An Explicit Inverse Based Fast Direct Volume Integral Equation Solver for Electromagnetic Analysis

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Outline

- Introduction
- Explicit Inverse Based Fast VIE Solver
- Numerical Results
- Conclusions

Introduction

- **Unlike PDE, IE based methods lead to dense matrices**

- **State-of-the-art *iterative* solutions**
  - Examples: FMM-based, Fast QR-based, FFT-based methods
  - Require iterations for each right hand side

- **State-of-the-art *direct* solutions**
  - Example: $\mathcal{H}^2$-based methods
  - Avoid iterations, require numerical inversion/factorization

Introduction

- Unlike Surface-IE, volume IE based methods offer flexibility of analyzing problems with complicated materials

- Resultant numerical system, however, is complicated by all four possible forms of double integrals

- Contribution of this work
  - Developed an explicit inverse for solving VIE system, which bypasses the need of numerical inversion and factorization
  - Further enhanced the efficiency of the proposed explicit inverse by error controlled $\mathcal{H}^2$-matrix-based fast computation

Explicit Inverse Based Fast VIE Solver

- **VIE System of Equations**
- **Explicit Inverse**
- $\mathcal{H}^2$ **-Matrix Representation and Its Error Bound**
- $\mathcal{H}^2$ **-Matrix Partition**
- **Rank Function**
- **Compact Storage**
- **Efficient Matrix-Vector Multiplication**

VIE System

- **VIE based formulation**

$$\mathbf{S}D = E \quad \text{where} \quad \mathbf{S} = (\mathbf{\Lambda} + \mathbf{G})$$

- $\mathbf{\Lambda}$ is **sparse** while **G** is **dense**

$$\mathbf{\Lambda}_{mn} = \frac{1}{\varepsilon_n^+} \int_{T_m^+} \vec{f}_m \cdot \vec{f}_n \, dv + \frac{1}{\varepsilon_n^-} \int_{T_m^-} \vec{f}_m \cdot \vec{f}_n \, dv \; ;$$

$$\mathbf{G}_{mn} = -\omega^2 \int_V \vec{f}_m \cdot \vec{A}_n(\vec{r}) dv - \int_V \phi_n(\vec{r}) \nabla \cdot \vec{f}_m dv + \int_S \phi_n(\vec{r}) \vec{f}_m \cdot \hat{n} \, ds; \quad \text{where}$$

$$\vec{A}_n(\vec{r}) = \frac{\mu_o a_n}{3} \left(\frac{\kappa_n^+}{V_n^+} \int_{T_n^+} \rho_n^+ \mathbf{g}(\vec{r}, \vec{r}') dv' + \frac{\kappa_n^-}{V_n^-} \int_{T_n^-} \rho_n^- \mathbf{g}(\vec{r}, \vec{r}') dv' \right) ; \phi_n(\vec{r}) = \frac{-a_n}{\varepsilon_o} \left(\frac{\kappa_n^+}{V_n^+} \int_{T_n^+} \mathbf{g}(\vec{r}, \vec{r}') dv' - \frac{\kappa_n^-}{V_n^-} \int_{T_n^-} \mathbf{g}(\vec{r}, \vec{r}') dv' - \frac{(\kappa_n^+ - \kappa_n^-)}{a_n} \int_{a_n} \mathbf{g}(\vec{r}, \vec{r}') ds' \right)$$

$$G_i = \mathrm{K} \int_P f_m(\vec{r}) \int_{Q'} g(\vec{r}, \vec{r}') h(\vec{r}') dq' dp \qquad \{P, Q\} \in \{Volume, Surface\}$$

Explicit Inverse

- **A straightforward approach**

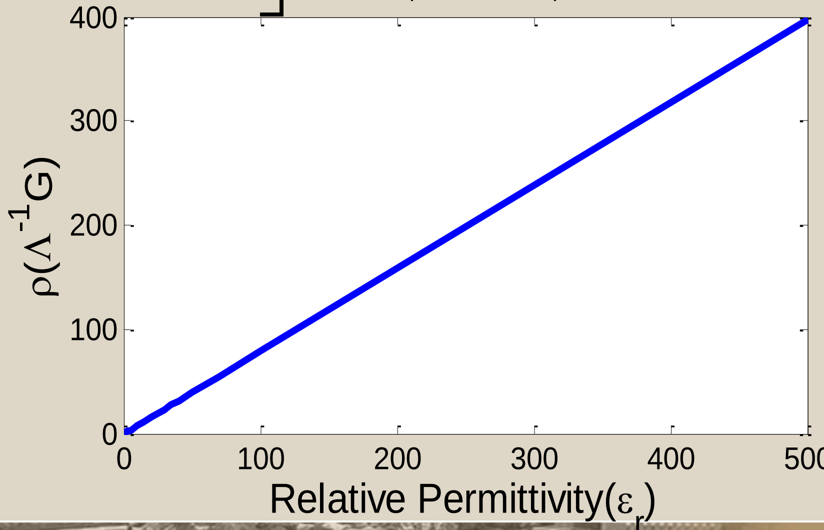
$$\mathbf{S} = \mathbf{\Lambda}\left[\mathbf{I} + \mathbf{\Lambda}^{-1}\mathbf{G}\right]$$

Then using Newman Series

$$\left[I + W\right]^{-1} = \left[I - W + W^2 - W^3 + W^4 - W^5 + W^6 \ldots\right] \quad ; \quad \left|\rho(W)\right| \le 1$$

Problem:

Resultant spectral radius depends on dielectric constant



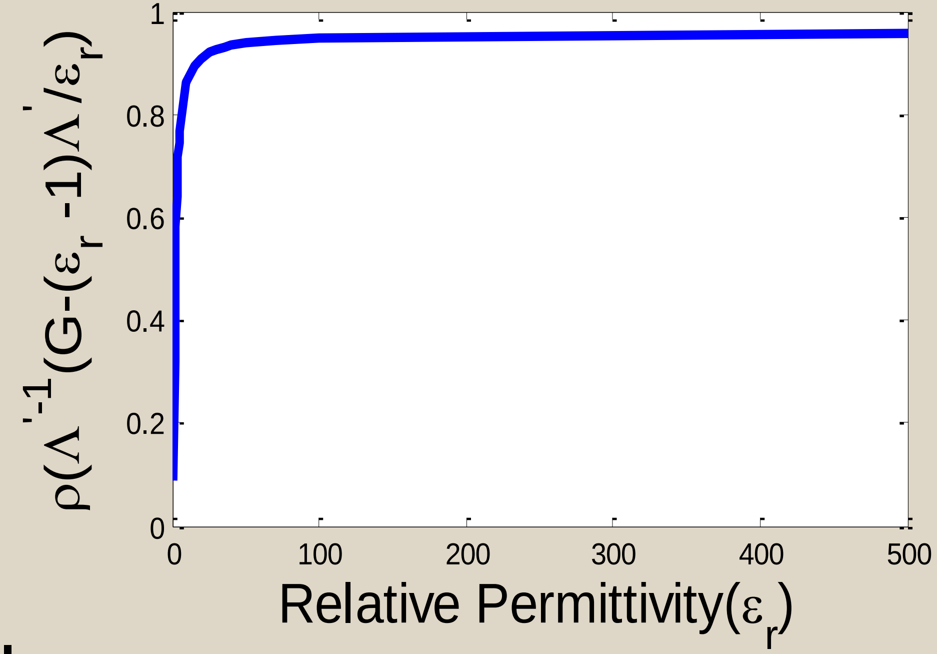
Explicit Inverse

- **A much better approach**

$$\mathbf{S} = \Lambda' \left[\mathbf{I} + \Lambda'^{-1} \left(\mathbf{G} - \mathbf{Z} \right) \right]$$

where

$$\Lambda' = \varepsilon_r \Lambda \qquad \mathbf{Z} = \Lambda' - \Lambda$$



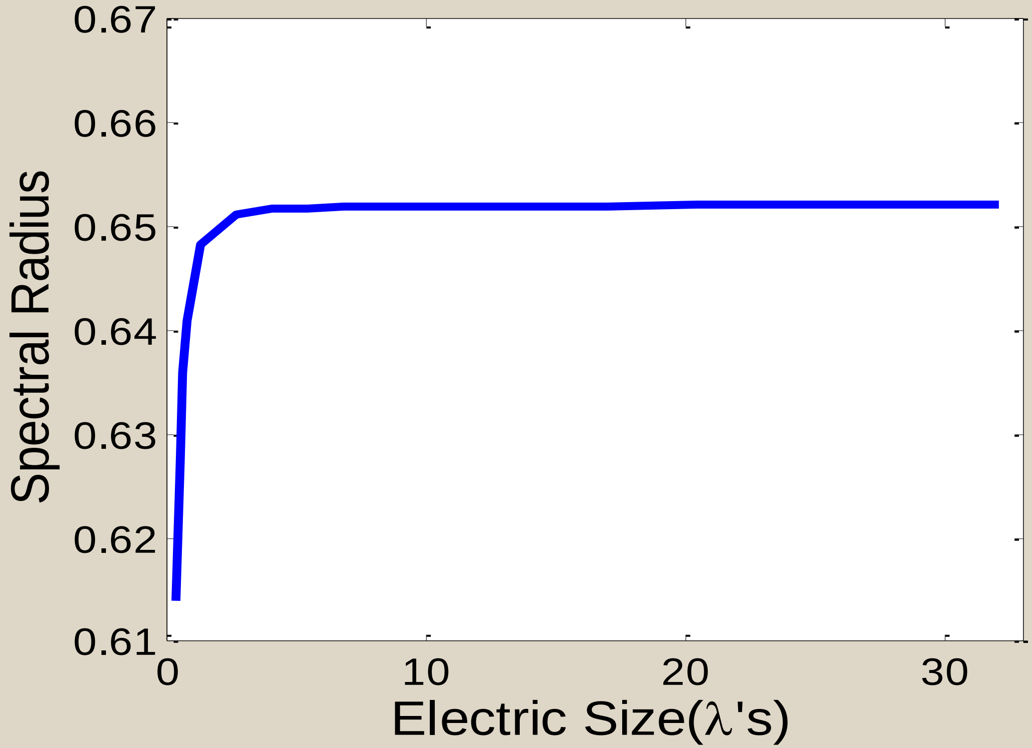
Comparison of Spectral Radius vs. Permittivity (length = 1λ)

Advantage:

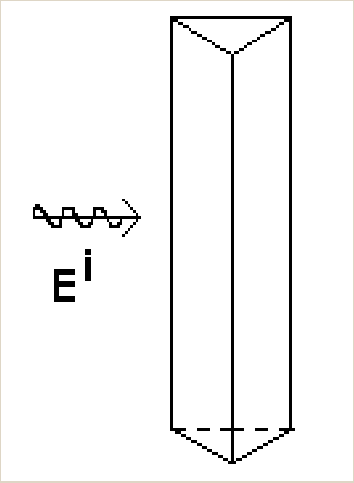
Resultant spectral radius has little dependence on dielectric constant.

Explicit Inverse

- **Spectral radius vs. electric size for a dielectric rod**



Spectral radius vs. electric size



Explicit Inverse

- **Direct solution based on explicit inverse**

$$\mathbf{D} = \left[\mathbf{I} + \Lambda'^{-1}(\mathbf{G}-\mathbf{Z}) \right]^{-1} \Lambda'^{-1} E$$

$$\text{if} \quad \left| \rho\left(\Lambda'^{-1}(\mathbf{G}-\mathbf{Z}) \right) \right| \le 1$$

$$= \left[\mathbf{I} - \Lambda'^{-1}(\mathbf{G}-\mathbf{Z}) + \Lambda'^{-1}(\mathbf{G}-\mathbf{Z})\,\Lambda'^{-1}(\mathbf{G}-\mathbf{Z}) - \Lambda'^{-1}(\mathbf{G}-\mathbf{Z})\,\Lambda'^{-1}(\mathbf{G}-\mathbf{Z})\,\Lambda'^{-1}(\mathbf{G}-\mathbf{Z}) + \ldots \right] \Lambda'^{-1} E$$

  - **The (*k*+1)-th term is $\Lambda'^{-1}(\mathbf{G}-\mathbf{Z})$ times the *k*-th term**
  - **Total cost is a few matrix-vector multiplications**

$$\Lambda'^{-1}(\mathbf{G}-\mathbf{Z})\, f^{(k)}, \quad k = 1, 2, \ldots, p$$

Explicit Inverse

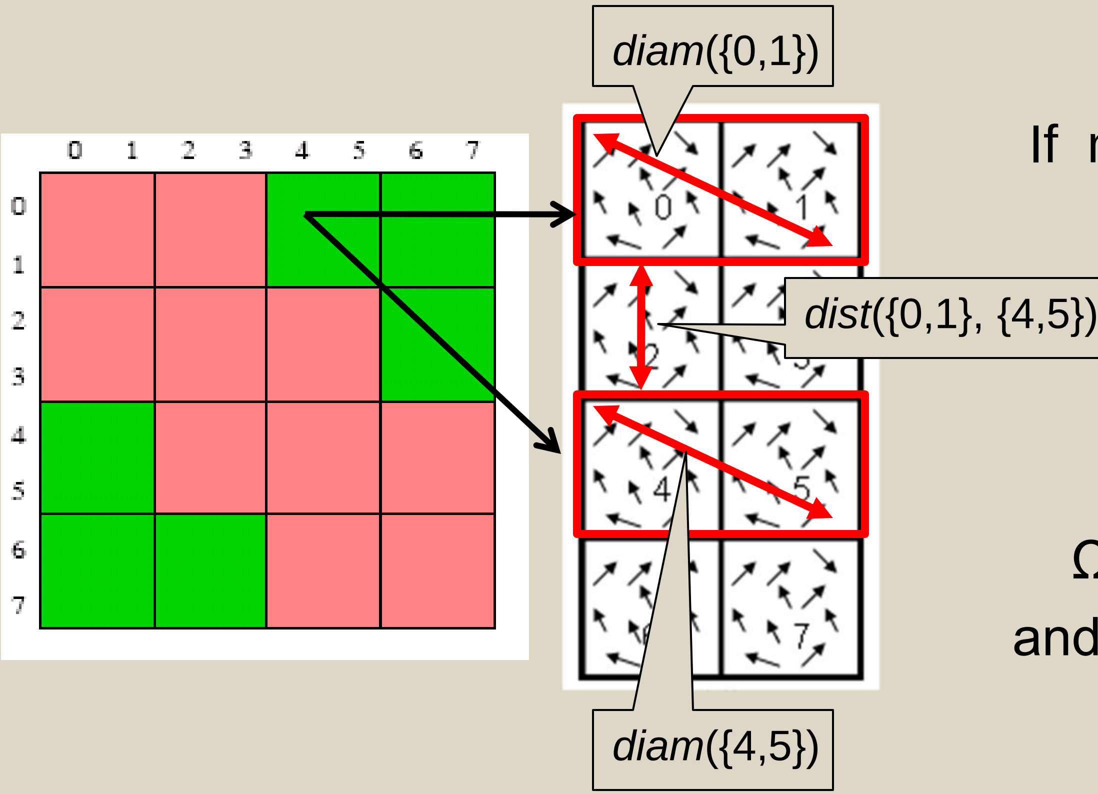
- **Accelerate** $\Lambda'^{-1}(\mathrm{G}-\mathrm{Z})f^{(k)},\ k=1, 2, \ldots, p$

  - **The sparse matrix inverse Λ'^{-1} can be performed in linear complexity by orthogonal bases [1]**
  - **The dense matrix-vector multiplication is accelerated by $\mathcal{H}^2$-matrix-based fast computation, the cost of which is *O*(*N*) from small to tens of wavelengths [2]**
  - **The overall cost of the proposed explicit inverse is *O*(*N*) from small to tens of wavelengths**

[1] D. Jiao and J. M. Jin, "Three-dimensional orthogonal vector basis functions for time-domain finite element solution of vector wave equations," *IEEE Trans. Antennas Propagat.*, vol. 51, no. 1, pp. 59-66, Jan. 2003.
[2] W. Chai and D. Jiao, "An $\mathcal{H}^2$-Matrix-Based Integral-Equation Solver of Reduced Complexity and Controlled Accuracy for Solving Electrodynamic Problems," *IEEE Trans. Antennas Propag.*, vol. 57, no. 10, pp. 3147–3159, 2009.

$\mathcal{H}^2$-Matrix Representation and Its Error Bound

Admissibility condition



If $\max\{diam(\Omega_t), diam(\Omega_s)\}$
$\leq \eta\ dist(\Omega_t, \Omega_s)$

Ω_t and Ω_s are admissible,
and $\Omega_t \times \Omega_s$ is admissible block.

[1] W. Hackbusch, B. Khoromskij, and S. Sauter, "On $\mathcal{H}^2$–matrices," Lecture on Applied Mathematics, H. Bun-gartz, R. Hoppe, and C. Zenger, eds., pp. 9-29, 2000.
[2] W. Chai and D. Jiao, "An $\mathcal{H}^2$-Matrix-Based Integral-Equation Solver of Reduced Complexity and Controlled Accuracy for Solving Electrodynamic Problems," *IEEE Trans. Antennas Propag.*, vol. 57, no. 10, pp. 3147–3159, 2009.

$\mathcal{H}^2$-Matrix Representation and Its Error Bound

Degenerate Approximation

If the admissibility condition is satisfied,

$$g(\vec{r},\vec{r}\,') = \exp(-jk\,|\vec{r}-\vec{r}\,'|)\,/\,4\pi\,|\vec{r}-\vec{r}\,'|$$

$$\tilde{g}^{t,s}(\vec{r},\vec{r}\,') = \sum_{\nu\in K^t}\sum_{\mu\in K^s} g(\xi_\nu^t,\xi_\mu^s)L_\nu^t(\vec{r})L_\mu^s(\vec{r}\,')$$

where, $K := \{\nu\in\mathbb{N}^d : \nu_i \le p \ \ for \ \ all \ \ i\in\{1,\dots,d\}\} = \{1,\dots,p\}^d$

d =1, 2, 3, for 1-, 2-, and 3-D problems respectively

p is the number of interpolation points in each dimension

ξ is a family of interpolation points

L are the corresponding Lagrange polynomials

$\mathcal{H}^2$-Matrix Representation and Its Error Bound

$$G_i = \mathrm{K}\int_P f_m(\vec{r}) \int_{Q'} g(\vec{r}, \vec{r}\,') h(\vec{r}\,') dq' dp$$

$$\{P, Q\} \in \{Volume, Surface\}$$

$$\tilde{\mathbf{G}}_i = \mathrm{K} \sum_{\nu \in K^t} \sum_{\mu \in K^s} g(\xi_\nu^t, \xi_\mu^s) \int_P f_m(\vec{r}) L_\nu^t(\vec{r}) dp \int_{Q'} h(\vec{r}\,') L_\mu^s(\vec{r}\,') dq'$$

$$\tilde{\mathbf{G}}_i := \mathbf{V}^t \mathbf{S}^{t,s} \mathbf{V}^{sT}$$

where,

$\mathbf{V}^t \in \mathbb{R}^{\#t \times \# K^t}$ is the integral over domain *P*

$\mathbf{V}^s \in \mathbb{R}^{\#s \times \# K^s}$ is the integral over domain *Q'*

$\mathbf{S}^{t,s} \in \mathbb{C}^{\# K^t \times \# K^s}$ represents the coupling matrix $\mathrm{K}\, g(\xi_\nu^t, \xi_\mu^s)$

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$\mathcal{H}^2$-Matrix Representation and Its Error Bound

- **If the admissibility condition is satisfied:**

$$\tilde{\mathbf{G}}_i \coloneqq \mathbf{V}^t \mathbf{S}^{t,s} \mathbf{V}^{s^T}$$ **($\mathcal{H}^2$ - matrix representation)**

$$\| g(\boldsymbol{r},\boldsymbol{r}') - \tilde{g}^{(t,s)}(\boldsymbol{r},\boldsymbol{r}') \|_{\infty, Q_t \times Q_s}$$

$$\leq \frac{4ed}{\pi} (\Lambda_p)^{2d} p \frac{1}{dist(Q_t, Q_s)} [1 + \sqrt{2}\kappa\eta\, dist(Q_t, Q_s) + \sqrt{2}\eta][1 + \frac{\sqrt{2}}{\kappa\eta\, dist(Q_t, Q_s) + \eta}]^{-p}$$

where,

κ is the wave number

Λ_p is interpolation scheme dependent constant

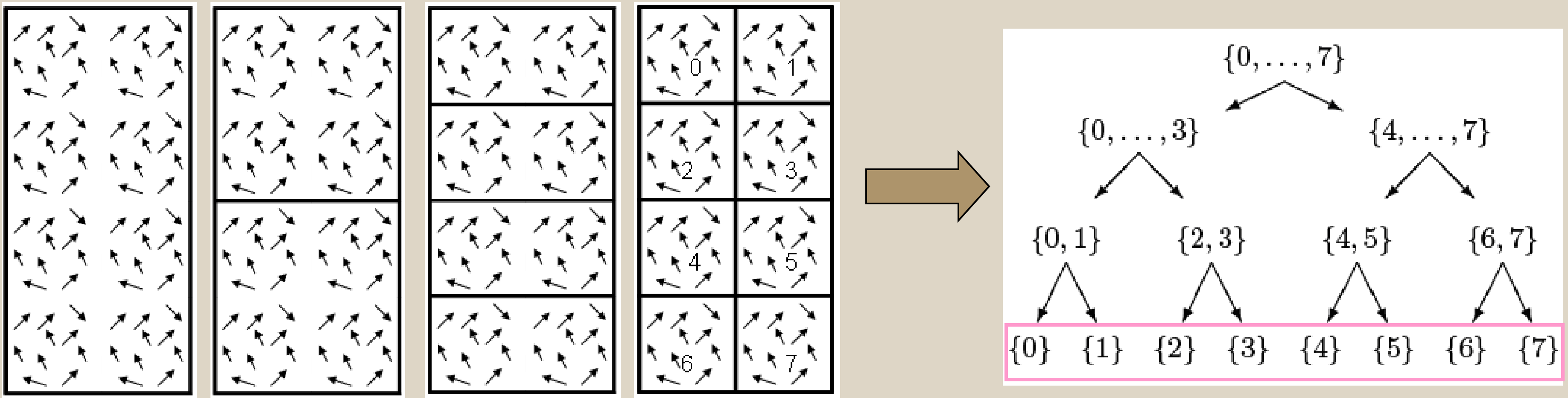
- **If not :**

$$\tilde{\mathbf{G}}_i^{(t,s)} \coloneqq \mathbf{G}_i^{(t,s)}$$ **(Full- matrix representation)**

$\mathcal{H}^2$-Matrix Partition

Build Cluster Tree

Geometric partitioning

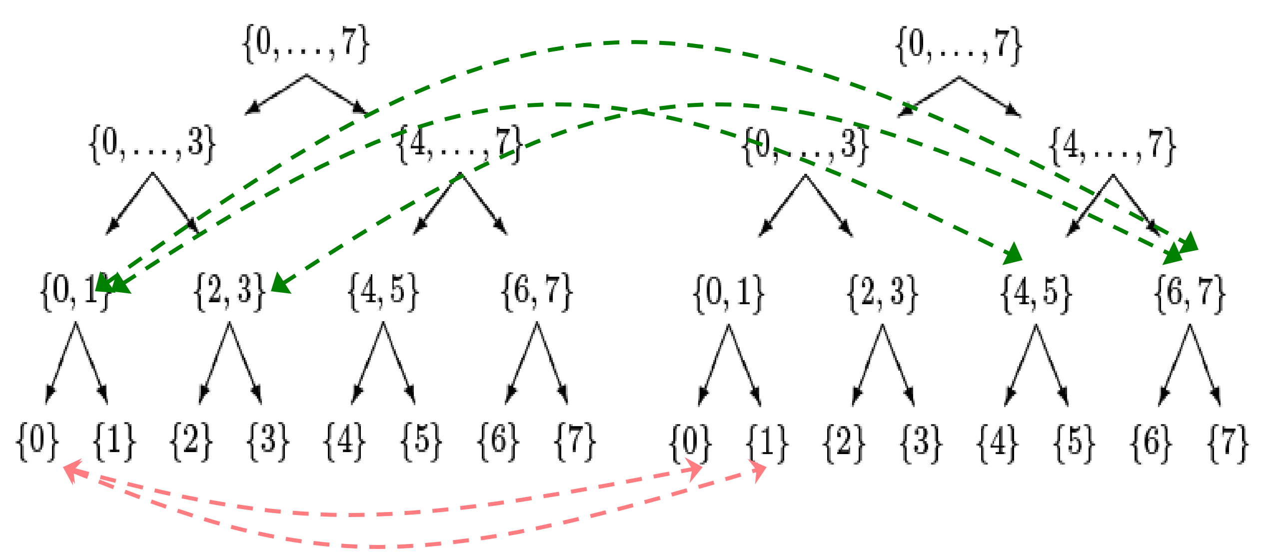


A cluster tree

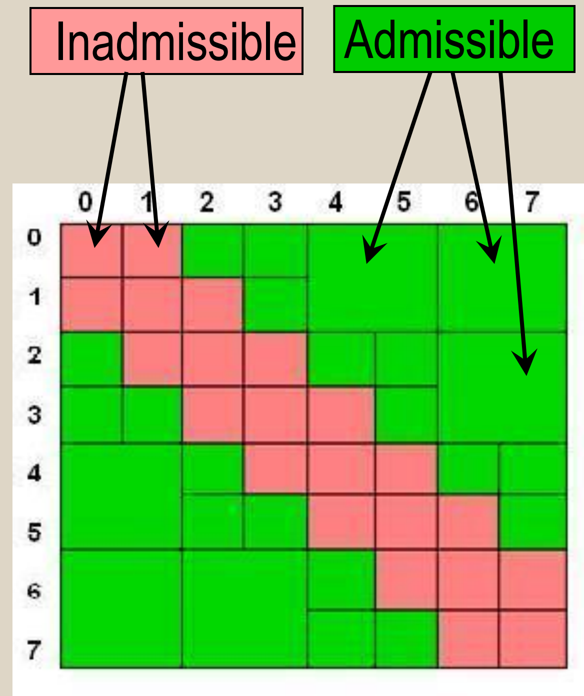
$$leafsize\ (n_{\min}) = 10$$

$\mathcal{H}^2$-Matrix Partition

Build Block Cluster Tree



Total number links in each level bounded by C_{sp}



$C_{sp} = 6$

Rank Function

Frequency is increased

$$k_{\mathrm{var}}(b) = p(b)^d$$

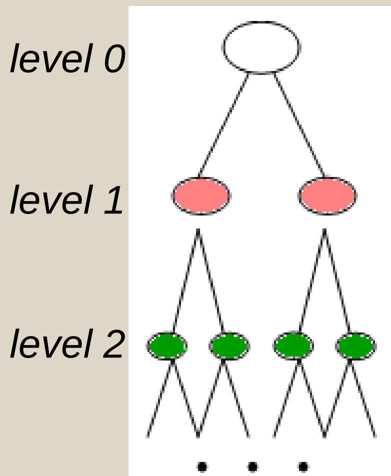
$$\left[1 + \frac{\sqrt{2}}{\kappa\, diam(Q_t \times Q_s) + \eta}\right]^{-p}$$

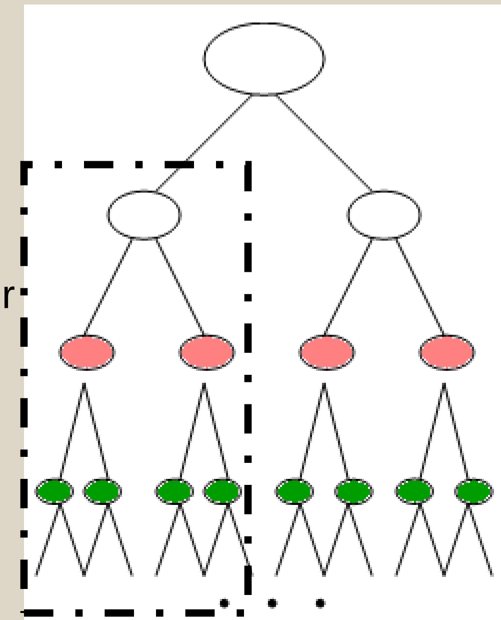
$$p(b) = \hat{a} + \hat{b}(L - l(b))$$

where

$$l(b) = level(t) = level(s)$$

$$p(b) = \hat{a} \quad \text{if} \quad L \le l(b)$$





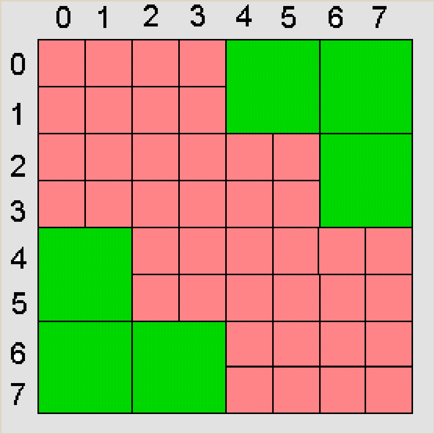
$\hat{a}, \hat{b}$ are constants

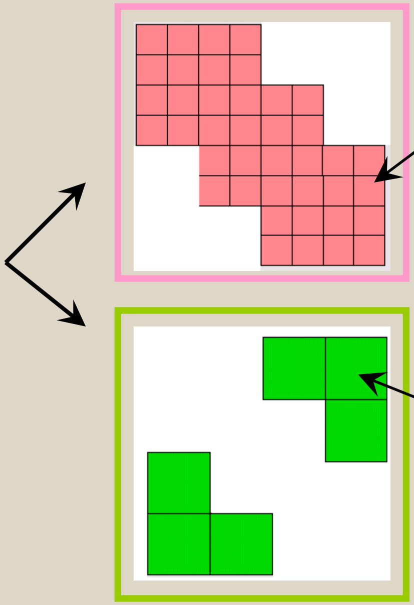
$$L = L_{\min} = \min\{level(\tau) : \tau \in \mathcal{L}_{\mathcal{I}}\}$$

☞**An effective means of increasing the rank for accuracy control in a finite electric size range (tens of wavelengths) without sacrificing the linear cost**

$O(N)$ Storage

- $\mathcal{H}^2$-matrix Representation





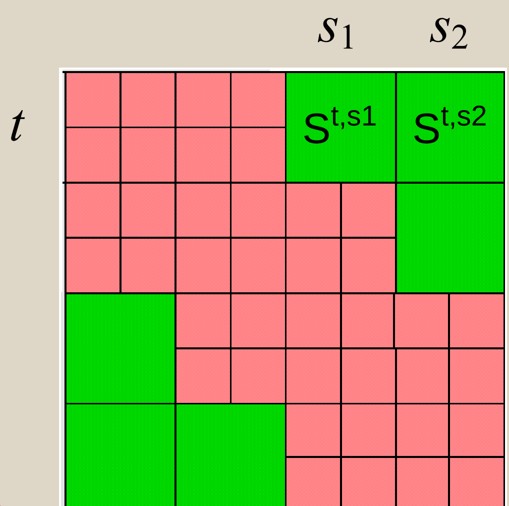
$\mathbf{G}^{t,s}$ Store full matrix

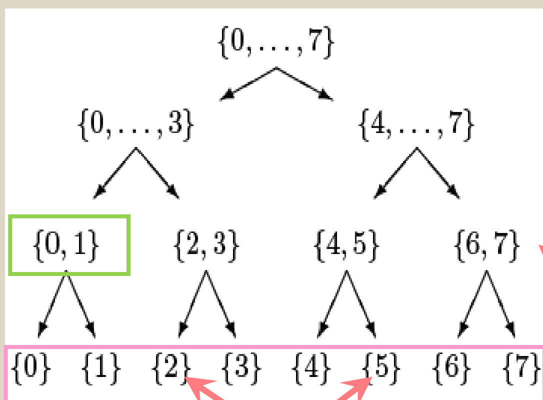
$\tilde{\mathbf{G}}^{t,s} := \mathbf{V}^t \mathbf{S}^{t,s} \mathbf{V}^{s^{\mathrm{T}}} \rightarrow$ Store V and S

O(*N*) Storage

- Nested property

$$\mathbf{V}^{t} = \begin{pmatrix} \mathbf{V}^{t1}\mathbf{E}^{t1} \\ \mathbf{V}^{t2}\mathbf{E}^{t2} \end{pmatrix} = \begin{pmatrix} \mathbf{V}^{t1} & \\ & \mathbf{V}^{t2} \end{pmatrix} \begin{pmatrix} \mathbf{E}^{t1} \\ \mathbf{E}^{t2} \end{pmatrix}$$

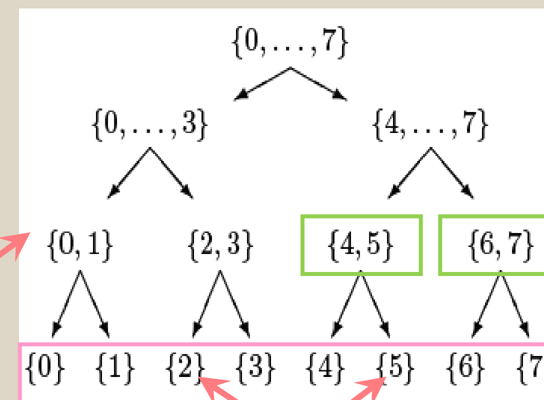




Store $\mathbf{V}$ for leaf clusters

Store $\mathbf{E}$ for all non-leaf clusters

$$\mathbf{E}^{t'}_{v'v} = L^{t}_{v}(\xi^{t'}_{v'})$$



Store $\mathbf{V}$ for leaf clusters

$O(N)$ Storage

- **Cost Analysis**

$$
\begin{aligned}
St(\mathcal{H}^2\text{-matrix}) &= \boxed{St(\textit{all leaf clusters})} + \boxed{St(\textit{all nonleaf clusters})} + \boxed{St(\textit{all admissible blocks})} + \boxed{St(\textit{all inadmissible blocks})} \\
&= \boxed{St(\mathbf{V}^t)} + \boxed{St(\textit{transfer matrix } \mathbf{E}^{t'})} + \boxed{St(\textit{coupling matrix } \mathbf{S}^b)} + \boxed{St(\textit{full matrix } \mathbf{G}^b)} \\
&= \sum_{t\in\mathcal{L}_\mathcal{I}} O(k_{\text{var}}(t))\#\hat{t} + \sum_{t\in T_\mathcal{I}\setminus\mathcal{L}_\mathcal{I}} \sum_{t'\in sons(t)} O(k_{\text{var}}(t)k_{\text{var}}(t')) + \sum_{b=(t,s)\in\mathcal{L}^+_{\mathcal{I}\times\mathcal{I}}} O(k_{\text{var}}(t)k_{\text{var}}(s)) + \sum_{b=(t,s)\in\mathcal{L}^-_{\mathcal{I}\times\mathcal{I}}} \#\hat{t}\,\#\hat{s} \\
&\le O(\hat{a}^d)\cdot N + \sum_{t\in T_\mathcal{I}} \sum_{t'\in sons(t)} O(k_{\text{var}}(t)k_{\text{var}}(t')) + \sum_{t\in T_\mathcal{I}} \sum_{s\in col(t)} O(k_{\text{var}}{}^2(t)) + \sum_{t\in T_\mathcal{I}} \sum_{s\in col(t)} n_{\min}^2 \\
&\le O(\hat{a}^d)\cdot N + 2\sum_{t\in T_\mathcal{I}} O(k_{\text{var}}{}^2(t)) + C_{sp}\sum_{t\in T_\mathcal{I}} O(k_{\text{var}}{}^2(t)) + C_{sp}\sum_{t\in T_\mathcal{I}} n_{\min}^2 \\
&\le \boxed{O(\hat{a}^d)\cdot N} + 2\boxed{\sum_{t\in T_\mathcal{I}} O(k_{\text{var}}{}^2(t))} + C_{sp}\boxed{\sum_{t\in T_\mathcal{I}} O(k_{\text{var}}{}^2(t))} + \boxed{2C_{sp}n_{\min}^2 N}
\end{aligned}
$$

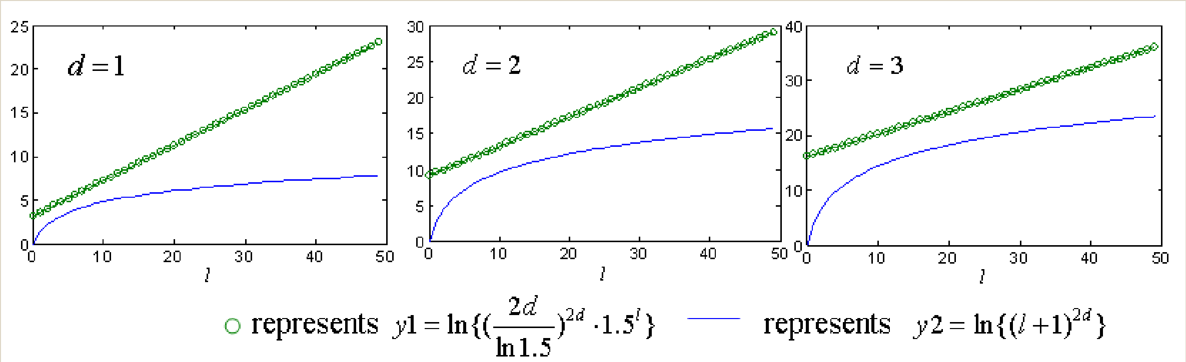
$O(N)$ Storage

$$\sum_{t\in T_I} O(k_{var}^2(t)) = O\Big(\sum_{l=0}^{L}\sum_{t\in T_I^{(l)}}[\hat a+\hat b(L-l)]^{2d}\Big) \le O\Big(\sum_{l=0}^{L}[\hat a+\hat b(L-l)]^{2d}\cdot 2^l\Big)$$

$$\le O((\hat a+\hat b)^{2d})\sum_{l=0}^{L}(1+L-l)^{2d}\cdot 2^l \le O((\hat a+\hat b)^{2d})\cdot 2^L\cdot\sum_{l=0}^{L}(l+1)^{2d}\cdot 2^{-l} \le O((\hat a+\hat b)^{2d})\cdot 2^L\cdot\sum_{l=0}^{L}\Big(\frac{2d}{\ln 1.5}\Big)^{2d}\cdot 1.5^l\cdot 2^{-l}$$

$$\le 2^{L+1}\cdot O\Big(\Big(\frac{2d(\hat a+\hat b)}{\ln 1.5}\Big)^{2d}\Big)\cdot\sum_{l=0}^{L}\Big(\frac{3}{4}\Big)^l \le 2^{L+1}\cdot O\Big(\Big(\frac{2d(\hat a+\hat b)}{\ln 1.5}\Big)^{2d}\Big)\cdot 4 \le 4O\Big(\Big(\frac{2d(\hat a+\hat b)}{\ln 1.5}\Big)^{2d}\Big)\cdot N$$

$$(l+1)^{2d} \le \Big(\frac{2d}{\ln 1.5}\Big)^{2d}\cdot 1.5^l$$



$$St(\mathcal{H}^2\text{-matrix}) \Leftarrow O(\hat a^d)\cdot N + 8O\Big(\Big(\frac{2d(\hat a+\hat b)}{\ln 1.5}\Big)^{2d}\Big)\cdot N + 4C_{sp}O\Big(\Big(\frac{2d(\hat a+\hat b)}{\ln 1.5}\Big)^{2d}\Big)\cdot N + 2C_{sp}O(n_{min}^2)\cdot N$$

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O(*N*) Matrix-Vector Multiplication

- If $\mathbf{G}^{t,s}$ is an **admissible** block

  Matrix-vector multiplication can be performed in **three** steps

$$\tilde{\mathbf{G}}_i \cdot \boldsymbol{x} := \mathbf{V}^t \mathbf{S}^{t,s} \mathbf{V}^{s^T} \cdot \boldsymbol{x}$$

(a) Forward transformation $\quad \boldsymbol{x}^s = \mathbf{V}^{s^T} \cdot \boldsymbol{x} \longrightarrow$ ***O*(*N*)**

(b) Coupling-matrix Multiplication $\quad \boldsymbol{y}^t = \sum_{s \in R^t} \mathbf{S}^{t,s} \boldsymbol{x}^s \longrightarrow$ ***O*(*N*)**

(c) Backward transformation $\quad y_i = \sum_{t, i \in \hat{t}} (\mathbf{V}^t \boldsymbol{y}^t)_i \longrightarrow$ ***O*(*N*)**

- If $\mathbf{G}^{t,s}$ is an **inadmissible** block

(d) Direct multiplication $\longrightarrow$ ***O*(*N*)**

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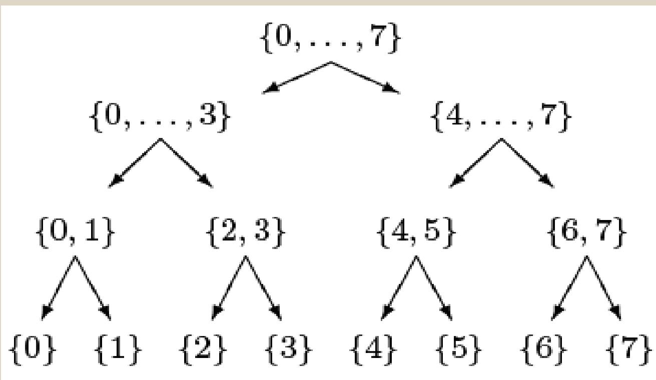
$\tilde{\mathbf{G}}^{t,s}\cdot x := \mathbf{V}^t\mathbf{S}^{t,s}\mathbf{V}^{s^{\mathrm{T}}}\cdot x$

(a) Forward transformation $\boldsymbol{x}^s = \mathbf{V}^{s^{\mathrm{T}}}\cdot \boldsymbol{x}$

$$\mathbf{V}^s = \begin{pmatrix} \mathbf{V}^{s1}\mathbf{E}^{s1} \\ \mathbf{V}^{s2}\mathbf{E}^{s2} \end{pmatrix} = \begin{pmatrix} \mathbf{V}^{s1} & \\ & \mathbf{V}^{s2} \end{pmatrix}\begin{pmatrix} \mathbf{E}^{s1} \\ \mathbf{E}^{s2} \end{pmatrix}$$



Non-leaf cluster ⟵ $\boldsymbol{x}^s = \mathbf{V}^{s^{\mathrm{T}}}\boldsymbol{x}|_{\hat{s}} = \sum_{s'\in sons(s)} (\mathbf{V}^{s'}\mathbf{E}^{s'})^{\mathrm{T}}\boldsymbol{x}|_{\hat{s}'} = \sum_{s'\in sons(s)} (\mathbf{E}^{s'})^{\mathrm{T}}\boldsymbol{x}^{s'}$

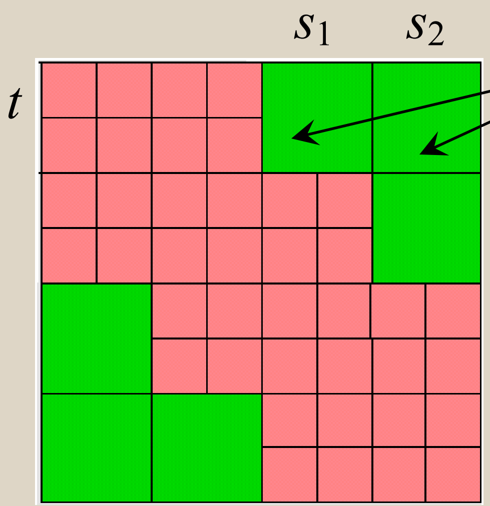
Leaf cluster ⟵ $\boldsymbol{x}^s = \mathbf{V}^{s^{\mathrm{T}}}\boldsymbol{x}|_{\hat{s}}$

$$Comp(\sum_{s\in T_{\mathcal{I}}} \mathbf{V}^{s^{\mathrm{T}}}\boldsymbol{x}|_{\hat{s}}) = \sum_{s\in\mathcal{L}(T_{\mathcal{I}})} O(k_{\mathrm{var}}(s))\#\hat{s} + \sum_{\substack{s\in T_{\mathcal{I}}\setminus\mathcal{L}_{\mathcal{I}} \\ s'\in sons(s)}} O(k_{\mathrm{var}}(s)k_{\mathrm{var}}(s')) \le O(\hat{a}^d)N + \sum_{s\in T_{\mathcal{I}}} O(k_{\mathrm{var}}^2(s))$$

$$\le O(\hat{a}^d)N + 2O((\frac{2d(\hat{a}+\hat{b})}{\ln 1.5})^{2d})N$$

$$\tilde{\mathbf{G}}^{t,s}\cdot x := \mathbf{V}^{t}\mathbf{S}^{t,s}\mathbf{V}^{s^{T}}\cdot x$$

(b) Coupling-matrix Multiplication $y^{t} = \sum_{s\in R^{t}} \mathbf{S}^{t,s} X^{s}$



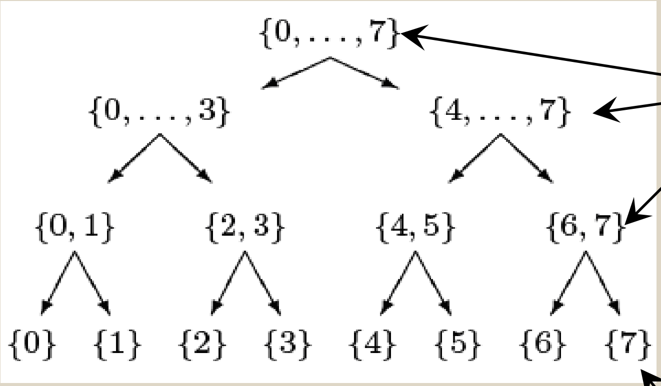
$$\boldsymbol{y}^{t} = \mathbf{S}^{t,s_1}\boldsymbol{x}^{s_1} + \mathbf{S}^{t,s_2}\boldsymbol{x}^{s_2}$$

$$Comp\Big(\sum_{b=(t,s)\in\mathcal{L}^{+}_{\mathcal{I}\times\mathcal{I}}} \mathbf{S}^{t,s}\boldsymbol{x}^{s}\Big) = \sum_{b=(t,s)\in\mathcal{L}^{+}_{\mathcal{I}\times\mathcal{I}}} O(k_{\mathrm{var}}^{2}(b)) \le \sum_{b=(t,s)\in\mathcal{L}_{\mathcal{I}\times\mathcal{I}}} O(k_{\mathrm{var}}^{2}(b)) \le \sum_{t\in T_{\mathcal{I}}}\sum_{s\in col(t)} O(k_{\mathrm{var}}^{2}(b))$$

$$\le C_{sp}\sum_{t\in T_{\mathcal{I}}} O(k_{\mathrm{var}}^{2}(b)) \le 2C_{sp}\,O\Big(\Big(\frac{2d(\hat{a}+\hat{b})}{\ln 1.5}\Big)^{2d}\Big)\cdot N$$

$\tilde{\mathbf{G}}^{t,s} \cdot x := \mathbf{V}^t \mathbf{S}^{t,s} \mathbf{V}^{s^{\mathrm{T}}} \cdot x$

(c) Backward transformation $y_i := \sum_{t, i\in\hat{t}} (\mathbf{V}^t y^t)_i$



$$y_i = (\mathbf{V}^{t1} y^{t1})_{i\in t1} + (\mathbf{V}^{t2} y^{t2})_{i\in t2} + (\mathbf{V}^{t3} y^{t3})_{i\in t3}$$

i

$$Comp\left(\sum_{t\in T_{\mathcal{I}}} \mathbf{V}^t y^t\right) = \sum_{t\in\mathcal{L}(T_{\mathcal{I}})} O(k_{\mathrm{var}}(t))\#\hat{t} + \sum_{\substack{t\in T_{\mathcal{I}}\setminus\mathcal{L}_{\mathcal{I}} \\ t'\in sons(t)}} O(k_{\mathrm{var}}(t)k_{\mathrm{var}}(t')) \le O(\hat{a}^d)N + 2O\left(\left(\frac{2d(\hat{a}+\hat{b})}{\ln 1.5}\right)^{2d}\right)N$$

$O(N)$ Matrix-Vector Multiplication

If $\mathbf{G}^{t,s}$ is **inadmissible** block

Since they are full matrices, multiplication can be done directly.

$$Comp(\sum_{b=(t,s)\in\mathcal{L}_{\mathcal{I}\times\mathcal{I}}} \mathbf{G}^{t,s} x|_{\hat{s}}) = \sum_{b=(t,s)\in\mathcal{L}_{\mathcal{I}\times\mathcal{I}}} O(n_{\min}^2) \le 2C_{sp} O(n_{\min}^2) N$$

$O(N)$ Matrix-Vector Multiplication

Add four steps

$$Comp(\mathcal{H}^2\text{-}Matrix\text{-}vector\ mult.) \to O(N)$$

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Numerical Results

- **Dielectric Sphere**
- **Triangular Dielectric Rod**

Case 1: Dielectric Sphere ($\varepsilon_r = 36$)

Parameters: $n_{\min} = 18, \quad \eta = 0.95, \quad \hat{a} = 4, \quad \hat{b} = 2$

Electric size: $k_0 a = 0.408$

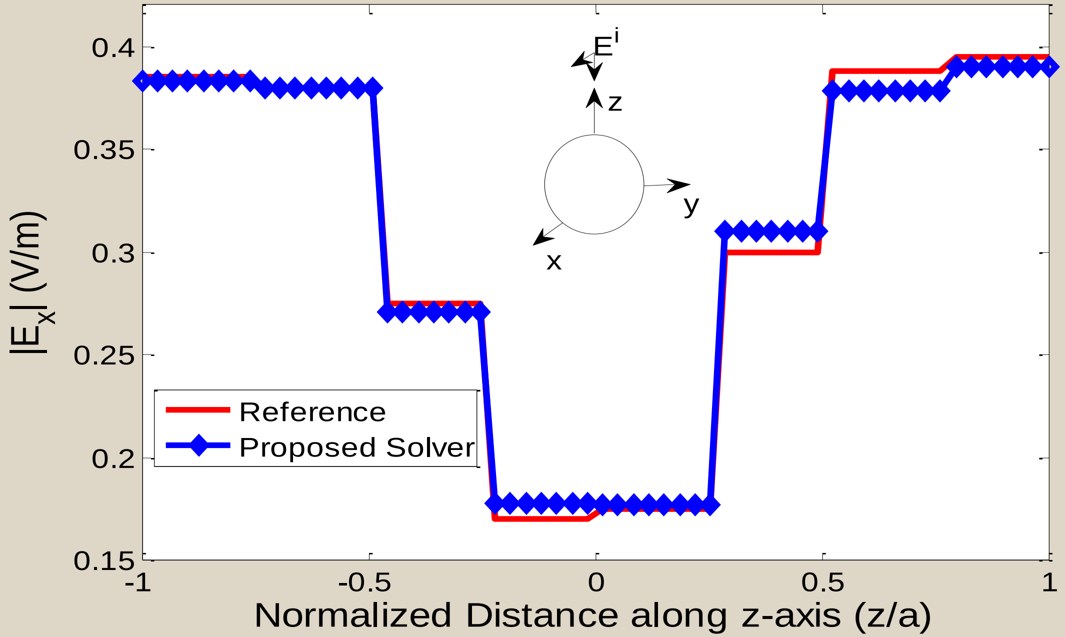


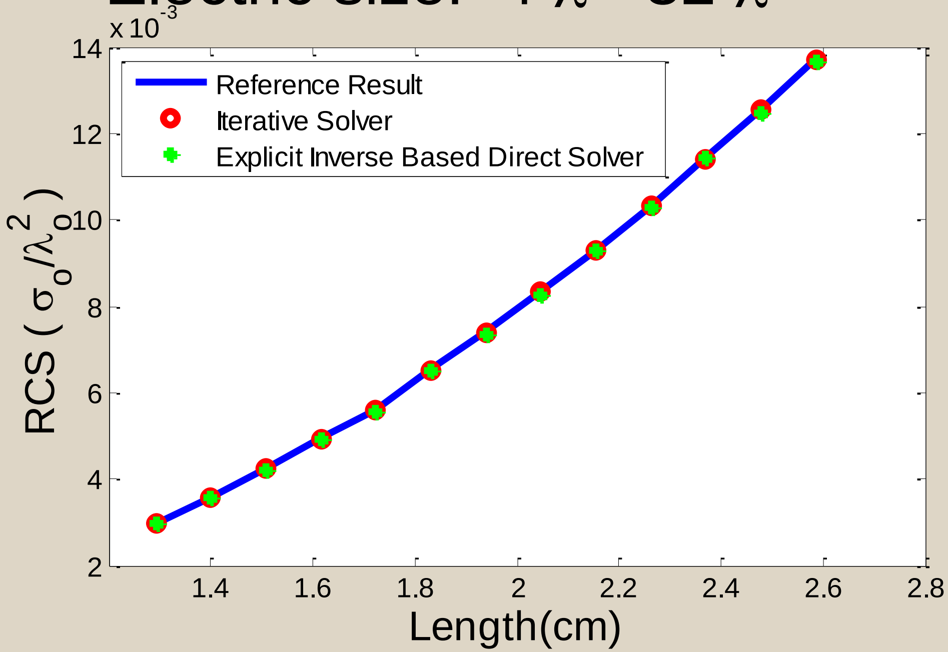
Fig. 1. Field along z-axis of a dielectric sphere with $\varepsilon_r = 36$ (Reference data is from [1]).

[1] D. H. Schaubert, D. R. Wilton, A. W. Glisson, "A tetrahedral modeling method for electromagnetic scattering by arbitrarily shaped inhomogeneous dielectric bodies," *IEEE Trans. Antennas Propagat.*, vol. 32, pp. 77-85, Jan. 1984.

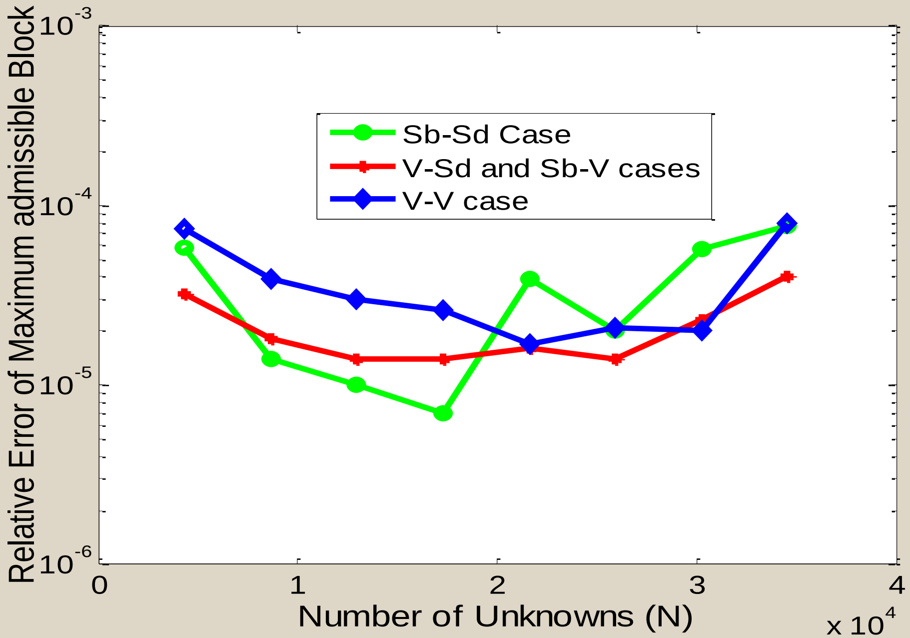
Case 2: Triangular Dielectric Rod ($\varepsilon_r = 2.54$)

Parameters: $n_{\min} = 80, \quad \eta = 0.95, \quad \hat{a} = 4(5), \quad \hat{b} = 4(5)$

Electric size: 4 λ - 32 λ



(a) RCS comparison



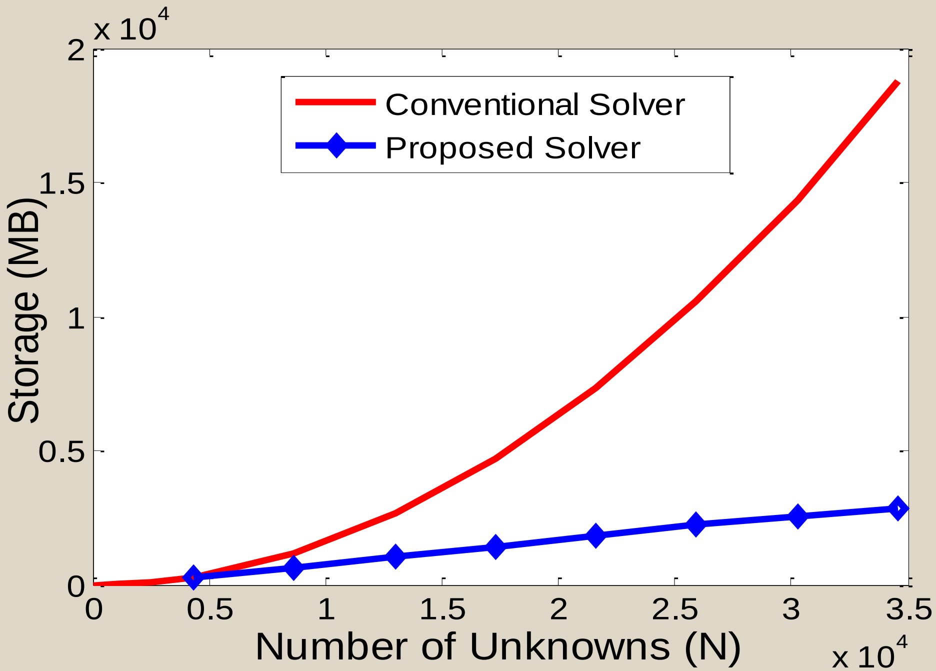
(b) Relative error of the maximal admissible block

[1] D. H. Schaubert, D. R. Wilton, A. W. Glisson, "A tetrahedral modeling method for electromagnetic scattering by arbitrarily shaped inhomogeneous dielectric bodies," *IEEE Trans. Antennas Propagat.*, vol. 32, pp. 77-85, Jan. 1984.

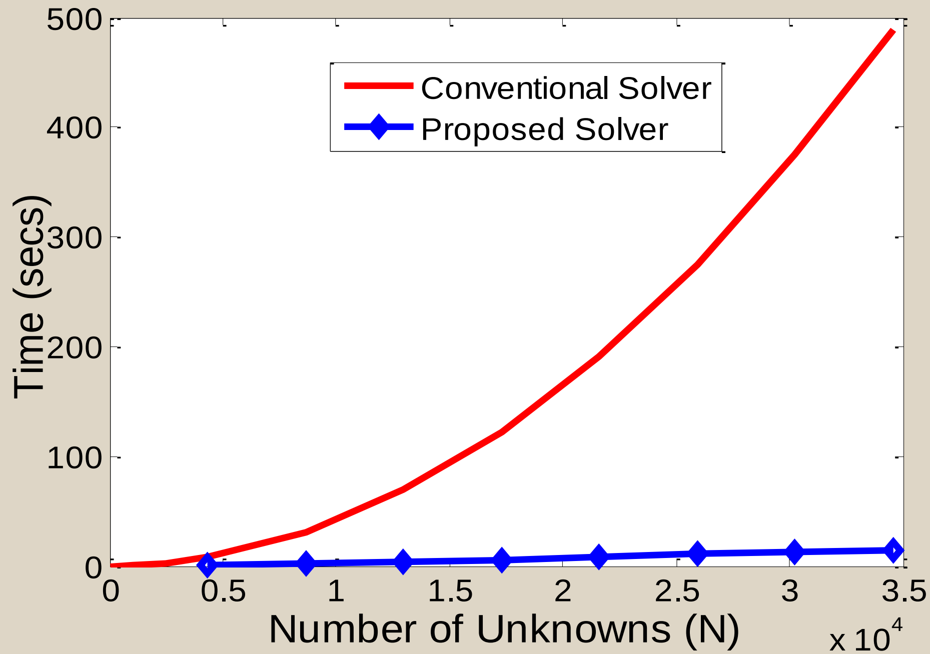
Case 2: Triangular Dielectric Rod ($\varepsilon_r = 2.54$)

Parameters: $n_{\min} = 80, \quad \eta = 0.95, \quad \hat{a} = 4(5), \quad \hat{b} = 4(5)$

Electric size: 4 λ - 32 λ



(c) Storage comparison



(d) CPU MVM time comparison

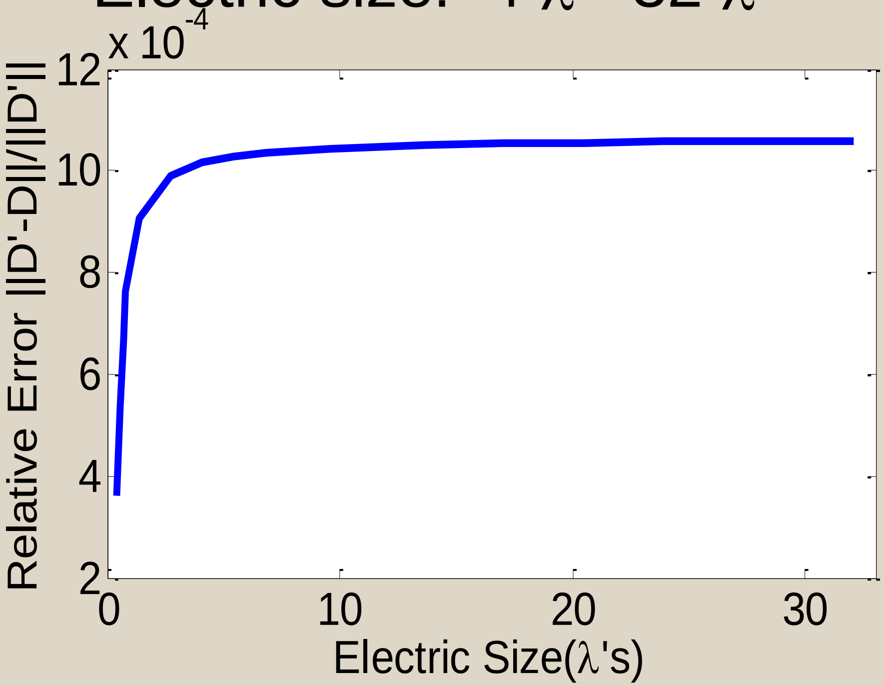
Fig. 2. Simulation of a dielectric rod with $\varepsilon_r = 2.54$.

Case 2: Triangular Dielectric Rod ($\varepsilon_r = 2.54$)

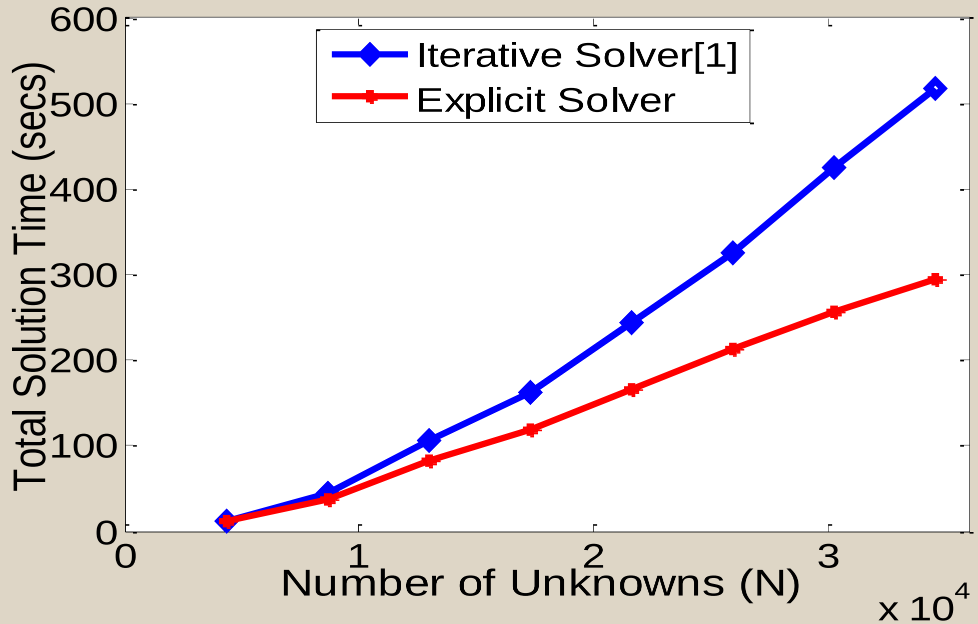
Parameters: $n_{\min} = 80,\quad \eta = 0.95,\quad \hat{a} = 4(5),\quad \hat{b} = 4(5)$

Electric size: 4 λ - 32 λ

No. of terms: 20



(e) Relative Error



(f) Solution time comparison with iterative solver[1]

Fig. 2. Simulation of a dielectric rod with ε_r = 2.54.

[1] S.Omar and D.Jiao, "An $\mathcal{H}^2$-Matrix Based Fast Volume Integral Equation Solver for Electrodynamic Analysis," ACES 2011.

Case 2: Triangular Dielectric Rod (ε_r = 2.54)

Parameters: $n_{\min} = 80, \quad \eta = 0.95, \quad \hat{a} = 7(8), \quad \hat{b} = 7(8)$

Electric size: 80 λ

Accuracy : 10^{-4}

No. of terms: 20

Memory : 9 GB (112 GB for conventional solver)

Solution Time : 1675.5 s (61096.3 s for conventional solver)

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Conclusions

Explicit Inverse Based Fast VIE Solver Developed

- Avoid numerical inversion and troubles of iterations
- Accelerated by $\mathcal{H}^2$-based fast matrix-vector multiplications
- Applicable to problems with large dielectric contrast ratio
- Numerical results demonstrate its accuracy and efficiency
- Future work: To explore its application to general problems and larger electric sizes.