

- Introduction
- Problem Identification
- Correct Derivation Using Standard Bilinear Concomitant
  - ✓ Method 1
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- Derivation Using Delta Function
- Numerical Implementation and Validation

Introduction

- The recent modeling of composite Volumetric materials employ some form of equivalent electric and magnetic volumetric currents.
- With finite volumes these currents take a derivative of the discontinuity corresponding to the boundary between the scatterer and the surrounding medium.
- This issue is not particularly mentioned till this paper and in some of the literature the representations disagree with the ones presented by this paper.

- Thus the major aim of this paper is to provide three different derivations which are consistent and to overcome this issue.

- The secondary issue of the paper is the presentation of integrals which lead to formulations involving either the electric or magnetic field alone which lead to the reduction in unknowns when solving the subject integral equation.

- The outline of the paper is as follows:

1. Section II identifies where the derivative of the discontinuity occurs.
2. Section III and IV derive the correct presentation using bilinear concomitants .
3. Section V depicts the derivation using delta function to arrive at the same equation as in section III and IV.
4. Section VI presents numerical results based on the volume integral equation that results from this paper's representation

Problem Identification

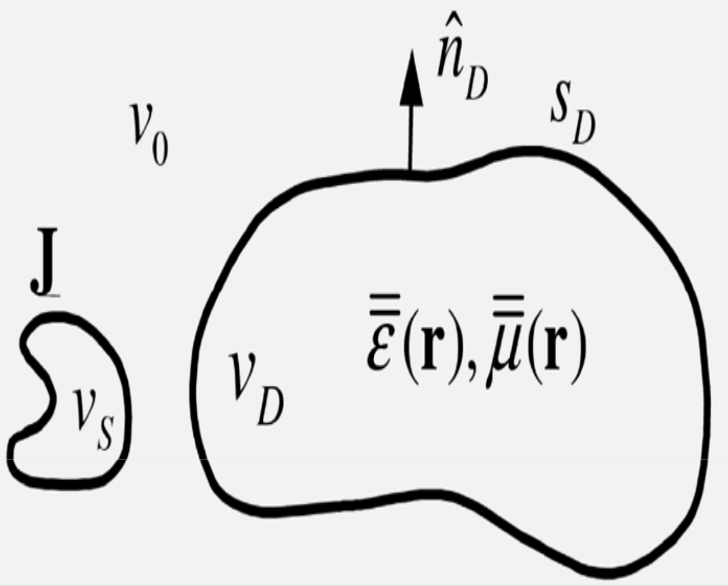
Maxwell's eq.'s:

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu_0 \mathbf{H}(\mathbf{r}), \qquad \mathbf{r} \in v_0 \qquad (1)$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\epsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \qquad \mathbf{r} \in v_0 \qquad (2)$$

$$\nabla \times \mathbf{E}(\mathbf{r}) = i\omega\mu_0 \overline{\overline{\mu}}_r(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r}), \qquad \mathbf{r} \in v_D \qquad (3)$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\epsilon_0 \overline{\overline{\epsilon}}_r(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}), \qquad \mathbf{r} \in v_D. \qquad (4)$$



$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k_0^2 \mathbf{E}(\mathbf{r}) = i\omega\mu_0 \mathbf{J}(\mathbf{r}) \qquad \mathbf{r} \in v_0. \qquad (5)$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k_0^2 \mathbf{E}(\mathbf{r}) = k_0^2 \left[\overline{\overline{\epsilon}}_r(\mathbf{r}) - \overline{\overline{\mathbf{I}}}\right] \cdot \mathbf{E}(\mathbf{r}) + \nabla \times \left[\nabla \times \mathbf{E}(\mathbf{r}) - \overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot (\nabla \times \mathbf{E}(\mathbf{r}))\right], \qquad \mathbf{r} \in v_D. \qquad (6)$$

where

$$k_0^2 = (i\omega\mu_0)(-i\omega\epsilon_0)$$

Eq.(5)+eq.(6) :

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k_0^2 \mathbf{E}(\mathbf{r}) = i\omega\mu_0 \mathbf{J}(\mathbf{r}) + \mathbf{J}_{eq}(\mathbf{r}), \qquad \mathbf{r} \in v_0 \cup v_D \qquad (7)$$

$$\mathbf{J}_{eq}(\mathbf{r}) = k_0^2 \left[\overline{\overline{\epsilon}}_r(\mathbf{r}) - \overline{\overline{\mathbf{I}}}\right] \cdot \mathbf{E}(\mathbf{r}) + \nabla \times \left[\nabla \times \mathbf{E}(\mathbf{r}) - \overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot (\nabla \times \mathbf{E}(\mathbf{r}))\right].$$

Using the identity:

$$\nabla \times \{\nabla \times \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r})\} - k_0^2 \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) = \mathcal{L}\overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) = \overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}_0) \quad (9)$$

eq.(8) and eq.(7):

$$\mathbf{E}(\mathbf{r}_0) = \int_{v_\infty} [i\omega\mu_0 \mathbf{J}(\mathbf{r}) + \mathbf{J}_{eq}(\mathbf{r})] \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv \quad (10)$$

$$\mathbf{E}(\mathbf{r}_0) = \mathbf{E}_{\text{inc}}(\mathbf{r}_0) + \int_{v_D} \{k_0^2 [\overline{\overline{\epsilon}}_r(\mathbf{r}) - \overline{\mathbf{I}}] \cdot \mathbf{E}(\mathbf{r}) + \nabla \times \left[\nabla \times \mathbf{E}(\mathbf{r}) - \overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{E}(\mathbf{r})\right]\} \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv. \quad (13)$$

where

$$\mathbf{E}_{\text{inc}}(\mathbf{r}_0) = i\omega\mu_0 \int_{v_s} \mathbf{J}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv$$

- Eq.(13) is the standard representation for $\underline{E}$ but it is Incorrect.
- The error is the term $\nabla \times [\overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{E}(\mathbf{r})]$ in eq.(8) which involves the differentiation of the discontinuous permeability.
- Thus in this paper the authors have derived the correct representation using three methods and verified their consistency with the known results.

Correct Derivation Using Standard Bilinear Concomitant

- Concomitants:

  An expression $B(u,v)$, where u,v are functions of x, satisfying $vL(u)$ - $u\bar{L}(v) = (d/dx)\cdot B(u,v)$, where L *and* $\bar{L}$ are given adjoint differential equations.

- Concomitants used in Method 1:

$$
\begin{aligned}
\overline{\mathbf{Q}}_0(\mathbf{r}_0,\mathbf{r}) = {} & \mathbf{E}_0(\mathbf{r}) \times \left[\nabla \times \overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r})\right] \\
& + \left[\nabla \times \mathbf{E}_0(\mathbf{r})\right] \times \overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r}) \\
\overline{\mathbf{Q}}_D(\mathbf{r}_0,\mathbf{r}) = {} & \mathbf{E}_D(\mathbf{r}) \times \left[\nabla \times \overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r})\right] \\
& + \left[\nabla \times \mathbf{E}_D(\mathbf{r})\right] \times \overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r}).
\end{aligned}
$$

- E and H are derived by integrating the divergence of the **Q** functions over *v*$_O$ and *v*$_D$
- When the integration and observation vectors are in the same region integration of δ(r-r$_o$) is encountered which gives rise to E(r$_o$).
- Integrating over *v*$_O$:

$$\int_{v_\infty} \nabla \cdot \overline{\mathbf{Q}}_0 dv = -\int_{s_D} \hat{n}_D(\mathbf{r}) \cdot \overline{\mathbf{Q}}_0 ds = \int_{v_0} \mathcal{L}\mathbf{E}_0(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv - \mathbf{E}_0(\mathbf{r}_0), \quad \mathbf{r}_0 \in v_0 \tag{16}$$

$$-\int_{s_D} \hat{n}_D(\mathbf{r}) \cdot \overline{\mathbf{Q}}_0 ds = \int_{v_0} \mathcal{L}\mathbf{E}_0(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv, \quad \mathbf{r}_0 \in v_D. \tag{17}$$

- Substituting the expression for $\mathbf{E}_{inc}$ gives:

$$\int_{v_\infty} \nabla \cdot \overline{\mathbf{Q}}_D dv = \int_{s_D} \hat{n}_D(\mathbf{r}) \cdot \overline{\mathbf{Q}}_D(\mathbf{r}_0, \mathbf{r}) ds = \mathbf{E}_{\text{inc}}(\mathbf{r}_0) - \mathbf{E}_0(\mathbf{r}_0), \quad \mathbf{r}_0 \in v_0 \tag{18}$$

$$\int_{s_D} \hat{n}_D(\mathbf{r}) \cdot \overline{\mathbf{Q}}_D(\mathbf{r}_0, \mathbf{r}) ds = \mathbf{E}_{\text{inc}}(\mathbf{r}_0), \quad \mathbf{r}_0 \in v_D \tag{19}$$

- Similarly integrating over v_D:

$$\int_{s_D} \hat{n}_D(\mathbf{r}) \cdot \overline{\mathbf{Q}}_D(\mathbf{r}_0, \mathbf{r}) ds = \int_{v_D} \mathcal{L}\mathbf{E}_D(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv, \quad \mathbf{r}_0 \in v_0 \tag{20}$$

$$\int_{s_D} \hat{n}_D(\mathbf{r}) \cdot \overline{\mathbf{Q}}_D(\mathbf{r}_0, \mathbf{r}) ds = \int_{v_D} \mathcal{L}\mathbf{E}_D(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv - \mathbf{E}_D(\mathbf{r}_0), \quad \mathbf{r}_0 \in v_D. \tag{21}$$

The key to elimination of error is that integration is done over v_o and v_D separately so no discontinuity in μ_r is encountered.

- Substituting eq.(6) in (21) and adding in (19):

$$\mathbf{E}_D(\mathbf{r}_0) = \mathbf{E}_{\text{inc}}(\mathbf{r}_0) + \int_{v_D} \left\{ k_0^2 \left[\overline{\overline{\epsilon}}_r(\mathbf{r}) - \overline{\mathbf{I}}\right] \cdot \mathbf{E}_D(\mathbf{r}) + \nabla \times \left[\nabla \times \mathbf{E}_D(\mathbf{r}) - \overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{E}_D(\mathbf{r})\right]\right\} \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv + \mathbf{I}(\mathbf{r}_0) \qquad (22)$$

where

$$\mathbf{I}(\mathbf{r}_0) = \int_{s_D} \hat{n}_D(\mathbf{r}) \cdot \left[\overline{\mathbf{Q}}_0(\mathbf{r}_0, \mathbf{r}) - \overline{\mathbf{Q}}_D(\mathbf{r}_0, \mathbf{r})\right] ds$$

which can be expanded as

$$\mathbf{I}(\mathbf{r}_0) = \int_{s_D} \{[\hat{n}_D(\mathbf{r}) \times (\mathbf{E}_0(\mathbf{r}) - \mathbf{E}_D(\mathbf{r}))] \cdot \nabla \times \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) + [\hat{n}_D(\mathbf{r}) \times (\nabla \times \mathbf{E}_0(\mathbf{r}) - \nabla \times \mathbf{E}_D(\mathbf{r}))] \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r})\} \, ds.$$

From the continuity of tangential E across s_D and Maxwell's Eq. :

$$\mathrm{I}(\mathbf{r}_0)=i\omega\mu_0\int_{s_D}\left\{\hat{n}_D(\mathbf{r})\times\left[\mathrm{H}_0(\mathbf{r})-\overline{\overline{\mu}}_r\cdot\mathrm{H}_D(\mathbf{r})\right]\right\}\cdot\overline{\mathrm{G}}(\mathbf{r}_0,\mathbf{r})ds.$$

- Since I(r_o) does not vanish unless μ_r=I this term is the main missing term in formerly derived VIE.
- So for the correct derivation of E considering the third term in (22):

$\mathrm{I}_v(\mathbf{r}_0)=\int_{v_D}[\nabla\times\mathbf{V}(\mathbf{r})]\cdot\overline{\mathrm{G}}(\mathbf{r}_0,\mathbf{r})dv$ where $\mathrm{V}(\mathbf{r})=\nabla\times\mathrm{E}_D(\mathbf{r})-\overline{\overline{\mu}}_r^{-1}(\mathbf{r})\cdot\nabla\times\mathrm{E}_D(\mathbf{r}).$

Integrating over v_D, using Divergence theorem and the identity

$$\nabla\cdot[\mathbf{V}(\mathbf{r})\times\overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r})]=[\nabla\times\mathbf{V}(\mathbf{r})]\cdot\overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r})-\mathbf{V}(\mathbf{r})\cdot[\nabla\times\overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r})]$$

$$\mathbf{I}_v(\mathbf{r}_0)=\int_{s_D}\hat{n}_D(\mathbf{r})\cdot[\mathbf{V}(\mathbf{r})\times\overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r})]\,ds+\int_{v_D}\mathbf{V}(\mathbf{r})\cdot[\nabla\times\overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r})]\,dv.$$

- substituting in eq.(22):

$$\mathbf{E}_D(\mathbf{r}_0) = \mathbf{E}_{\text{inc}}(\mathbf{r}_0) + \int_{v_D} \Big\{ k_0^2 \left[\overline{\overline{\epsilon}}_r(\mathbf{r}) - \overline{\mathbf{I}}\right] \cdot \mathbf{E}_D(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) + \left[\nabla \times \mathbf{E}_D(\mathbf{r}) - \overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{E}_D(\mathbf{r})\right] \cdot \left[\nabla \times \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r})\right] \Big\}\, dv + \mathbf{I}_{\text{comb}}(\mathbf{r}_0). \quad (30)$$

where I_{comb} is the sum of $\mathrm{I}(r_o)$ and the surface integral of $\mathrm{I}_v(r_o)$ which can be simplified to zero due to continuity of tangential H across S_D.

Thus,

$$\mathbf{E}_D(\mathbf{r}_0) = \mathbf{E}_{\mathrm{inc}}(\mathbf{r}_0) + \int_{v_D} \Big\{ k_0^2 \left[\overline{\overline{\epsilon}}_r(\mathbf{r}) - \overline{\mathbf{I}}\right] \cdot \mathbf{E}_D(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) + \left[\nabla \times \mathbf{E}_D(\mathbf{r}) - \overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{E}_D(\mathbf{r})\right] \cdot \left[\nabla \times \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r})\right] \Big\} \, dv \tag{33}$$

or equivalently

$$\mathbf{E}_D(\mathbf{r}_0) = \mathbf{E}_{\mathrm{inc}}(\mathbf{r}_0) + \int_{v_D} \Big\{ k_0^2 \left(\left[\overline{\overline{\epsilon}}_r(\mathbf{r}) - \overline{\mathbf{I}}\right] \cdot \mathbf{E}_D(\mathbf{r}) \right) \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) + i\omega\mu_0 \left(\left[\overline{\overline{\mu}}_r(\mathbf{r}) - \overline{\mathbf{I}}\right] \cdot \mathbf{H}_D(\mathbf{r}) \right) \cdot \left[\nabla \times \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r})\right] \Big\} \, dv \tag{34}$$

Method 2

- Concomitants used:

$$\overline{\mathbf{Q}}_\alpha(\mathbf{r}_0,\mathbf{r}) = \mathbf{E}_\alpha(\mathbf{r}) \times \left[\nabla \times \overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r})\right] + i\omega\mu_0 \mathbf{H}_\alpha(\mathbf{r}) \times \overline{\mathbf{G}}(\mathbf{r}_0,\mathbf{r}), \quad \alpha = 0, D.$$

- Proceeding in the same fashion the authors end up with the same expressions as eq.(33) thus validating their claim that eq.(13) is not valid without the integral in eq.(22)

Derivation Using Delta Function

- This section also builds on the same idea of discontinuity of μ_r in J_{eq} but deals it using the delta function. Rewriting the equation as:

$$\begin{aligned}\mathbf{E}(\mathbf{r}_0) = \mathbf{E}_{\text{inc}}(\mathbf{r}_0) & \\ & + \int_{v_D} k_0^2 \left\{ \left[\overline{\overline{\epsilon}}_r(\mathbf{r}) - \overline{\mathbf{I}} \right] \cdot \mathbf{E}_D(\mathbf{r}) \right\} \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv \\ & + \int_{v_\infty} \left\{ \nabla \times \mathbf{F}(\mathbf{r}) \right\} \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv \qquad (47)\end{aligned}$$

where $\mathbf{F}(\mathbf{r}) = \left[\nabla \times \mathbf{E}_D(\mathbf{r}) - \overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{E}_D(\mathbf{r}) \right]$

- Because F falls to zero outside the material boundary and has finite discontinuity so

$$\mathbf{F}(\mathbf{r}) = u_D(\mathbf{r}) \mathbf{F}_D(\mathbf{r})$$

- So in order to avoid the step function in eq.(47) the following identity is used to give eq.(52)

$$\nabla \times \mathbf{F}(\mathbf{r}) = u_D(\mathbf{r}) \nabla \times \mathbf{F}_D(\mathbf{r}) - \delta_s(\mathbf{r} - \mathbf{r}_s) \left[\hat{n}(\mathbf{r}) \times \mathbf{F}_D(\mathbf{r})\right]$$

$$\begin{aligned}\mathbf{E}(\mathbf{r}_0) = {} & \mathbf{E}_{\text{inc}}(\mathbf{r}_0) \\ & + \int_{v_D} \left\{ k_0^2 \left[\overline{\overline{\epsilon}}_r(\mathbf{r}) - \overline{\mathbf{I}}\right] \cdot \mathbf{E}_D(\mathbf{r}) + \nabla \right. \\ & \left. \times \left[\nabla \times \mathbf{E}_D(\mathbf{r}) - \overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{E}_D(\mathbf{r})\right] \right\} \\ & \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv + \mathbf{I}_\delta(\mathbf{r}_0) \qquad (52)\end{aligned}$$

where $\mathbf{I}_\delta(\mathbf{r}_0) = -\int_{v_\infty} \delta_s(\mathbf{r} - \mathbf{r}_s) \left[\hat{n}(\mathbf{r}) \times \mathbf{F}_D(\mathbf{r})\right] \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) dv.$

- $\mathrm{I}_\delta(\mathrm{r}_o)$ is the missing term in erroneous eq.(13) which can be further reduced using delta functions properties as

$$\mathbf{I}_\delta(\mathbf{r}_0) = \int\limits_{s_D} \hat{n}(\mathbf{r}) \times \left[\overline{\overline{\mu}}_r^{-1}(\mathbf{r}) \cdot \nabla \times \mathbf{E}_D(\mathbf{r}) - \nabla \times \mathbf{E}_D(\mathbf{r})\right] \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) ds \quad (54)$$

which further can be written as:

$$\mathbf{I}_\delta(\mathbf{r}_0) = i\omega\mu_0 \int\limits_{s_D} \hat{n}(\mathbf{r}) \times \left[\mathbf{H}_D(\mathbf{r}) - \overline{\overline{\mu}}_r(\mathbf{r}) \cdot \mathbf{H}_D(\mathbf{r})\right] \cdot \overline{\mathbf{G}}(\mathbf{r}_0, \mathbf{r}) ds$$

which gives the same expression as in eq.(22) from where eq.(34) is derived as before.

- Thus all the three methods used in this paper yield the same integral representation.
- Next it is further validated by the numerical implementation in the last section of the paper.

Numerical Implementation and Validation

- Using the **Method of Moments** eq.(34) is validated by casting it in the form:

$$[Z_{ji}]\{x_i\} = \{b_j\}$$

- **Discretization:** The volumetric scatterer geometry is subdivided using hexahedral elements whose volume is specified by the parametric position vector:

$$\mathbf{r}(u,v,w) = \sum_{m=0}^{2}\sum_{n=0}^{2}\sum_{p=0}^{2}\mathbf{r}_{mnp}L_{mnp}(u,v,w)$$

where r_{mnp} are the 27 defining points of each hexahedron and $L_{mnp}(u,v,w)$ are the Lagrange interpolation polynomials.

- **Basis Functions:** The field within the volume is expanded using 12 basis functions each associated with one of the 12 edges of the hexahedron. Hence, the total discretized field in each hexahedron is defined as:

$$\mathbf{E}_{hex}\left(\mathbf{r}(u,v,w)\right) = \sum_{i=1}^{4} \mathbf{e}_i^{(u)}\left(\mathbf{r}(u,v,w)\right) + \sum_{i=1}^{4} \mathbf{e}_i^{(v)}\left(\mathbf{r}(u,v,w)\right) + \sum_{i=1}^{4} \mathbf{e}_i^{(w)}\left(\mathbf{r}(u,v,w)\right)$$

- The four basis functions for each of the four edges along the u, v *and* w parametric directions respectively are:

$$\mathbf{e}^{(u)}\left(\mathbf{r}(u,v,w)\right) = \frac{1}{\sqrt{G}}\begin{Bmatrix}1-v\\ v\end{Bmatrix}\begin{Bmatrix}1-w\\ w\end{Bmatrix}\mathbf{a}_u$$

$$\mathbf{e}^{(w)}\left(\mathbf{r}(u,v,w)\right) = \frac{1}{\sqrt{G}}\begin{Bmatrix}1-u\\ u\end{Bmatrix}\begin{Bmatrix}1-v\\ v\end{Bmatrix}\mathbf{a}_w$$

$$\mathbf{e}^{(v)}\left(\mathbf{r}(u,v,w)\right) = \frac{1}{\sqrt{G}}\begin{Bmatrix}1-u\\ u\end{Bmatrix}\begin{Bmatrix}1-w\\ w\end{Bmatrix}\mathbf{a}_v$$

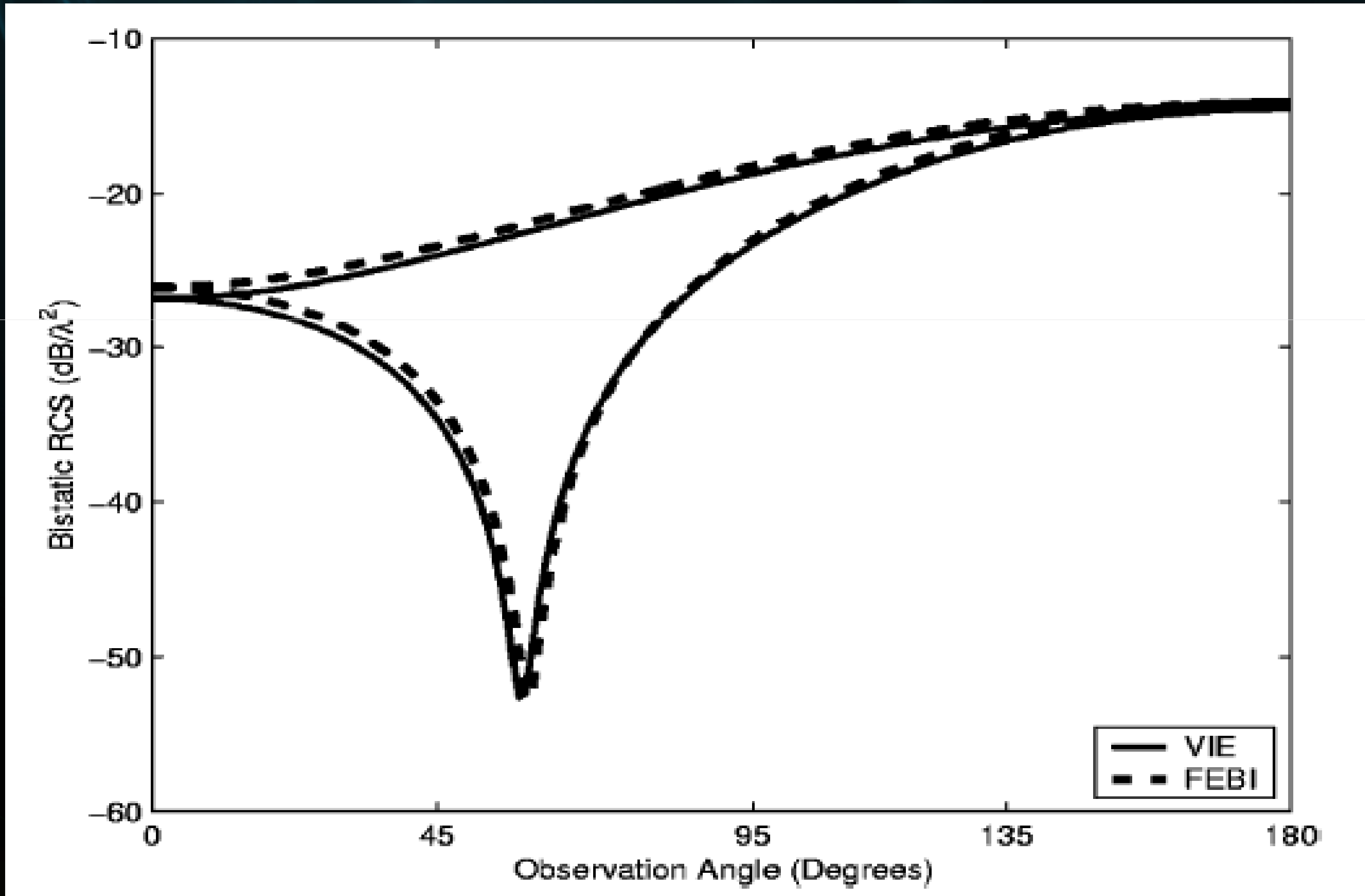
- The resulting Matrix elements are given by:

$$
\begin{aligned}
Z_{ji} = {} & \langle \mathbf{e}_j(\mathbf{r}), \mathbf{e}_i(\mathbf{r}) \rangle \\
& - k_0^2 \left\langle \mathbf{e}_j(\mathbf{r}), \int_v d\mathbf{r}' \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot (\overline{\epsilon}_r - \overline{\mathbf{I}}) \cdot \mathbf{e}_i(\mathbf{r}') \right\rangle \\
& + \left\langle \mathbf{e}_j(\mathbf{r}), \int_v d\mathbf{r}' \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \right. \\
& \qquad \left. \cdot (\overline{\mu}_r^{-1} - \overline{\mathbf{I}}) \cdot \nabla \times \mathbf{e}_i(\mathbf{r}') \right\rangle.
\end{aligned}
$$

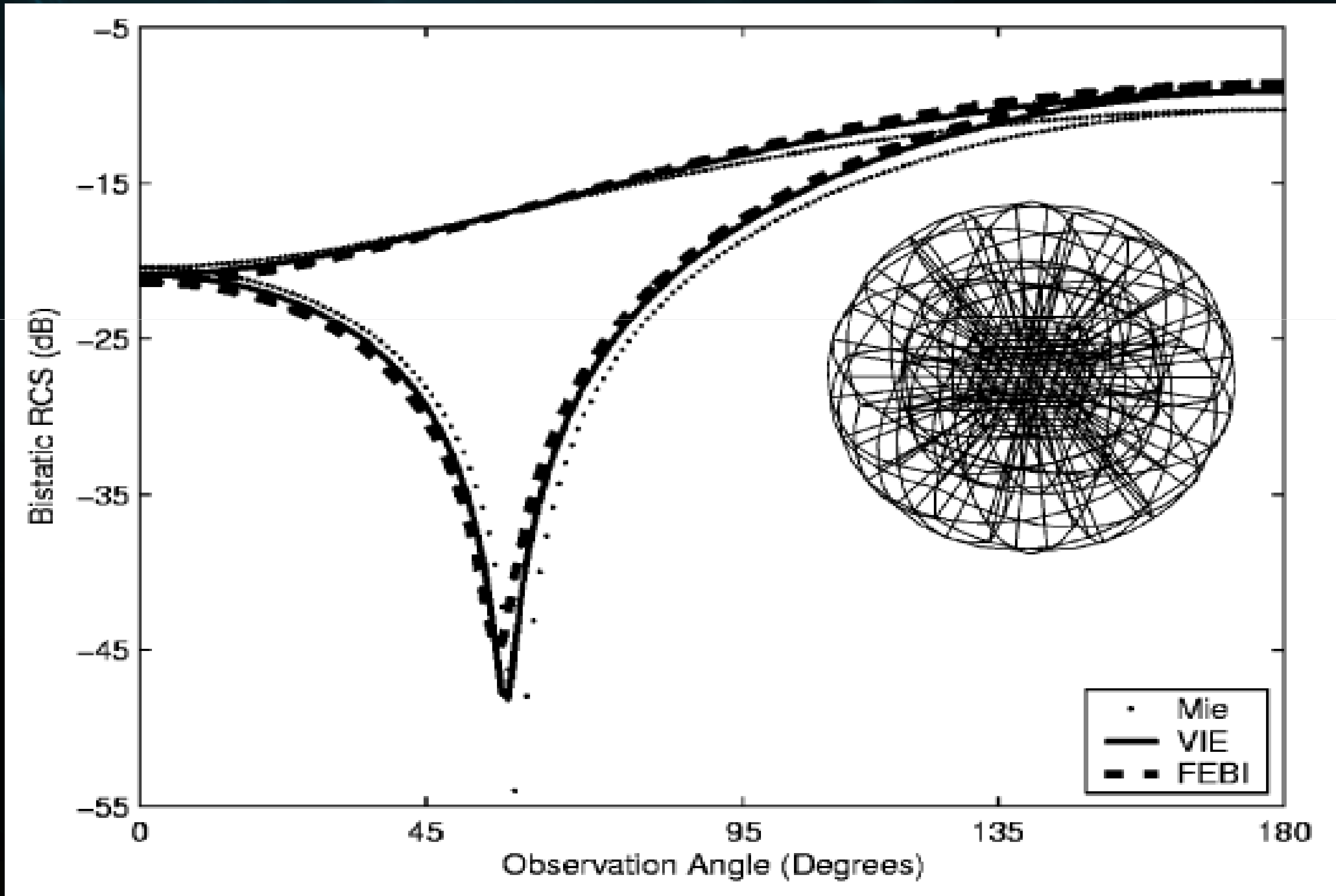
- whereas the RHS of the matrix system takes the form:

$$
b_j = \langle \mathbf{e}_j(\mathbf{r}), \mathbf{E}_{\text{inc}}(\mathbf{r}) \rangle
$$

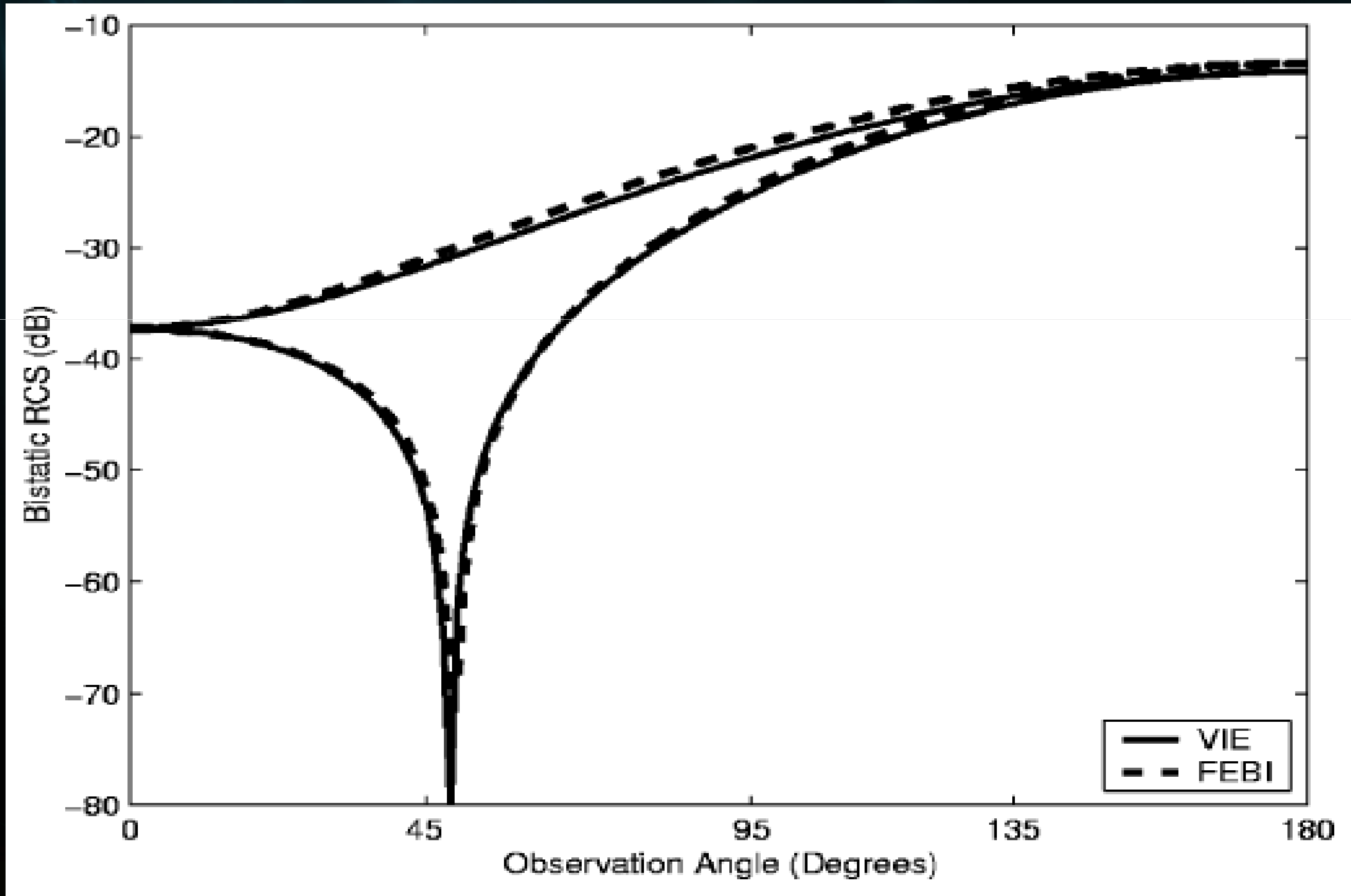
RCS of homogeneous composite cube of side length a = 0.2λ, ε_r = 1.5 and μ_r = 2.2



RCS of homogeneous composite sphere of radius= 0.15λ, ε_r = 1.5 and μ_r = 2.2



RCS of homogeneous composite shell of outer radius = 0.2λ , thickness = 0.02λ , ε_r = 1.5 and μ_r = 2.2



Efficiency of VIE

- Since VIE is a second kind Integral equation so its iterative solution converges much faster than FE-BI system.

- For a system of 1082(2nd example) VIE unknowns, CGS(Conjugate Gradient Squared) solver converged in 5 iterations to achieve a relative error of 1% and the solution was completed in 0.7 seconds on a 1GHz P3 processor.
- For the 3rd example with 880 VIE unknowns , the iterative CGS solver converged in 13 iterations and took 1.1 seconds.

Conclusions

- The examples presented demonstrate the validity of eq.(34).
- In contrast to using two sets of unknowns for modeling both the electric and magnetic field intensities for a magneto-dielectric scatterer, this numerical procedure discretizes only the electric field intensity, thus generating a VIE system half the size of those of traditional methods.

Thank You…..