FUEL-OPTIMAL, LOW-THRUST TRANSFERS BETWEEN LIBRATION POINT ORBITS

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Jeffrey R. Stuart

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“If I have seen further than others, it is by standing upon the shoulders of giants.” - Isaac Newton

This work is dedicated to Charles and Jane “Gong-gong” Hoffmann and Tom and Mary Stuart. A child cannot wish for better grandparents.
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SYMBOLS

$t$  
-time

d  
-relative displacement vector between two particles

d  
magnitude of relative displacement vector

$m, M$  
-mass

$\mathbf{F}$  
-force vector

$G$  
-Universal Gravitational Constant

$r$  
-position vector with respect to inertially fixed point

$r$  
-magnitude of position vector

$P_1$  
-larger primary

$P_2$  
-smaller primary

$P_3$  
-particle of interest

$B$  
-system barycenter

$I$  
-inertially fixed frame

$\hat{X}, \hat{Y}, \hat{Z}$  
-unit vectors defining inertial frame

$\hat{x}, \hat{y}, \hat{z}$  
-unit vectors defining rotating frame

$x, y, z$  
-position coordinates with respect to rotating frame

$\theta$  
-angle between inertial and rotating frames

$l^*$  
-characteristic length of system

$m^*$  
-characteristic mass of system

$t^*$  
-characteristic time scale of system, in seconds

$t^*_d$  
-characteristic time scale of system, in days

$\mu$  
-primary mass ratio

$M_e$  
-mass of Earth

$M_m$  
-mass of Moon
\( \mathbf{v} \) velocity vector of \( P_3 \) with respect to system barycenter

\( \mathbf{x} \) state vector of \( P_3 \), composed of \( \mathbf{r} \) and \( \mathbf{v} \)

\( \dot{\mathbf{x}}, \mathbf{f}(\mathbf{x}) \) equations of motion of particle \( P_3 \)

\( \mathbf{f}_1(\mathbf{x}) \) natural acceleration terms of \( P_3 \)

\( U^* \) pseudo-potential function of CR3BP

\( U_q^* \) partial derivative of pseudo-potential function with respect to variable \( q \)

\( U_{qw}^* \) second partial of pseudo-potential function with respect to variables \( q \) and \( w \)

\( K \) kinetic energy

\( U \) gravitational potential energy

\( C \) Jacobi Constant

\( \mathbf{x}_{eq} \) equilibrium solution

\( L_i \) libration point \( i \)

\( \iota_i \) displacement of \( L_i \) from nearest primary

\( \epsilon, \delta \) small tolerance or perturbation

\( \delta \mathbf{x} \) perturbation vector

\( \mathbf{A}(t) \) matrix containing linearized equations of motion

\( \phi(t), \psi(t) \) solution to equations of motion

\( \mathbf{p}(t) \) periodic solution to equations of motion

\( P \) period of orbit

\( \gamma_i \) eigenvalue

\( \hat{\gamma} \) eigenvector

\( n_s, n_c, n_u \) number of stable, center, and unstable eigenvalues

\( E_S, E_C, E_U \) stable, center, and unstable eigenspaces

\( \mathbf{W}_{loc}^S, \mathbf{W}_{loc}^C, \mathbf{W}_{loc}^U \) stable, center, and unstable local invariant manifolds

\( \hat{\gamma}^s, \hat{\gamma}^u \) stable and unstable eigenvectors

\( \hat{\mathbf{v}}^s, \hat{\mathbf{v}}^u \) normalized stable and unstable eigenvectors

\( \mathbf{x}_s, \mathbf{x}_u \) initial conditions of stable and unstable invariant manifolds
$X$ general design vector

$X_0$ initial guess of solution

$X_S$ single shooting design vector

$X_M$ multiple shooting design vector

$F(X)$ constraint vector

$F_S(X_S)$ single shooting constraint vector

$F_M(X_M)$ single shooting constraint vector

$DF(X)$ gradient of constraint vector

$x^*$ baseline/reference solution to equations of motion

$\Phi, \Psi$ state transition matrices

$I$ identity matrix

$h$ small step size for finite differencing

$x_0, x_I$ initial state

$x_T$ target state

$x^t$ state after propagation time $t$

$t_e$ half-period of orbit

$\tau$ time-like parameter defining state on periodic orbit

$\alpha$ time-like parameter defining propagation time along invariant manifold

$J$ cost functional

$J_e$ end cost

$L$ Lagrangian (indirect optimization)

$H$ Hamiltonian

$\nu$ Lagrange multipliers

$\nu$ vector of control variables

$\psi$ boundary condition vector

$\lambda$ co-state vector

$\dot{\lambda}$ time derivative of co-state vector

$\chi$ state vector incorporating position, velocity, and spacecraft mass
\( \dot{\chi} \) equations of motion for thrusting spacecraft

\( T \) thrust magnitude

\( P \) engine power

\( P_{\text{max}} \) maximum engine power

\( u \) thrust direction unit vector

\( I_{sp} \) specific impulse

\( g_0 \) gravitational acceleration at Earth’s surface

\( TD \) thrust duration

\( m_f \) final spacecraft mass

\( \lambda_r \) position co-state vector

\( \lambda_v \) velocity co-state vector

\( \lambda_1 \) initial condition of position and velocity co-state vectors

\( \lambda_v \) magnitude of velocity co-state vector

\( S \) switching function

\( \Upsilon \) initial state change of manifold along orbit

\( V \) spacecraft centered frame

\( \hat{V}, \hat{n}, \hat{b} \) unit vectors defining frame \( V \)

\( \rho, \beta \) spherical angles defining thrust direction within frame \( V \)

\( D \) DCM from rotating frame \( R \) to frame \( V \)

\( \dot{D} \) DCM rate of change

\( X_{SACT} \) single shooting design vector using adjoint control transformation

\( \xi \) combined state and co-state vector

\( a, \dot{\xi} \) EOM vector of combined state and co-state

\( W \) sub-matrix of \( 14 \times 14 \) linearized equations of motion

\( P_S \) phase shift

\( X_{PS} \) phase-shifting, single shooting design vector

\( F_{PS}(X_{PS}) \) phase-shifting, single shooting constraint vector

\( X_{Imp} \) impulsive maneuver design vector

\( F_{Imp}(X_{Imp}) \) impulsive maneuver constraint vector
\( \Delta v \) instantaneous change in velocity
\( v_{orb} \) velocity on orbit
\( v_{Imp} \) velocity on impulsive transfer arc
\( g \) inequality constraint vector
\( h \) equality constraint vector
\( x^L, x^U \) lower and upper bounds on variables
\( \Lambda \) Lagrangian function (direct optimization)
\( \nabla \) gradient of a scalar
\( \zeta \) step size of SQP
\( s^g \) step direction of SQP
\( Q \) quadratic function of SQP
\( B \) approximation to Hessian matrix of Lagrangian \( \Lambda \)
\( \kappa_j \) slack variable
\( X_C \) state vector with slack variables
\( F_C(X_C) \) constraint vector including path constraints
ABBREVIATIONS

2PBVP  two-point boundary value problem
3BP    three-body problem
ACT    adjoint control transformation
CoV    calculus of variations
CR3BP  circular restricted three body problem
DCM    direction cosine matrix
EOM    equation of motion
H.O.T.  higher order terms
NLP    non-linear programming
ODE    ordinary differential equation
SQP    sequential quadratic programming
STM    state transition matrix
VSI    variable specific impulse
ABSTRACT


Mission design requires the efficient management of spacecraft fuel to reduce mission cost, increase payload mass, and extend mission life. High efficiency, low-thrust propulsion devices potentially offer significant propellant reductions. Periodic orbits that exist in a multi-body regime and low-thrust transfers between these orbits can be applied in many potential mission scenarios, including scientific observation and communications missions as well as cargo transport. In light of the recent discovery of water ice in lunar craters, libration point orbits that support human missions within the Earth-Moon region are of particular interest. This investigation considers orbit transfer trajectories generated by a variable specific impulse, low-thrust engine with a primer-vector-based, fuel-optimizing transfer strategy. A multiple shooting procedure with analytical gradients yields rapid solutions and serves as the basis for an investigation into the trade space between flight time and consumption of fuel mass. Path and performance constraints can be included at node points along any thrust arc. Integration of invariant manifolds into the design strategy may also yield improved performance and greater fuel savings. The resultant transfers offer insight into the performance of the variable specific impulse engine and suggest novel implementations of conventional impulsive thrusters. Transfers incorporating invariant manifolds demonstrate the fuel savings and expand the mission design capabilities that are gained by exploiting system symmetry. A number of design applications are generated.
1. INTRODUCTION

An eventual human return to Earth’s Moon is likely. In the near term, there is an increase in the number of scientific missions to the lunar region, including NASA’s Lunar Reconnaissance Orbiter [1] and JAXA’s KAGUYA mission [2]. For future extended human missions, a constant communications link is also necessary between the Earth and a lunar outpost. To support these scientific and communications missions over the next few decades, non-traditional periodic orbits in the vicinity of the Moon, for example, the Lissajous orbits associated with the Earth-Moon equilibrium points, offer intriguing possibilities. But, the trajectory costs, in terms of both time and fuel, are always a high priority. Non-Keplerian trajectories, including transfers between such orbits, could expand the options in the Earth-Moon system but have not been thoroughly explored. In addition, low-thrust propulsion systems potentially offer significant propellant mass savings, as demonstrated by Sovey et al. [3] and, thus, are often proposed to reduce fuel costs or increase spacecraft payload. In recent years, much effort has been expended on novel low-thrust propulsion systems and the implications for trajectory design. This investigation combines these technologies to explore low-thrust orbit transfer design in a multi-body regime, specifically applied to the Earth-Moon region.

1.1 Problem Definition

The primary focus of this work is the generation and analysis of fuel-optimal transfers between periodic orbits accomplished via low-thrust propulsion produced by a variable specific impulse engine. The periodic orbits under investigation exist in the vicinity of the libration points of the Earth-Moon system, modeled as a circular restricted three-body problem, and the invariant manifolds associated with these orbits
aid in transfer design and fuel conservation. Optimization of the transfers is accomplished via application of the calculus of variations and, when path constraints are required, the use of non-linear programming tools. The optimal transfers are investigated for insight into the natural system dynamics and mission design applications of both conventional and future propulsion systems.

1.2 Previous Work

Trajectory design incorporating low-thrust propulsion systems has been a subject of intense research for many years. The various studies have covered a wide variety of topics, from the design of the propulsion system hardware to the optimization of various mission architectures. A brief background of previous work relevant to this investigation is useful.

1.2.1 Natural Dynamical Behavior due to Gravitational Fields

The natural motion in a gravitational field as a subject of formal mathematical analysis dates to the time of Newton and his publication of the *Philosophiæ Naturalis Principia Mathematica* [4]. Newton explored the general three-body problem in his attempts to model the motion of the Moon, relative to the Earth, as perturbed by the Sun. Euler first formulated the restricted three-body problem (R3BP) and demonstrated the existence of the three collinear libration points while Lagrange later described the locations of all five equilibrium solutions including the equilateral points [5]. Jacobi furthered the investigation of the problem of three bodies by the determination of an integral of motion (that bears his name) when the system equations are formulated relative to a rotating frame [6]. Hill continued the work of Jacobi and demonstrated that the Jacobi integral defines regions of exclusion [6]. Poincaré investigated the qualitative behavior in the circular restricted three-body problem (CR3BP) and hypothesized that an infinite number of periodic solutions exist [7]. Additionally, Poincaré proved that the general $n$-body problem does not possess a
closed-form solution using analytical integration techniques such as integrals of the motion. Szebehely, in 1967, compiled the then-current knowledge of the R3BP into his seminal work, *Theory of Orbits* [8]. Since the advent of high speed computers, there has been an explosive increase in the volume of work related to the R3BP.

### 1.2.2 Low-Thrust Transfer Design

Efficient design of low-thrust trajectories requires several key components including a model of the dynamical behavior, selection of a thrust history, as well as the computation and optimization of the solution. Since the 1960’s, two approaches for low-thrust trajectory design and local optimization have been developed: *indirect* methods based on the calculus of variations and *direct* methods including parameter optimization.

Reformulating an optimization problem into a less computationally intensive boundary or initial value problem is a hallmark of indirect optimization. In 1963, Lawden applied the Euler-Lagrange Theory to formulate trajectory optimization within the context of a two-point boundary value problem (2PBVP) [9]. At roughly the same time, engine power limits were first incorporated into the design techniques for optimal orbit transfers [10,11]. In more recent years, similar approaches have been applied to spacecraft rendezvous [12] and transfers [13–19] in the CR3BP. Although indirect methods are typically low-dimensioned and computationally inexpensive, noted disadvantages include numerical difficulties in enforcing path constraints and a requirement for extensive redervation when the fundamental problem is even slightly altered [20]. With the availability of modern computers, the indirect optimization 2PBVPs are commonly solved using explicit numerical integration and shooting methods [21].

Direct optimization algorithms, in contrast to indirect methods, extremize a cost function in its given form and apply path constraints without far-reaching reformulations of the problem. In this investigation, the term *direct* is further restricted to include calculus-based parameter optimization methods. For example, Non-Linear
Programming (NLP) [22] techniques such as Sequential Quadratic Programming (SQP) [23] approximate the original optimization problem as a more tractable sub-problem but still, via the Karush-Kuhn-Tucker (KKT) conditions, ensure optimality [24, 25]. Several commercially-available algorithms, such as SNOPT [26, 27], are now accessible as computation packages for applying SQP techniques. Recently, hybrid optimization algorithms have also been developed that combine many of the advantages of both indirect and direct optimization techniques [19, 28]. Additionally, direct transcription techniques have recently also been applied to optimize low-thrust spacecraft trajectories for coverage of the lunar south pole [29, 30].

1.2.3 Variable Specific Impulse Engines

The development of low-thrust propulsion hardware systems has proceeded apace with the application of low-thrust propulsion to expand the options in trajectory design. In 1916, Robert Goddard began a series of experiments focused on ion propulsion [31]. Over 50 years later, Hermann Oberth published the possible fuel mass savings due to the electrostatic propulsion of charged gases [32]. In the 1960’s and 70’s, extensive tests of electric rockets and solar electric propulsion were conducted, with the SERT II flight as the first major in-flight demonstration [33]. A more contemporary example of the successful implementation of electric propulsion is the Deep Space 1 spacecraft [34], launched in October 24, 1998 and logging over 9,241 hours of continuous operation by February 17, 2001 [35]. The European Space Agency’s SMART-1 spacecraft employed a Hall effect thruster to transfer from Earth orbit to a lunar trajectory [36].

Historically, low-thrust electromagnetic propulsion systems possess a single operational engine efficiency value for the given electrical power. However, in recent years, low-thrust engines have emerged that are characterized by a variable specific impulse, independent of operating power. Some designs, such as the VIPER version of the NEXIS thruster proposed by Goebel et al. [37], are modifications of existing tech-
nologies. An alternative approach achieves specific impulses over a range of 200-3,600 seconds using a laser-powered liquid propellant system \[38\]. The Variable Specific Impulse Magnetoplasma Rocket (VASIMR) \[39\] and the Electron and Ion Cyclotron Resonance engine (EICR) \[40\] are electric propulsion systems that generate thrust via accelerated plasma and offer the potential for high-power, rapid transport within the solar system.

1.3 Overview of Present Work

Within this work, a variety of optimal trajectory applications are examined. The body of this investigation is organized as follows:

- Chapter 2: The dynamical motion in a force field dominated by the gravity from multiple bodies is described. The derivation originates with the \(N\)-body problem and reduces the model to the circular restricted three-body problem. The differential equations of motion are summarized with accompanying integrals of motion and equilibrium solutions. Stability characteristics and flow structures associated with the equilibrium points are introduced.

- Chapter 3: A general corrections procedure for targeting solutions is detailed. An algorithm for implementing the shooting scheme is presented, along with numerical techniques that incorporate the state transition matrix as well as finite differencing. The generation of periodic orbits in the vicinity of the libration points is based on a straightforward shooting method. Then, the stability of the periodic orbits is addressed and flow structures related to the orbits are produced.

- Chapter 4: The thrust force from the variable specific impulse engine is added to the gravitational terms to produce a full dynamical system model. Minimization of fuel is formulated as a two-point boundary value problem via the calculus of variations. The corrections procedure yields a general targeting scheme to
generate unconstrained transfers between periodic orbits associated with the libration points.

• Chapter 5: Fuel optimal, low-thrust transfers are computed for a variety of initial and target orbits. These sample transfers incorporate a mix of coast segments in addition to the thrust arcs. The thrust profile and performance of the VSI engine is related to the duration of the thrust arc, and the low-thrust behavior is compared to conventional propulsion systems. The effects of system symmetry in the CR3BP on optimal transfer architectures are detailed.

• Chapter 6: The indirect formulation of the optimization problem is combined with non-linear programming techniques to produce a hybrid optimization algorithm that addresses path constraints directly. A restriction on engine performance is simulated by placing limits on the allowable value of engine specific impulse. The hybrid optimization procedure is tested by bounding the minimum and maximum values of the specific impulse profile.

• Chapter 7: A summary of the investigation and concluding remarks are presented. Potential avenues for future investigation are offered.

With the scope of the investigation defined, the next step is to develop the components of the hybrid optimization scheme. The algorithm is then applied to various mission scenarios to produce both unconstrained and constrained low-thrust trajectories.
2. SYSTEM MODEL

Three essential elements are required to generate fuel optimal spacecraft transfers: a mathematical model incorporating the natural dynamics as well as thrust effects, the desired departure and arrival conditions, and an optimization strategy. Of course, the specific application drives the development of each element. The general focus of this investigation is the design of transfers between periodic orbits in a regime comprised of multiple gravitational bodies. Thus, the mathematical representation of the dynamical model leads directly into the generation of the initial and target states that correspond to departure and arrival. Accordingly, the dynamical model is formulated in terms of the circular restricted three-body problem.

2.1 The N-body Problem

Though humanity has puzzled for millenia on the motion of the stars and planets, Newton’s Law of Universal Gravitation finally supplied a firm physical and mathematical basis for Kepler’s earlier kinematical observations of planetary motion. Indeed, Newton’s Law is fundamental and remains a widely applied concept in preliminary mission design, even though Einstein’s theory of general relativity, a more accurate model of gravity, is now readily available. Using modern vector notation, the model for universal gravitation is expressed

\[ F_g = -\frac{G M m d}{d^3} \]  

(2.1)

where \( F_g \) is the gravitational force due to a point mass \( M \) acting on point mass \( m \), the relative position vector is \( d \) (directed from \( M \) to \( m \)), and \( G \) is the Universal Gravitational Constant. Note that vectors are indicated by bold-type. Also included in Newton’s *Philosophiæ Naturalis Principia Mathematica* [4] are his equally famous
three laws of motion. The well-known second law states that the change in the momentum of a particle is directly proportional to the sum of forces acting on the particle. This relationship is mathematically represented as

$$F = m\ddot{r}$$  \hspace{1cm} (2.2)$$

where $F$ is the vector force resultant, $r$ is the position vector of the particle with respect to an inertially fixed base point, and dots indicate a derivative with respect to time. From the law of motion in Eq. (2.2) and incorporating the gravity force representation in Eq. (2.1), the system is expanded to include $N$ point masses. As a result, the motion of a particle $i$, that is $P_i$, in inertial space is written

$$m_i\ddot{r}_i = -G \sum_{j=1}^{n} \frac{m_im_j}{d_{ji}^3} d_{ji}. \hspace{1cm} (2.3)$$

In Eq. (2.3), the vector $r_i$ denotes the position of particle $i$ with respect to an inertially fixed base point and the vectors $d_{ji}$ denote relative displacements such that $d_{ji} = r_i - r_j$. This system of $N$ particles is represented in Fig. 2.1, with the vectors $r_i$ and $d_{ji}$ annotated.

Figure 2.1. Diagram of $N$-body problem
For centuries, mathematicians have sought solutions to the vector differential equation in Eq. (2.3). Unfortunately, a general closed-form analytical solution is not available. However, in the special case that the system is reduced to only two bodies, then the relative motion is conic and a complete solution is known [41]. Fortunately, such two-body (i.e., Keplerian) solutions are very useful for a wide variety of scenarios. In many instances, the effects of multiple bodies and gravitational harmonics are incorporated as perturbations to Keplerian two-body motion. But, circumstances also exist where a Keplerian model does not suffice and the gravitational influence of multiple bodies must be directly modeled. In compensation for modeling and computational difficulties, these multi-body systems afford solutions that emerge only when more than one gravity field is present.

2.2 Circular Restricted Three-Body Problem

Given the many successful results from applications of the two-body problem, the next step is an expansion of the trajectory options by exploring the motion of three bodies under their mutual gravitational attraction. In 1967, Victor Szebehely authored an in-depth analysis of the Three-Body Problem (3BP) in his book, Theory of Orbits [8]. Much of the problem formulation here follows that in Szebehely.

The model for the general equation of motions (EOM), Eq. (2.3), is initially focused on $N = 3$. So, reducing Eq. (2.3) to three particles and selecting the point $P_3$ as the body of interest results in

\[
\ddot{r}_3 = -G \frac{m_1}{d_{13}^3} d_{13} - G \frac{m_2}{d_{23}^3} d_{23}
\]

(2.4)

where the mass $m_3$ has been factored from both sides of the equation. To solve this equation analytically requires knowledge of the motion of $P_1$ and $P_2$, and without a priori time histories for $r_1$ and $r_2$, the motion for all three bodies must be evaluated simultaneously. Such a complete set of vector differential equations represents 18 scalar first-order differential equations (DE). Thus, 18 integrals are required for a complete closed-form solution. However, only ten integrals of motion are known:
six from conservation of system linear momentum, three from conservation of system angular momentum, and one from conservation of system energy. Since an insufficient number of analytical constants are available, the three-body problem must, of necessity, be investigated primarily through numerical methods. But, a few simplifying assumptions produce much insight without the loss of significant fidelity.

The first simplifying assumption in the 3BP involves the mass of the third particle \( P_3 \); the mass of \( P_3 \), i.e. \( m_3 \), is assumed negligible in comparison to the masses of the other two bodies, \( m_1 \) and \( m_2 \). Therefore, the motion of the massive points \( P_1 \) and \( P_2 \), generally termed the “primaries”, are unaffected by the motion of \( P_3 \). Such an assumption reflects a wide variety of interesting systems. For example, \( P_3 \) might be a spacecraft while \( P_1 \) and \( P_2 \) are more massive bodies forming a two-body system (e.g., Sun/Earth/spacecraft or Jupiter/Io/spacecraft). For convenience, it is further assumed that \( m_1 > m_2 \); then, \( P_1 \) is labeled the larger primary while \( P_2 \) denotes the primary of smaller mass. Since the motions of \( P_1 \) and \( P_2 \) no longer depend on \( P_3 \), they represent a two-body system with a known conic solution. For most systems of interest, this conic is closed and, for simplicity, can be approximated as circular. So, one additional simplification is that \( P_1 \) and \( P_2 \) move in circular orbits around their common barycenter, \( B \). This selection of a circular orbit, in contrast to an elliptical path for the primaries, is not necessary but reasonably represents many cases of interest. One notable representative example is a primary system comprised of the Sun and one of the planets. These simplifications reduce the general three-body problem to the Circular Restricted Three-Body Problem (CR3BP).

The differential equation in Eq. (2.4) specifies the motion of \( P_3 \) relative to an inertially fixed observer, but an inertially fixed view is not always the most appropriate for problem formulation or visualization. An inertial frame, \( I \), is defined with the unit vectors \( \hat{X}, \hat{Y}, \hat{Z} \) forming a right-hand coordinate system with origin at the barycenter of the primaries \( B \). The symbol ‘\(^\wedge\)’ denotes a unit vector. For convenience, frame \( I \) is constructed such that the plane containing the conic motion of the primaries is spanned by the unit vectors \( \hat{X} \) and \( \hat{Y} \). The unit vector \( \hat{Z} \) is then directed along
the orbital angular momentum vector of the primary orbit. A rotating frame, \( R \), is defined relative to the inertial frame by a rotation through an angle \( \theta \) about the \( \hat{Z} \) axis; unit vectors \( \hat{x}, \hat{y}, \hat{z} \) represent this coordinate frame, with \( \hat{Z} \parallel \hat{z} \). The angle \( \theta \) is constrained such that the unit vector \( \hat{x} \) remains directed from \( P_1 \) to \( P_2 \) at all times. Therefore, a coordinate transformation from frame \( R \) to frame \( I \) has the form

\[
\begin{pmatrix}
\hat{X} \\
\hat{Y} \\
\hat{Z}
\end{pmatrix} =
\begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{pmatrix}
\]

(2.5)

where the rotation matrix is labeled the Direction Cosine Matrix (DCM). Because the transformation from \( R \) to \( I \) consists of only a simple rotation about the common \( \hat{z}, \hat{Z} \) axis, the \( XY \)-plane is the same as the \( xy \)-plane. Figure 2.2 supplies a schematic representing the system in the CR3BP, with the rotating and inertial frames and the paths of the primaries illustrated.

Figure 2.2. Schematic representing the circular restricted three-body problem
With the previous simplifications, it is convenient to non-dimensionalize the equations of motion in Eq. (2.4). The fundamental properties of length, mass, and time generate the characteristic quantities. To denote length, define

\[ l^* = r_1 + r_2 \]  

(2.6)

where \( r_i = |r_i| \), the magnitude of the position vector locating the primaries relative to \( B \). Because the primaries move in closed circles, the quantities \( r_1, r_2, \) and \( l^* \) are constant. A second characteristic quantity involves mass, that is, the sum of the primary masses

\[ m^* = m_1 + m_2 \]  

(2.7)

where \( m_1 \) and \( m_2 \) are the dimensional masses of the larger and smaller primaries, respectively. Finally, the characteristic time, \( t^* \), is constructed such that the value of the non-dimensional universal gravitational constant, \( G \), becomes unity, thus

\[ t^* = \sqrt{\frac{(r_1 + r_2)^3}{G(m_1 + m_2)}} = \sqrt{\frac{l^3}{Gm^*}}. \]  

(2.8)

Given the characteristic quantities, some useful non-dimensional parameters are then defined, i.e.

\[ \mu = \frac{m_2}{m^*} = \frac{r_1}{l^*} \]  

(2.9)

\[ 1 - \mu = \frac{m_1}{m^*} = \frac{r_2}{l^*} \]  

(2.10)

\[ r = \frac{r_3}{l^*} \]  

(2.11)

\[ d_i = \frac{d_{3i}}{l^*} \]  

(2.12)

where Eq. (2.9) yields the non-dimensional distance of the larger primary with respect to the system barycenter and Eq. (2.10) defines the non-dimensional distance of \( P_2 \) from the barycenter. The characteristic quantities in the Earth-Moon system appear in Table 2.1. Note that the primary mass ratio \( \mu \) for the Earth-Moon system is relatively high; the \( \mu \) values corresponding to most other CR3BPs in the solar system are at least two orders of magnitude lower (e.g., \( \mu = 0.0009538 \) for the Sun-Jupiter system or \( \mu = 0.0002366 \) for the Saturn-Titan system).
Table 2.1 Characteristic quantities in the Earth-Moon system

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth mass ($M_e$)</td>
<td>$5.972 \times 10^{24}$</td>
<td>kg</td>
</tr>
<tr>
<td>Lunar mass ($M_m$)</td>
<td>$7.346 \times 10^{22}$</td>
<td>kg</td>
</tr>
<tr>
<td>Mass parameter ($\mu$)</td>
<td>0.01215</td>
<td>NA</td>
</tr>
<tr>
<td>Earth moon distance ($r^*$)</td>
<td>384400</td>
<td>km</td>
</tr>
<tr>
<td>Characteristic Time ($t^*$)</td>
<td>$3.752 \times 10^5$</td>
<td>sec</td>
</tr>
<tr>
<td>Characteristic Time ($t_d^*$)</td>
<td>4.342</td>
<td>days</td>
</tr>
</tbody>
</table>

The equations of motion for $P_3$ are then reformulated within the context of a rotating reference frame. Because the characteristic time $t^*$ is also the inverse of the mean motion, $n$, of the primaries in their respective orbits, the angle $\theta$ between the inertial frame and the rotating reference frame is simply the non-dimensional time $t$, thus $\theta = nt = t$. The frame $R$ is centered on the system barycenter, so the vector $\mathbf{r} = (x, y, z)^T$ is easily defined as the non-dimensional displacement vector of particle $P_3$ from the system barycenter, represented in terms of the rotating coordinates. Therefore, the variables in $\mathbf{r}$ are displacements in the directions of the corresponding unit vectors ($\hat{x}, \hat{y}, \hat{z}$), i.e., $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$. Then, the relative displacement vectors

$$d_1 = \begin{pmatrix} x + \mu \\ y \\ z \end{pmatrix} \quad d_2 = \begin{pmatrix} x + \mu - 1 \\ y \\ z \end{pmatrix}$$

reflect the distances of $P_3$ from the two primaries, with corresponding distances

$$d_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}$$

$$d_2 = \sqrt{(x + \mu - 1)^2 + y^2 + z^2}$$
in non-dimensional units. Now, the six-dimensional state vector corresponding to the particle $P_3$, expressed in terms of the rotating reference frame is defined

$$x = \begin{bmatrix} r \\ v \end{bmatrix}$$

(2.16)

where $r$ has been previously defined. The three-element vector $v$ is the velocity of body $P_3$ with respect to the barycenter, as viewed by an observer in $R$, so $v = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$. The equations of motion (EOM), derived by non-dimensionalizing Eq. (2.4), are then written in the form

$$\dot{x} = f(x) = \begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ f_1(r, v) \end{bmatrix}$$

(2.17)

where $f_1$ represents the acceleration of the spacecraft. The scalar elements of $f_1$, in terms of the rotating coordinates, that is

$$f_1 = \begin{bmatrix} 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{d_1^2} - \frac{\mu(x+\mu-1)}{d_2^2} \\ -2\dot{x} + y - \frac{(1-\mu)y}{d_1^2} - \frac{\mu y}{d_2^2} \\ -\frac{(1-\mu)x}{d_1^2} - \frac{\mu z}{d_2^2} \end{bmatrix}$$

(2.18)

are relative to a rotating observer and, thus, include centrifugal and Coriolis terms in addition to the gravitational terms. The acceleration representing these natural dynamics is formulated in terms of a pseudo-potential function, that is

$$U^* = \frac{1 - \mu}{d_1} + \frac{\mu}{d_2} + \frac{1}{2}(x^2 + y^2)$$

(2.19)

containing only position, not velocity, terms. Equation (2.18) can then be expressed

$$f_1 = \begin{bmatrix} 2\dot{y} + \frac{\partial U^*}{\partial x} \\ -2\dot{x} + \frac{\partial U^*}{\partial y} \end{bmatrix}$$

(2.20)

where the dynamical terms containing only position states have been replaced by partials of the pseudo-potential.
The pseudo-potential function is particularly useful. The function $U^*$ supplies a succinct expression for an integral of motion, that is, the Jacobi Constant $C$. To determine an expression for Jacobi Constant, begin with expressions for the non-dimensional kinetic energy

$$K = \frac{1}{2}(\ddot{x}^2 - 2\dot{x}\dot{y} + y^2 + \dot{y}^2 + 2x\dot{y} + x^2 + \dot{z}^2) \quad (2.21)$$

and the non-dimensional gravitational potential energy

$$U = -\frac{1}{d_1} - \frac{\mu}{d_2} \quad (2.22)$$

corresponding to the particle $P_3$. Split the kinetic energy expression into separate terms such that

$$K = K_0 + K_1 + K_2 \quad (2.23a)$$

$$K_0 = \frac{1}{2}(x^2 + y^2) \quad (2.23b)$$

$$K_1 = x\dot{y} - \dot{x}y \quad (2.23c)$$

$$K_2 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \quad (2.23d)$$

Since the potential energy in Eq. (2.22) includes no time derivatives, then, according to Lanczos [42], there exists an integral of motion

$$C_I = K_2 + U - K_0 = \frac{1}{2}(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2) - \frac{1}{d_1} - \frac{\mu}{d_2} - \frac{1}{2}(x^2 + y^2) \quad (2.24)$$

where $C_I$ is a constant of integration. Recall the definition of the pseudo-potential from Eq. (2.19), and this constant is also expressed

$$v^2 = 2U^* - C, \quad (2.25)$$

where $C$ is the Jacobi Constant.

### 2.3 Equilibrium Solutions and Stability

Computation of equilibrium solutions is a sensible first step in the analysis of non-linear dynamical systems such as the CR3BP. Equilibrium solutions are often
available analytically and may be the only particular solutions that are immediately apparent. If the velocity of $P_3$ relative to the rotating frame is set equal to zero, equilibrium points are then determined as the locations where relative acceleration is also zero. Thus, from Eq. (2.20), the partials

$$\frac{\partial U^*}{\partial x} = \frac{\partial U^*}{\partial y} = \frac{\partial U^*}{\partial z} = 0$$

(2.26)
determine the location of any points with zero relative motion. As a matter of notation, the symbol $U^*_q$ indicates a partial of the psuedo-potential with respect to the variable $q$ (e.g., $U^*_x = \frac{\partial U^*}{\partial x}$). Five equilibrium points, typically labelled the libration or Lagrange points, exist in the CR3BP, all lying in the $xy$-plane of primary motion, such that the state vector for these equilibrium solutions is $x_{eq} = (x_i, y_i, 0, 0, 0, 0)^T$ where $x_i$ and $y_i$ are the coordinates of the $i$-th equilibrium point. These five points appear in Fig. 2.3 relative to the Earth-Moon rotating frame, with locations to scale.

Figure 2.3. Libration points in the Earth-Moon system (Earth and Moon size not to scale)
The points $L_4$ and $L_5$ are located analytically such that

\[ x_{4,5} = \frac{1}{2} - \mu \]  
\[ y_{4,5} = \pm \frac{\sqrt{3}}{2} \]

and are commonly termed the equilateral or triangular points since they form equilateral triangles with the two primaries. Unfortunately, the collinear points $L_1$, $L_2$, and $L_3$ do not possess such closed-form solutions and must be determined numerically in an iterative process. Because the collinear points lie on the $x$-axis, the coordinates $y$ and $z$ are equal to zero, while the $x$ coordinates are typically expressed as the functions

\[ x_1 = 1 - \mu - \iota_1 \]
\[ x_2 = 1 - \mu + \iota_2 \]
\[ x_3 = -\mu + \iota_3 \]

where the variables $\iota_i$ are defined as displacements from the nearest primary (Earth for $L_3$ and the Moon for $L_1$ and $L_2$). The precise locations of the collinear points are then determined by solving the following non-linear relationships

\[ f_1(\iota_1) = \iota_1 + \mu - 1 + \frac{1 - \mu}{(1 - \iota_1)^2} - \frac{\mu}{\iota_1^2} = 0 \]  
\[ f_2(\iota_2) = \iota_2 + \mu - 1 - \frac{1 - \mu}{(1 - \iota_2)^2} - \frac{\mu}{\iota_2^2} = 0 \]  
\[ f_3(\iota_3) = \iota_3 + \mu - 1 - \frac{1 - \mu}{\iota_3^2} - \frac{\mu}{(1 - \iota_3)^2} = 0. \]

Given a reasonably accurate initial guess, Newton’s method [43] is used to iteratively solve for $\iota_i$ using the following update equation

\[ \iota_i^{(k+1)} = \iota_i^{(k)} - \frac{f_i(\iota_i^{(k)})}{f_i'(\iota_i^{(k)})} \]  

where $\iota_i^{(k)}$ is the approximation of $\iota_i$ associated with the $k$-th iteration and iteration is terminated when $|\iota_i^{(k+1)} - \iota_i^{(k)}|$ falls below a certain tolerance $\epsilon$. The computed
locations, in both dimensional and non-dimensional units, of the five libration points in the Earth-Moon system are summarized in Table 2.2.

The five libration points are equilibrium solutions and, by definition, possess zero relative motion with respect to the primaries. Thus, they are often proposed as potential sites for communications or observational spacecraft. However, the stability of the equilibrium solutions is a key component in evaluating the station-keeping and transfer costs. Before stability is determined, the motion near an equilibrium solution is formulated in a systematic way. To this end, define a state \( x \) near an equilibrium solution \( x_{eq} \) as

\[
x = x_{eq} + \delta x
\]

(2.35)

where the vector \( \delta x \) is a small perturbation. Given the EOMs in Eq. (2.17), expanding \( f \) in terms of the state \( x \) in a Taylor series yields

\[
f(x) = f(x_{eq} + \delta x) = f(x_{eq}) + \frac{\partial f(x)}{\partial x_{eq}} \delta x + H.O.T.
\]

(2.36)

but since \( \dot{x} = \dot{x}_{eq} + \delta \dot{x} \) and \( f(x) = \dot{x} \), Eq. (2.36) reduces to

\[
\delta \dot{x} = \frac{\partial f(x)}{\partial x_{eq}} \delta x = A_6(t) \delta x + H.O.T.
\]

(2.37)

<table>
<thead>
<tr>
<th>Point</th>
<th>Non-dimensional</th>
<th>Dimensional, km</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_1 )</td>
<td>0.83691531</td>
<td>321710.2452</td>
</tr>
<tr>
<td>( L_2 )</td>
<td>1.15568202</td>
<td>444244.1690</td>
</tr>
<tr>
<td>( L_3 )</td>
<td>-1.00506263</td>
<td>-386346.0751</td>
</tr>
<tr>
<td>( L_4 )</td>
<td>0.487849452</td>
<td>187529.3293</td>
</tr>
<tr>
<td>( L_5 )</td>
<td>0.487849452</td>
<td>-332900.1652</td>
</tr>
</tbody>
</table>
where the higher-order terms converge to zero as \( x \) approaches \( x_{eq} \). The 6 \( \times \) 6 matrix \( A_6(t) \) contains the linearized dynamics relative to the reference

\[
A_6(t) = \left. \frac{\partial f(x_{eq})}{\partial x} \right|_{x=x_{eq}} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
U_{xx}^* & U_{xy}^* & U_{xz}^* & 0 & 2 & 0 \\
U_{xy}^* & U_{yy}^* & U_{yz}^* & -2 & 0 & 0 \\
U_{xz}^* & U_{yz}^* & U_{zz}^* & 0 & 0 & 0
\end{bmatrix}
\] (2.38)

where \( U_{qw}^* \) indicates the second-partial derivatives with respect to the variables \( q \) and \( w \); that is, \( U_{xy}^* = \frac{\partial^2 U^*}{\partial x \partial y} \), etc. Evaluated at the libration points, this matrix is constant since the reference solution \( x_{eq} \) is constant and does not evolve with time.

The stability of the libration points is investigated using Lyapunov stability analysis. A solution is considered stable when it remains bounded in a small region as the system evolves in time. Mathematically, this criterion is equivalent to stating that a solution to Eq. (2.17), \( \psi(t) \), is stable if for any small initial perturbation \( \delta > 0 \) there is a solution \( \phi(t) \) such that

\[
|\phi(t_0) - \psi(t_0)| < \delta
\] (2.39)

that stays bounded with respect to \( \psi(t) \)

\[
|\phi(t) - \psi(t)| < \epsilon
\] (2.40)

for \( t \geq t_0 \) and a small perturbation \( \epsilon > 0 \). Since the equilibrium points are stationary, the solutions \( x_{eq} \) are Lyapunov stable if there is a \( \epsilon > 0 \) for a specified \( \delta > 0 \) such that

\[
|x(t_0) - x_{eq}| < \delta
\] (2.41)

and

\[
|x(t) - x_{eq}| < \epsilon
\] (2.42)

for \( t \geq t_0 \). If this condition is not satisfied, then the point is unstable. Alternatively, if

\[
\lim_{t \to \infty} x(t) = x_{eq}
\] (2.43)
the point is asymptotically stable.

The Lyapunov stability of the libration points is then determined via linear analysis and the constant matrix $A_6$. Define the eigenvalues $\gamma_i$ for $i = 1, 2, \ldots, 6$ with associated eigenvectors $\bar{\gamma}_i$. In general, the eigenvalues are complex with the real part determining the stability of the equilibrium point:

- if $\Re[\gamma_i] < 0$ for all $i$, the point is asymptotically stable,
- if $\Re[\gamma_i] > 0$ for any $i$, the point is unstable,
- if $\Re[\gamma_i] = 0$ for some $i$ and $\Re[\gamma_i] < 0$ for the remaining $i$, then the linear motion is oscillatory, the point is “marginally stable” in the linear system, and the non-linear behavior is indeterminate.

If more than one of these conditions on the real part of the eigenvalues is satisfied for any given $A_6$, then motion near the equilibrium point will exhibit some combination of decaying, growing, and oscillatory terms. As an example from the Earth-Moon system, motion near the collinear points $L_1$, $L_2$, and $L_3$ can exhibit stable, unstable, and oscillatory behavior, since the matrix $A_6$ has one negative real eigenvalue, one positive real eigenvalue, and four purely imaginary eigenvalues associated with each of these locations. In contrast, the Earth-Moon equilateral points possess purely imaginary eigenvalues so the linear motion is marginally stable and the non-linear behavior must be determined using higher-order stability analysis. Szebehely [8] discusses the stability of the triangular points in detail and concludes that the two points are stable for $0 < \mu < 0.03852$. The Earth-Moon libration point stability behavior is typical of most of the solar system CR3BPs because of their relatively low values for the parameter $\mu$.

### 2.4 Invariant Manifolds Associated with the Collinear Equilibrium Points

A geometrical investigation into the flow structures in any region of the phase space in the CR3BP is based on the work of Guckenheimer and Holmes [44], Parker and
Chua [45], and Perko [46]. In addition to determining the stability of the equilibrium points, the eigenvalues $\gamma_i$ of the matrix $A_6$ supply valuable insight into the local motion. The solution to the linear differential equation in Eq. (2.37) is a sum of exponential terms [47]

$$\delta x = \sum_i c_ie^{\gamma_it}$$

(2.44)

where the constants $c_i$ depend on the initial conditions. Then, individual components of a solution grow or decay based upon the real parts of the eigenvalues $\gamma_i$. So, let the matrix $A_6$ have $n_u$ associated eigenvalues with a positive real part, $n_c$ eigenvalues with real components equal to zero, and $n_s$ eigenvalues that contain only negative real terms. All eigenvalues satisfy one of these three conditions, thus, for an $n$-dimensional space, $n = n_s + n_c + n_u$. The eigenvectors corresponding to these eigenvalues, because of their linear independence, span the full space $\mathbb{R}^n$. The eigenspaces containing these eigenvectors of $A_6$ are defined $E^S$, $E^C$, and $E^U$ in correspondence to the stable, center, and unstable spectra of $A_6$, that is

$$\mathbb{R}[^\gamma_i] \begin{cases} 
< 0 & \gamma_i \in n_s \\
= 0 & \gamma_i \in n_c \\
> 0 & \gamma_i \in n_u.
\end{cases}$$

(2.45)

The subspaces $E^S$, $E^C$, and $E^U$ are denoted “invariant” because a point contained in one of the subspaces at an initial time $t_0$ remains in that subspace as time progresses.

Then, consistent with the Center Manifold Theorem [44], there exist stable $W^S_{loc}$, center $W^C_{loc}$, and unstable $W^U_{loc}$ local invariant manifolds tangent to the respective subspaces $E^S$, $E^C$, and $E^U$. These manifolds are invariant in the flow $f$, so a particle on the manifold will remain on the manifold for all time. The invariant manifolds $W^S_{loc}$ and $W^U_{loc}$ are unique while $W^C_{loc}$ is not necessarily so. However, the existence of a center invariant manifold at an equilibrium point indicates that the vicinity is rich with possibilities for periodic motion.

To generate the stable and unstable invariant manifolds $W^S_{loc}$ and $W^U_{loc}$, initial conditions obtained from the local linear behavior near the equilibrium point are
numerically propagated. As already noted, the collinear libration points have one stable and one unstable eigenvalue, so the stable and unstable eigenspaces are each one-dimensional. The eigenvectors composing the subspaces \( E^S \) and \( E^U \) are defined

\[
\tilde{\gamma}^s = \left( x_s \ y_s \ z_s \ \dot{x}_s \ \dot{y}_s \ \dot{z}_s \right)^T \\
\tilde{\gamma}^u = \left( x_u \ y_u \ z_u \ \dot{x}_u \ \dot{y}_u \ \dot{z}_u \right)^T
\]

(2.46)

where \( \tilde{\gamma}^s \) is the eigenvector defining the stable subspace and \( \tilde{\gamma}^u \) composes the unstable subspace. The eigenvectors are then normalized to

\[
\hat{\gamma}^s = \frac{\tilde{\gamma}^s}{|\tilde{\gamma}^s|} \\
\hat{\gamma}^u = \frac{\tilde{\gamma}^u}{|\tilde{\gamma}^u|}
\]

(2.47)

Stepping from the equilibrium solution in the direction of the normalized eigenvectors produces the required initial condition for numerical propagation; however, to generate the full manifold, steps in the eigenspace must be in both directions from \( x_{eq} \). Accordingly, the initial conditions for the stable and unstable invariant manifolds are generated from

\[
x_s = x_{eq} \pm \epsilon \hat{\gamma}^s \\
x_u = x_{eq} \pm \epsilon \hat{\gamma}^u
\]

(2.48)

where \( \epsilon \) is a scaling factor that ensures the initial condition is within the region where the linear approximation remains valid. The initial conditions are then propagated in the non-linear CR3BP: in forward time, the unstable manifold departs and the stable manifold asymptotically approaches the equilibrium point, while the opposite behavior occurs in reverse time. In Figure 2.4, the stable and unstable manifolds originating near the collinear \( L_1 \) and \( L_2 \) points are plotted in the Earth-Moon system. The libration points are marked with black diamonds. The solid manifolds originate from \( L_1 \) and the dashed manifolds are associated with \( L_2 \). The black, solid arrows are the directions of the linearized system eigenvectors, while the black circle in the center is the location of the Moon.

The mathematical description of the dynamical motion of a particle of negligible mass in a gravitational field generated by two massive bodies in now complete. The
Figure 2.4. Stable (green) and unstable (red) manifolds emanating from the regions near the Earth-Moon $L_1$ and $L_2$ points

Earth-Moon gravitational field is approximated as a CR3BP with the five equilibrium solutions identified. An investigation of the motion near the $L_1$ and $L_2$ collinear points indicates the existence of flow structures such as invariant manifolds and periodic orbits that offer possibilities for mission design. The next logical step is to generate these periodic orbits and integrate them into mission design.
3. DIFFERENTIAL CORRECTIONS METHODS

Spacecraft mission design requires the determination of trajectories that meet specified criteria as well as transfers that arrive at and depart from such orbits. In the two-body problem, this design process consists of selecting conic orbits that satisfy particular mission objectives or constraints. Transfers are designed analytically using conic solutions as intermediate transfer orbits and impulsive maneuvers allow shifts from one arc to the next. In contrast, the CR3BP and ephemeris models necessitate more extensive use of numerical techniques to design orbits and transfers; thus, the solutions do not possess analytical representations. Therefore, a general differential corrections procedure based on Newton’s method and exploiting dynamical sensitivities is developed and then employed to create periodic families of orbits. This procedure is also the basis for computing optimal low-thrust transfers that are applicable in any model. Because these orbit transfers incorporate intervals when the engine is off and the spacecraft is allowed to coast, dynamical structures that reflect the natural flow associated with periodic orbits are also key design components.

3.1 A General Corrections Procedure

The determination of orbits and transfers within a multi-body regime is a natural example of a two-point boundary value problem (2PBVP). Though many techniques for numerically computing solutions to 2PBVPs exist, an algorithm comprised of constraints and free variables [21], based on a generalization of Newton’s method, is employed here because of its simplicity in formulation and a recent history of
successful applications [48], [19]. Begin with a general design vector $\mathbf{X}$ of $n$ free variables $X_i$, that is, a vector of design variables,

$$
\mathbf{X} = \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix} \quad (3.1)
$$

where the objective is to determine a solution vector that satisfies a set of $m$ constraints of the form

$$
\mathbf{F}(\mathbf{X}) = \begin{bmatrix}
F_1(\mathbf{X}) \\
F_2(\mathbf{X}) \\
\vdots \\
F_m(\mathbf{X})
\end{bmatrix} = \mathbf{0}. \quad (3.2)
$$

One possible set of design variables that is applicable in this problem is position, velocity, and time while potential constraints include path restrictions on position and velocity as well as limits on the energy level of an orbit. When the constraint vector $\mathbf{F}(\mathbf{X})$ includes quantities downstream from $\mathbf{X}$ as a result of numerical simulation, the algorithm can be used to implement a shooting process [49].

After the free variables are selected and the constraints are defined, an algorithm is developed that iterates from an initial guess of the solution, $\mathbf{X}_0$, toward a true solution. To construct this algorithm, the constraint vector $\mathbf{F}(\mathbf{X})$ is expanded to first order in a Taylor series about the initial guess $\mathbf{X}_0$

$$
\mathbf{F}(\mathbf{X}) \approx \mathbf{F}(\mathbf{X}_0) + \mathbf{D}\mathbf{F}(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0) \quad (3.3)
$$

where $\mathbf{X}$ is assumed close to $\mathbf{X}_0$. The Jacobian matrix $\mathbf{D}\mathbf{F}(\mathbf{X})$ is an $m \times n$ gradient matrix

$$
\mathbf{D}\mathbf{F}(\mathbf{X}) = \frac{\partial \mathbf{F}(\mathbf{X})}{\partial \mathbf{X}} = \begin{bmatrix}
\frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \cdots & \frac{\partial F_1}{\partial X_n} \\
\frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \cdots & \frac{\partial F_2}{\partial X_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_m}{\partial X_1} & \frac{\partial F_m}{\partial X_2} & \cdots & \frac{\partial F_m}{\partial X_n}
\end{bmatrix} \quad (3.4)
$$
that relates changes in the constraint vector \( F(X) \) to changes in the state vector \( X \). Recalling the goal to satisfy the constraints such that \( F(X) = 0 \), Eq. (3.3) is reformulated as an iterative function, that is

\[
F(X_i) + DF(X_i)(X_{i+1} - X_i) = 0
\]  

(3.5)

and \( X_{i+1} \) is computed until the constraint vector falls below a certain specified tolerance \( \| F(X_{i+1}) \| < \epsilon \). The norm of the full constraint vector may not always yield the best performance and such criteria can be adjusted as necessary. When the number of constraints and free variables are equal, \( m = n \), then the update process is straightforward and the multi-dimensional Newton’s method [49] is employed so that at each step

\[
X_{i+1} = X_i - DF(X_i)^{-1}F(X_i)
\]  

(3.6)

where the Jacobian matrix \( DF(X_i) \) is square and, so long as it is non-singular, invertible. If \( n > m \), in practice there are infinitely many solutions, so a single solution is determined from among many options by computing the vector \( X_{i+1} \) nearest to \( X_i \). This Minimum-Norm solution [50] is evaluated

\[
X_{i+1} = X_i - DF(X_i)^T\left[ DF(X_i)DF(X_i)^T \right]^{-1}F(X_i)
\]  

(3.7)

where Eq. (3.7) uses an orthogonal projection of \( DF(X_i) \) to compute \( X_{i+1} \) and, thus, the solution \( X_{i+1} \) typically retains the properties of the previous solution \( X_i \).

Differential corrections is a powerful tool in orbit and transfer design and the implementation of the process in terms of a general constraint-variable corrections procedure is the primary algorithm used in this analysis. The general algorithm is summarized as follows:

1. Determine all free variables and all functional dependencies, construct the variable vector \( X \), and generate an initial guess \( X_0 \).

2. Define all constraints in the problem, formulate them in terms of equality constraints, and fill the constraint vector \( F(X) = 0 \).
3. Calculate the Jacobian matrix \( DF(X) = \frac{\partial F(X)}{\partial X} \), that is, the gradient of the constraints with respect to the free variables.

4. Use Newton’s method (Eq. (3.6)), a Minimum-Norm solution (Eq. (3.7)), or an acceptable alternative, to iterate until the constraint falls below an acceptable tolerance, \( \| F(X_{i+1}) \| < \epsilon \).

Assuming all constraints are formulated as equality constraints and the initial guess \( X_0 \) is sufficiently close to a true solution, the algorithm presented converges quadratically [49]. The constraint-variable procedure is used to generate all orbits and unconstrained low-thrust transfers in this analysis.

### 3.1.1 Variational Equations and the State Transition Matrix

For applications in the CR3BP, the corrections algorithm includes the evaluation of the elements of a constraint vector from states obtained via numerical integration. The update equation is then based on a Jacobian matrix which, for its construction, requires some method to map variations in the initial states to variations in the final states. The first step in determining these variational relationships is a baseline solution \( \mathbf{x}^*(t) \) to the equations of motion (the symbol * here indicates an association with a reference trajectory). For non-linear systems, determination of a baseline solution entails numerical integration of a set of \( n \) first-order ordinary differential equations (ODE). Thus, for the CR3BP, the equations of motion \( \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \) are propagated for some time interval \([t_0, t]\) from an initial state \( \mathbf{x}^*_0 \) to obtain \( \mathbf{x}^*(\mathbf{x}^*_0, t) \).

Several ODE solvers are available, for example, MATLAB®’s \texttt{ode45} and \texttt{ode113} functions or Adams-Bashforth-Moulton integrators in Fortran [51].

With the known trajectory \( \mathbf{x}^*(\mathbf{x}^*_0, t) \), nearby solutions are generated by employing a continuation process. Variations between the final state along the known trajectory and the positions and velocity vectors corresponding to the nearby solutions are de-
terminated by approximating the effects of variations in the initial state on $x^*(x^*_0, t)$. To model a nearby path, define a new initial state

$$ x_0 = x^*_0 + \delta x_0 $$

(3.8)

where $\delta x_0$ is a small variation. The variation at the terminal time $t$ is then

$$ \delta x(t) = x(x_0, t) - x^*(x^*_0, t) = x(x^*_0 + \delta x_0, t) - x^*(x^*_0, t). $$

(3.9)

The process of representing a neighboring arc relative to a reference trajectory is illustrated in Fig. 3.1. Expanding the term $x(x^*_0 + \delta x_0, t)$ and linearizing about the reference trajectory for an assumed small initial variation yields

$$ \delta x(t) = \frac{\partial x}{\partial x_0} \delta x_0 $$

(3.10)

as in Kahlil [52]. The matrix derivative $\frac{\partial x}{\partial x_0}$ is evaluated on the reference path and linearly maps variations in the initial state $x_0$ to changes in the state downstream, $x(x_0, t)$. This transformation matrix is denoted the State Transition Matrix (STM) and is defined

$$ \Phi(t, t_0) = \frac{\partial x}{\partial x_0}. $$

(3.11)
When \( t = t_0 \) and thus \( x = x_0 \), Eq. (3.11) indicates that \( \Phi(t_0, t_0) = I \), that is the initial STM, is the identity matrix. Differentiating Eq. (3.10) with respect to time produces

\[
\frac{d}{dt} \left( \frac{\partial x}{\partial x_0} \right) = \frac{\partial}{\partial x_0} \frac{dx}{dt} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial x_0} = \frac{\partial \dot{x}}{\partial x} \frac{\partial x}{\partial x_0} \tag{3.12}
\]

and, recall from Eq. (2.38), that \( \frac{\partial \dot{x}}{\partial x} = A(t) \). Substituting Eq. (3.11) into Eq. (3.12) produces the matrix differential equation

\[
\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0). \tag{3.13}
\]

Thus, a system with \( n \) state variables requires the integration of \( n^2 \) additional differential equations for the availability of the STM. However, if the system is time-invariant, the matrix \( A \) is constant, the analysis simplifies and the STM is evaluated as \( \Phi(t) = e^{At} \) [53].

### 3.1.2 Finite Differencing

In some instances, numerical simulations using the STM to evaluate the elements of the Jacobian matrix are not productive, prohibitively difficult in a large-scale problem, or computationally expensive. Therefore, an alternative approach to calculate these values is useful. Numerical finite-differencing is common when a slight loss in accuracy is offset by an increase in computational efficiency which is acceptable. Central differencing is the most popular of the finite-differencing options because of the balance between accuracy and the number of function evaluations as well as the ease of implementation [49]. In central differencing, the derivative of the \( j^{\text{th}} \) scalar constraint \( F_j \) with respect to the \( i^{\text{th}} \) state variable \( X_i \) is approximated as

\[
\frac{\partial F_j}{\partial X_i} \approx \frac{F_j(X_i + h) - F_j(X_i - h)}{2h} \tag{3.14}
\]

where \( h \) is a “small” step. In this investigation, a step size of \( h = 10^{-5} \) supplies an acceptable trade-off between truncation error and round-off error.
3.1.3 Single and Multiple Shooting

In astrodynamics trajectory design problems, targeting or shooting methods are common. A shooting scheme typically involves numerical integration along one or more arcs; the initial state along each arc is generally corrected to achieve a specified objective. Any shooting procedure can be implemented in a corrections algorithm. The general corrections procedure used in this investigation does not place restrictions on the selection of either the variables in the state vector $X$ or the constraints in the vector $F(X)$. However, some terminology to distinguish between various applications of the process is helpful. Consider the vector $X = \{x_0, t\}^T$ that includes the initial state $x_0$ and the propagation time $t$ along a trajectory. Then, the arc possesses a target terminal point, $x_T$, such that the constraint is expressed $F(X) = x_t - x_T$. All free variables are associated with a solitary initial state and time and all constraints are determined by a propagation from that specified initial state for the predetermined time interval $t$ (denoted by the superscript $t$). This application of the corrections algorithm is labeled “single shooting”; a diagram appears in Fig. 3.2.

![Diagram of single shooting](image-url)
The general corrections procedure is not restricted to single states and one time interval to define a trajectory; in fact, segmenting a single arc into several sub-arcs offers many advantages. For example, an end-to-end trajectory might require one or many close passages of a primary, necessitating a complex path with high numerical sensitivity. Then, the linear corrections procedure is more successfully applied to shorter arcs that each represent a small time interval along the entire trajectory. Corrections of the state and, possibly time corresponding to each arc, are evaluated simultaneously using a shooting method. Simultaneously updating the interior points is denoted “multiple shooting” and the procedure can be implemented using a free variable-constraint algorithm. The scheme is implemented in this analysis by adding the interior points to the design vector $X$. For example, collect a set of $n$ points $x_i$ that represent a sequential series of arcs and define the initial states corresponding to each arc to seed $n$ numerical propagations of possibly varying duration $t_i$. The goal is the construction of a smooth trajectory connecting all $n$ points in series to a terminal point $x_T$; this case is illustrated in Fig. 3.3. Then, the constraint vector $F(X)$ includes elements of the form $x_i - x_{i-1}^t$ that ensure continuity between the terminal point of one arc and the initial point of the following arc. Thus, the algorithm corrects a state vector $X$ including the $n$ state vectors $x_i$ and, if desired, the propagation times $t_i$.

3.2 Generating Families of Periodic Orbits

An infinite number of periodic orbits exist in the vicinity of the equilibrium solutions, that is, the libration points, in the CR3BP. Certain periodic orbits near the collinear libration points, namely planar Lyapunov orbits and three-dimensional halo orbits, are determined via an exploitation of the CR3BP symmetry across the $xz$-plane. From a perpendicular crossing of the $x$-axis, another perpendicular $x$-axis crossing is targeted precisely one half-period later. Because of the CR3BP symmetry, a full periodic orbit is mirrored acrossed the $xz$-plane [8]. Families of planar Lyapunov
Figure 3.3. Illustration of a series of arcs in a multiple shooting scheme
punov orbits are generated in the vicinity of the $L_1$ and $L_2$ points and then used to develop families of three-dimensional halo orbits.

### 3.2.1 Planar Lyapunov Orbit Families

The general corrections procedure from Section 3.1 is applied in the CR3BP to compute the planar Lyapunov periodic orbits. The first step in developing the targeter is the selection of the elements that comprise the design vector $\mathbf{X}$. Recall that the orbits of interest in this example are planar. In this case, the given initial state is a perpendicular $x$-axis crossing with a specified $x$ position, so the variables that are available for adjustment include the initial velocity component in the $y$-direction as well as the time corresponding to a half-period such that $\mathbf{X} = [\dot{y}_0, t_e]^T$ where $t_e$ is half the period of the full orbit. The target point is the next perpendicular crossing, thus, at the half period, the state vector should appear in the form $\mathbf{x} = (x, 0, 0, 0, \dot{y}, 0)^T$.

Thus, if the crossing is perpendicular, $y = \dot{x} = 0$, and the constraint vector is $\mathbf{F}(\mathbf{X}) = [y, \dot{x}]^T = 0$. The elements of $D\mathbf{F}(\mathbf{X})$ corresponding to the time variable are simply the second and fourth EOM from Eq. (2.17) evaluated at the end of the computed trajectory. In contrast, the elements of $D\mathbf{F}(\mathbf{X})$ that are related to $\dot{y}_0$ are determined from the variational relationships. Upon examining the STM of the CR3BP

\[
\begin{bmatrix}
\delta x \\
\delta y \\
\delta z \\
\delta \dot{x} \\
\delta \dot{y} \\
\delta \dot{z}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} & \frac{\partial x}{\partial \dot{x}_0} & \frac{\partial x}{\partial \dot{y}_0} \\
\frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} & \frac{\partial y}{\partial \dot{x}_0} & \frac{\partial y}{\partial \dot{y}_0} \\
\frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} & \frac{\partial z}{\partial \dot{x}_0} & \frac{\partial z}{\partial \dot{y}_0} \\
\frac{\partial \dot{x}}{\partial x_0} & \frac{\partial \dot{x}}{\partial y_0} & \frac{\partial \dot{x}}{\partial z_0} & \frac{\partial \dot{x}}{\partial \dot{x}_0} & \frac{\partial \dot{x}}{\partial \dot{y}_0} \\
\frac{\partial \dot{y}}{\partial x_0} & \frac{\partial \dot{y}}{\partial y_0} & \frac{\partial \dot{y}}{\partial z_0} & \frac{\partial \dot{y}}{\partial \dot{x}_0} & \frac{\partial \dot{y}}{\partial \dot{y}_0} \\
\frac{\partial \dot{z}}{\partial x_0} & \frac{\partial \dot{z}}{\partial y_0} & \frac{\partial \dot{z}}{\partial z_0} & \frac{\partial \dot{z}}{\partial \dot{x}_0} & \frac{\partial \dot{z}}{\partial \dot{y}_0}
\end{bmatrix} \begin{bmatrix}
\delta x_0 \\
\delta y_0 \\
\delta z_0 \\
\delta \dot{x}_0 \\
\delta \dot{y}_0 \\
\delta \dot{z}_0
\end{bmatrix}
\]

\[(3.15)\]

the appropriate selection for $D\mathbf{F}(\mathbf{X})$ is located in the second and fourth rows of the fifth column of $\Phi(t, t_0)$, $\frac{\partial y}{\partial y_0}$ and $\frac{\partial \dot{z}}{\partial y_0}$. The update equation is then based on

\[
\begin{bmatrix}
y \\
\dot{y}
\end{bmatrix}_j = \begin{bmatrix}
\frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial \dot{y}_0} \\
\frac{\partial \dot{y}}{\partial y_0} & \frac{\partial \dot{y}}{\partial \dot{y}_0}
\end{bmatrix} \begin{bmatrix}
\dot{y}_0 \\
t_e
\end{bmatrix}_j + \begin{bmatrix}
\dot{y}_0 \\
t_e
\end{bmatrix}_{j+1}
\]

\[(3.16)\]
where the subscript $j$ is the iteration number in the corrections algorithm. Initial guesses for the state vector $X_0$ corresponding to the planar Lyapunov orbits are originally produced from the well-known elliptical approximations of Szebehely [8]. Because there are two free variables and two constraints, Newton’s method from Eq. (3.6) is employed in the iteration process to compute the periodic orbit. A planar periodic orbit near $L_1$ is a good originating orbit. The targeted planar periodic Lyapunov orbit in the Earth-Moon system that is offset from the $L_1$ point by $-3844$ km appears in Fig. 3.4. The actual period of the targeted non-linear orbit is $P = 11.799$ days.

![Figure 3.4. Planar periodic orbit near $L_1$; green arrow indicates initial velocity direction](image)

Given one periodic orbit, a family of planar Lyapunov orbits is created with a natural parameter continuation process. Since $x_0$ is pre-determined, step along the $x$-axis increasingly further from the equilibrium point, using the iterative equation in Eq. (3.16) to converge to a new periodic orbit after each step. The initial velocity from the previous non-linear orbit in the family is employed as the initial guess in the targeting algorithm for the next orbit. Members of the Earth-Moon planar Lyapunov
families about $L_1$ (blue) and $L_2$ (red) are plotted in Fig. 3.5, where the green arrows indicate the direction of motion around the periodic orbit. The orbits in black signify a change in the stability properties of the planar Lyapunov orbits, to be discussed in Section 3.3; this change in stability also highlights the bifurcation from the planar Lyapunov family to the three-dimensional halo orbit family [48].

Figure 3.5. Earth-Moon $L_1$ (blue) and $L_2$ (red) planar Lyapunov orbits

3.2.2 Three-Dimensional Halo Orbits

The planar example that involves targeting periodic orbits is extended to generate periodic orbits in the full three-dimensional spatial coordinates of the CR3BP. To determine three-dimensional periodic orbits from the planar-to-halo bifurcation orbit in Fig. 3.5, perpendicular crossings of the $xz$-plane are targeted from a state with one fixed position state and three free variables. The members of the halo family of periodic orbits are targeted using three of the four design variables $\mathbf{X} = [x_0, z_0, \dot{y}_0, t_e]^T$ with either $x_0$ or $z_0$ specified for a particular orbit. A third constraint on $\dot{z}$ is added to ensure a perpendicular crossing of the $xz$-plane, so the constraint vector becomes
\( \mathbf{F}(\mathbf{X}) = [y, \dot{x}, \dot{z}]^T = 0 \). The gradient matrix \( \mathbf{D}\mathbf{F}(\mathbf{X}) \) is then evaluated with appropriate elements from the EOMs Eq. (2.17) or the STM Eq. (3.15). The update equations that are employed in the iteration process to target periodic halo orbits with the general corrections procedure are based on the relationship between the initial and final states, thus, for fixed \( x_0 \)

\[
\begin{align*}
\begin{cases}
  y \\
  \dot{x} \\
  \dot{z}
\end{cases}
  &=
\begin{bmatrix}
  \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \dot{y} \\
  \frac{\partial \dot{z}}{\partial x_0} & \frac{\partial \dot{z}}{\partial y_0} & \ddot{z} \\
  \frac{\partial \dot{z}}{\partial x_0} & \frac{\partial \dot{z}}{\partial y_0} & \ddot{z}
\end{bmatrix}
\begin{cases}
  z_0 \\
  \dot{y}_0 \\
  t_e
\end{cases}
  -
\begin{cases}
  z_0 \\
  \dot{y}_0 \\
  t_e
\end{cases},
\end{align*}
\]

(3.17)

or, for specified \( z_0 \)

\[
\begin{align*}
\begin{cases}
  y \\
  \dot{x} \\
  \dot{z}
\end{cases}
  &=
\begin{bmatrix}
  \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \dot{y} \\
  \frac{\partial \dot{z}}{\partial x_0} & \frac{\partial \dot{z}}{\partial y_0} & \ddot{z} \\
  \frac{\partial \dot{z}}{\partial x_0} & \frac{\partial \dot{z}}{\partial y_0} & \ddot{z}
\end{bmatrix}
\begin{cases}
  x_0 \\
  \dot{y}_0 \\
  t_e
\end{cases}
  -
\begin{cases}
  x_0 \\
  \dot{y}_0 \\
  t_e
\end{cases}.
\end{align*}
\]

(3.18)

Once again, families of orbits are produced via natural parameter continuation by either stepping in \( x_0 \) and using the update equation Eq. (3.17) or fixing \( z_0 \) and implementing Eq. (3.18). A number of trajectories in the northern halo orbit families near the Earth-Moon \( L_1 \) and \( L_2 \) libration points appear in Fig. 3.6. The colors in the plots indicate the value of the Jacobi constant that corresponds to each orbit. The term “northern” reflects the fact that the periodic orbit family initial conditions are constructed by using positive steps in \( z \), that is, above the fundamental plane. A “southern” family is generated by stepping in the negative direction for initial \( z \) value. See Grebow [48] for a more complete discussion of periodic orbit family generation, family bifurcations, and the types of periodic orbits available in the CR3BP.

### 3.3 The Stability of Periodic Solutions

Similar to the examination of the libration points, periodic orbits can also be analyzed in terms of their stability. Unfortunately, the Lyapunov criteria of bounded motion does not translate well to a study of periodic motion, so a new stability
condition based on Floquet theory is introduced. Assume that a periodic solution \( p(t) \) with period \( P \) is known. Recall the linearization of the system dynamics relative to a reference solution in Eq. (2.20) resulting in the matrix \( A_6(t) \) as defined in Eq. (2.38). In the case of an equilibrium point, the matrix \( A_6 \) is constant. If the reference solution is a periodic orbit, i.e. \( p(t) \), then \( A_6(t) \) is time-varying. In fact, the time-varying matrix \( A_6(t) \) is then periodic with period \( P \) when evaluated on the solution \( p(t) \). From Floquet’s theorem, the linear system \( \dot{x} = A_6(t)x \) has a minimum of one non-trivial solution \( \phi(t) \) that satisfies

\[
\phi(t + P) = \gamma \phi(t)
\]

where \( \gamma \) is a constant \([54]\). The constant \( \gamma \) specifies the rate of growth or decay of the solution \( \phi(t) \) as time progresses.

To complete a stability analysis of a periodic solution \( p(t) \) via the associated linear system \( \dot{x} = A_6(t)x \), the elements of Eq. (3.19) must be determined. The

Figure 3.6. Earth-Moon \( L_1 \) and \( L_2 \) Northern Families of Halo Orbits
matrix differential equation Eq. (3.13) governing the state transition matrix depends on $A_6(t)$. For a periodic orbit, the analog of an equilibrium point is a “fixed point”. Selecting any point along a periodic orbit as the fixed point, the state and STM are propagated for exactly one period. The STM evaluated after precisely a full period is denoted the monodromy matrix. From Floquet’s theorem, stability of the periodic orbit is assessed from examination of the phase space associated with the fixed point and the eigenvalues of the monodromy matrix. These eigenvalues correspond to $\gamma$ in Eq. (3.19) and, thus, the stability of the periodic orbit [46]. So, using the eigenvalues $\gamma_i$ of the monodromy matrix as the constant term in Eq. (3.19), Floquet’s theorem demonstrates that the constants $\gamma_i$ are characteristic multipliers corresponding to the linear system $\dot{x} = A_6(t)x$ [54]. The eigenvalues then, in a linear sense, predict whether a nearby solution diverges from, asymptotically approaches, or simply remains near the periodic trajectory. However, since the linear stability of periodic solutions is determined as though the evolution of the system occurs in discrete time intervals of duration $P$, the conditions for stability are slightly different from the continuous time analysis applied to the equilibrium points in Section 2.3. From Parker and Chua [45], if the magnitude of any eigenvalue of the monodromy matrix is greater than unity, then the periodic orbit is unstable. Alternatively, if the magnitude of an associated eigenvalue is less than one, then the corresponding eigenvector of the monodromy matrix asymptotically approaches the orbit as time progresses. Finally, for a real or complex eigenvalue that possesses unit magnitude, the corresponding eigenvector remains in a bounded region near the periodic orbit. Since the CR3BP is Hamiltonian and the reference orbit is periodic, all of the eigenvalues appear in reciprocal pairs. Thus, if the periodic solution has an associated asymptotically stable eigenvalue, then the monodromy matrix also contains an unstable eigenvalue [55]. Therefore, a periodic orbit is only “stable” if all the associated eigenvalues of the monodromy matrix are of unit magnitude.
3.4 Invariant Manifolds Associated with Periodic Orbits

Periodic orbits serve as the originating trajectory as well as the target destinations in this investigation. However, significant fuel savings are achieved by incorporating coasts along structures associated with the orbits that result from the natural system dynamics. The invariant flow structures, i.e., manifolds, allow asymptotic departure from and arrival at the periodic orbits. As is the case with equilibrium solutions, there is a center manifold theory for invariant flow structures associated with periodic orbits [44,45]. The eigenvalues of the monodromy matrix are again sorted consistent with their stability properties, that is

\[
\begin{align*}
|\gamma_i| &= 1 & \gamma_i \in n_c \\
< 1 & \gamma_i \in n_s \\
> 1 & \gamma_i \in n_u
\end{align*}
\]

(3.20)

where the collection of \(n_s\), \(n_c\), \(n_u\) eigenvalues and the companion eigenvectors define the linear stable \(E^S\), center \(E^C\), and unstable \(E^U\) subspaces, respectively. The subspaces are associated with the local invariant manifolds, defined as stable \(W^S_{loc}\), center \(W^C_{loc}\), and unstable \(W^U_{loc}\) manifolds. Again, the center subspace is not unique. Periodic orbits that exist in the CR3BP possess a wide range of stability characteristics, from orbits with only a center subspace to orbits with a neighboring phase space that includes multi-dimensional stable and unstable subspaces [48,56]. As a matter of convention, for the periodic orbits in this analysis, the terms “stable manifold” and “unstable manifold” refer to the invariant manifolds corresponding to the most stable (i.e., lowest magnitude) and most unstable (largest magnitude) real eigenvalues respectively.

Recall, from Section 2.4, that the eigenvectors of the variational matrix are employed to construct an initial condition for numerical integration to compute a manifold trajectory in the nonlinear system. In contrast, for a periodic orbit, any point along the orbit can serve as a fixed point. The eigenvalues of the corresponding monodromy matrix \(\Phi(t_0 + P, t_0)\) are independent of the value selected for the initial time.
$t_0$, but the corresponding eigenvectors do vary with the associated location along the periodic orbit. It is convenient to identify a location along the reference periodic orbit relative to a marker. Therefore, introduce a time parameter $\tau$ that identifies the state along the periodic orbit, such that $\tau = 0$ corresponds to the state along the orbit at the initial $xz$-plane crossing (denoted $x_0$) and $\tau = P$ corresponds to one full period of the orbit (note that $x(x_0, 0) = x(x_0, P)$). Recall that the six-dimensional state vector $x$ contains the three spatial and three velocity coordinates of the CR3BP. To simplify notation, a function with the single independent variable $\tau$ is associated with the specified location along a periodic orbit. Accordingly, a state vector along the orbit, $x(\tau)$, is evaluated by propagating forward for a duration $\tau$, given an initial condition on the orbit, i.e., $x_0$. Then, the monodromy matrix associated with a specific value of $\tau$, $\Phi(\tau + P, \tau)$, is computed from a numerical integration of duration $P$ from the state $x(\tau)$. However, rather than recomputing the full monodromy matrix for each value of $\tau$, a similarity transformation preserves the eigenvalues of the monodromy matrix while the eigenvectors at $\tau$ are computed via a matrix multiplication [47]. Once the eigenvectors $\tilde{\gamma}(0)$ of the monodromy matrix $\Phi(P, 0)$ are determined, an eigenvector at any point downstream is computed from

$$\tilde{\gamma}(\tau) = \Phi(\tau, 0)\tilde{\gamma}(0)$$  \hspace{1cm} (3.21)

where the only requirement is the STM from the initial point to the desired location along the orbit.

As in Section 2.4, the eigenvector is used to define an initial condition for propagation along the manifold. The eigenvector supplies the appropriate step direction, but the actual step must be of sufficiently small magnitude to preserve the characteristics of the associated eigenspace. So, after the eigenvector is determined, it is again normalized by its full magnitude, consistent with Eqs. (2.47), such that

$$\hat{v}^s(\tau) = \frac{\tilde{\gamma}^s(\tau)}{|\tilde{\gamma}^s(\tau)|}$$

$$\hat{v}^u(\tau) = \frac{\tilde{\gamma}^u(\tau)}{|\tilde{\gamma}^u(\tau)|}.$$  \hspace{1cm} (3.22)
Then, the perturbation step off the orbit that defines the initial state for the manifold arc, is constructed

\[ \mathbf{x}_s(\tau) = \mathbf{x}(\tau) \pm \epsilon \hat{v}^s(\tau) \]
\[ \mathbf{x}_u(\tau) = \mathbf{x}(\tau) \pm \epsilon \hat{v}^u(\tau) \]  

(3.23)

for the stable and unstable invariant manifolds, respectively. Recall that functions of the independent variable \( \tau \) are associated with the periodic orbit, so the initial conditions \( \mathbf{x}_s(\tau) \) and \( \mathbf{x}_u(\tau) \) that generate the manifold trajectories are uniquely specified given \( \tau \). The manifold arc is propagated for a non-dimensional time duration \( \alpha \), such that the parameter pair \( (\tau, \alpha) \) defines a point \( \mathbf{x}(\tau, \alpha) \) on the manifold surface. Generalizing this notation, a function of the independent variables \( \tau \) and \( \alpha \) is linked to the invariant manifold of a periodic orbit. A diagram illustrating the parameters \( \tau \) and \( \alpha \) appears in Fig. 3.7, where the arrows indicate the direction of propagation. Figure 3.8 displays trajectories from the unstable manifold of an \( L_1 \) northern halo orbit along with stable manifold trajectories from an \( L_2 \) southern halo orbit. The \( L_1 \) periodic orbit possesses a \( z \)-amplitude of \( z_0 = 34596 \) km and the \( L_2 \) halo orbit is characterized by \( z_0 = -36997 \) km. The signs in Eqs. (3.23) are selected such that both invariant manifolds pass through the region near the Moon. The unstable manifold trajectories corresponding to the \( L_1 \) halo are propagated for \( \alpha = 13.68 \) days while the stable manifold arcs possess a duration of \( \alpha = -16.28 \) days.

The initial and destination states of the low-thrust trajectories are now specified in terms of periodic orbits in the CR3BP. The stability properties of these orbits also lead to natural flow structures that reduce fuel costs for spacecraft transfers by incorporating coast arcs along the manifolds when appropriate. The shooting method previously introduced is subsequently employed to optimize trajectories when thrust terms are added to the dynamical model.
Figure 3.7. Representation of manifold trajectory with parameters $\tau$ and $\alpha$.

Figure 3.8. Unstable (red) and stable (green) Manifold Trajectories from Earth-Moon Halo Orbits in the CR3BP.
4. INDIRECT OPTIMIZATION AND UNCONSTRAINED OPTIMAL TRANSFERS

To generate optimal transfers in the CR3BP that incorporate a thrusting spacecraft, it remains necessary to merge the thrust model into the differential equations and introduce an optimization methodology. The spacecraft is assumed to possess a variable specific impulse (VSI) low-thrust engine, so terms arising from the engine operation are combined with the gravitational terms to create a complete model for spacecraft motion. Indirect optimization techniques originate with the calculus of variations and, when applied to the spacecraft, produce unconstrained locally optimal transfers. Determination of optimal transfers is posed as a two-point boundary value problem where solutions are numerically computed using either single or multiple shooting.

4.1 The Euler-Lagrange Theory

The first-order variational approach that approximates system behavior relative to a reference trajectory arc, as discussed in Section 3.1.1, essentially serves as an introduction to a powerful analytical approach, the Calculus of Variations (CoV). The rich history of CoV in the development of dynamical analysis and optimization techniques, as well as the insight offered into the workings of the physical world, creates great interest. A brief discussion of the CoV and its relationship to optimization is presented here, but more general and complete treatments are available in Bolza [57] and Lanczos [42]. In this investigation, the CoV is used to convert a functional optimization problem into a two-point boundary value problem (2PBVP).

Functional optimization is associated with a class of problems where the optimal solution is a continuous time history rather than a discrete set of variables. The
The objective of functional optimization is the determination of an extremum (maximum or minimum) of a scalar cost functional of the general scalar form

\[ J = J_e(x_f, t_f) + \int_{t_0}^{t_f} L(x, t, \eta) dt \]  

(4.1)

where the function \( J_e \) is the end cost and the evaluation of the integral yields the path cost. The integrand \( L \) is denoted the Lagrangian, and \( \eta \) is an \( m \)-dimensional vector of control variables. With a cost functional of the form in Eq. (4.1), the optimization problem is labeled the Bolza Problem with two special cases. If only the end-cost term is present, the Problem of Mayer emerges; while, if only the path cost appears, the problem is, appropriately, the Problem of Lagrange. The functional optimization problem, in addition to dynamical equations such as those in Eq. (2.17), usually includes boundary conditions of the form

\[ \psi(x, t) = 0 \]  

(4.2)

with the time and state \( x \) evaluated at the beginning or end of the trajectory arc.

Then, an optimal control history \( \eta^*(t) \) and an optimal set of free initial conditions \( x^*_0 \) is determined using indirect optimization.

Lawden [9], as well as Bryson and Ho [58], each offer a full development and proof of the Euler-Lagrange Theorem; another source, with more modern notation, is available in Longuski [59]. The Euler-Lagrange Theorem states that, for an extremum of the scalar cost functional \( J \), when \( dJ = 0 \), the following necessary conditions must follow, that is

\[ \dot{x} = f(x, t, \eta) = H_\lambda \]  

(4.3a)

\[ \dot{\lambda} = -H_x \]  

(4.3b)

\[ H_\eta = 0 \]  

(4.3c)

where \( H \) represents the Hamiltonian and the elements of the vector \( \lambda \) reflect the co-states that correspond to the states in \( x \). Subscripts on \( H \) indicate a gradient of
the Hamiltonian with respect to one of the vectors $x$, $\eta$, or $\lambda$ (i.e., $H_{\eta} = \frac{\partial H}{\partial \eta}$). The Hamiltonian, $H$, for the problem is then written as

$$H(x, t, \eta, \lambda) = L(x, t, \eta) + \lambda^T f(x, t, \eta).$$

(4.4)

Additional transversality conditions, that is

$$H dt - \lambda^T dx + dJ_e = 0$$

(4.5)

are constructed to apply at the beginning or the end of the trajectory arc. These transversality relationships match the number of free variables (e.g., states and times) to the number of boundary conditions. Because the boundary conditions $\psi$ are specified end states for the optimization problem, the variational vector relationship

$$d\psi = \psi_{tf} dt_f + \psi_{xf} dx_f = 0$$

(4.6)

is used to reduce Eq. (4.5) to a sum of differentials whose coefficients are set equal to zero. The boundary value problem then possesses a full set of parameters and boundary conditions; thus, extremizing $H$ with respect to $\eta$ extremizes $J$. For the optimal path $x^*(x^*_0, t)$, the Hamiltonian then satisfies one of the following two statements,

$$H [x^*(t), t, \eta^*(t), \lambda(t)] \leq H [x^*(t), t, \eta(t), \lambda(t)]$$

(4.7a)

$$H [x^*(t), t, \eta^*(t), \lambda(t)] \geq H [x^*(t), t, \eta(t), \lambda(t)]$$

(4.7b)

representing the minimum or maximum cost functional, respectively. The statement in Eq. (4.7), i.e., the Minimum or Maximum Principle depending on the problem formulation, is proven by, and often named for, Pontryagin [60].

4.2 Formulation of the Optimal VSI Problem

The natural dynamics in the CR3BP yield a wide variety of behaviors including the planar periodic Lyapunov orbits, three-dimensional periodic halo trajectories, and other flow structures, e.g., the invariant manifolds. Natural heteroclinic connections between appropriate periodic orbits do exist for use as transfers. For other
combinations of periodic orbits, transfers are effected via impulsive velocity shifts that implement necessary orbit maneuvers, though often at a prohibitive cost. Alternatively, mission flexibility is greatly increased by the inclusion of a continuous thrust engine that adds controllable forces to the existing natural dynamics. These engines may also allow for smoother insertion and departure options that can reduce fuel cost. Of particular interest here are fuel optimal transfers between periodic orbits associated with the libration points using a variable specific impulse (VSI) low-thrust engine. To expand the design options, the transfers allow the possibility of coast arcs along the invariant manifolds as well. When the engine is operating, the appropriate states are added and the dynamical model in the CR3BP is modified to include thrust terms. Equations (2.16) and (2.17) are therefore augmented as

\[
\chi = \begin{pmatrix}
    r \\
    v \\
    m
\end{pmatrix}
\]  

(4.8)

with the variable \( m \) representing the instantaneous mass of the spacecraft. The differential equations become

\[
\dot{\chi} = \begin{pmatrix}
    \dot{r} \\
    \dot{v} \\
    \dot{m}
\end{pmatrix} = \begin{pmatrix}
    v \\
    f_1(r, v) + \frac{T}{m} u \\
    -\frac{T^2}{2P}
\end{pmatrix}
\]  

(4.9)

where \( T \) is thrust magnitude, \( P \) is engine power, the control \( u \) is a unit vector defining the thrust direction, and the natural dynamics \( f_1 \) are defined in Eq. (2.18) of Section 2.2. The power \( P \) is restricted to be a scalar value between zero and a maximum corresponding to a power level specified by the engine model, i.e.,

\[
0 \leq P \leq P_{\text{max}}.
\]  

(4.10)

The thrust \( T \) of the VSI engine is evaluated via

\[
T = \frac{2P}{I_{\text{sp}} g_0}
\]  

(4.11)
where $I_{sp}$ is the engine specific impulse and $g_0 = 9.80665 \frac{m}{s^2}$, the gravitational acceleration at the surface of the Earth. Consistent with the position and velocity states, as well as the non-dimensional equations representing the natural dynamics, the spacecraft mass and thrust terms are also non-dimensionalized using the characteristic length $l^*$, the initial spacecraft mass $m_0$, and the characteristic time $t^*$.

With the inclusion of thrust terms, techniques from the CoV are now applied to convert the optimization problem to a 2PBVP. However, the thrust duration ($TD$) must be included since a VSI engine is employed. If no limit is specified for either the thrust duration or, alternatively, the minimum mass consumed, the optimization process drives $TD$ and $I_{sp}$ to infinity while consuming zero propellant mass. For this application, the goal is maximization of the spacecraft final mass for a specified thrust duration, thus, the performance index $J$ is the Mayer function

$$\max J = m_f. \quad (4.12)$$

The performance index is modified by the addition of the boundary conditions and the Hamiltonian, so that Eq. (4.12) evolves into the Bolza function

$$\max J' = m_f + \mathbf{v}_0^T \psi_0 + \mathbf{v}_f^T \psi_f + \int_{t_0}^{t_f} [H - \lambda^T \dot{x}] dt \quad (4.13)$$

where $H$, $\lambda$, and $\psi$ have the same definitions, and the terms $\mathbf{v}$ are Lagrange multipliers associated with the boundary conditions. The co-state vector is written

$$\lambda = \begin{bmatrix} \lambda_r \\ \lambda_v \\ \lambda_m \end{bmatrix} \quad (4.14)$$

where $\lambda_r = (\lambda_x, \lambda_y, \lambda_z)^T$ and $\lambda_v = (\lambda_{vx}, \lambda_{vy}, \lambda_{vz})^T$ are vectors comprised of the position and velocity co-states, respectively, and $\lambda_m$ is the scalar mass co-state. The initial and final boundary conditions are expressed as

$$\psi_0 = x_0(t_0) - x_I(\tau_0, \alpha_0) = 0 \quad (4.15)$$

and

$$\psi_f = x_f(t, x_0, \lambda) - x_T(\tau_f, \alpha_f) = 0 \quad (4.16)$$
with the subscripts $I$ and $T$, respectively, indicating the initial and target states along the invariant manifolds or periodic orbits. Again, $\tau$ is the time parameter that identifies the fixed point along the periodic trajectory that is employed to generate the manifold and $\alpha$ is the propagation duration along the manifold trajectory. The design strategy for a particular transfer might not incorporate a manifold arc corresponding to the departure and/or arrival orbit. In such a scenario, the corresponding manifold parameter $\alpha$ is set equal to zero and the parameter $\tau$ then uniquely specifies the state on the initial or target orbit. For example, assume that the transfer from one libration point orbit to another does not employ the unstable invariant manifold along the departure orbit. Recall from Fig. 3.7 that $x$ is the position and velocity on the periodic orbit at the point represented by the parameter $\tau$. The initial boundary condition in Eq. (4.15) is implicitly satisfied by defining $x_I$ as a state along the unstable manifold as defined by $\tau_0$ and $\alpha_0$.

The Euler-Lagrange Theorem from Section 4.1 is employed to define several properties of the 2PBVP and acquire the derivatives of the co-states. From Eq. (4.4), the Hamiltonian is written

$$H = \lambda^T r v + \lambda^T v \left[ f_1 (r, v) + \frac{T}{m} u \right] - \lambda_m \frac{T^2}{2P}$$

(4.17)

and the value of $H$ is constant over the entire trajectory because Eq. (4.17) does not explicitly include time [59]. The optimal controls are obtained by maximizing the Hamiltonian with respect to the controls $T$, $P$, and $u$, such that

$$P = P_{\text{max}}$$

(4.18)

$$T = \frac{\lambda_v P_{\text{max}}}{\lambda_m m}$$

(4.19)

$$u = \frac{\lambda_v}{\lambda_v}$$

(4.20)

where $\lambda_v = |\lambda_v|$. Equation (4.20) originates with Lawden’s primer vector theory [9] and is designated the primer vector. From these control results, the Hamiltonian, Eq. (4.17), is reformulated to be expressed as

$$H = \lambda_v^T v + \lambda_v^T f_1 + S \cdot T$$

(4.21)
where $S$ is the switching function

$$S = \frac{\lambda_v}{m} - \frac{\lambda_m T}{2P_{\text{max}}} \quad (4.22)$$

corresponding to the control. Application of the Euler-Lagrange conditions for optimality to the modified performance index in Eq. (4.13), with the reformulated Hamiltonian, Eq. (4.21), yields the following equations of motion for the co-states

$$\dot{\lambda} = - \left( \frac{\partial H}{\partial \chi} \right)^T = \begin{cases} -\lambda_v^T \left( \frac{\partial f_1}{\partial r} \right) \\
-\lambda_r^T - \lambda_v^T \left( \frac{\partial f_1}{\partial v} \right) \\
\lambda_v \frac{T}{m^2} \end{cases} \quad (4.23)$$

where the derivative of $\lambda_m$ is always positive so that the mass co-state continually increases. Because $\lambda_m$ is monotonically increasing and the problem formulation ensures that the value of either the initial or final mass co-state is unity [12], the initial mass co-state is set as $\lambda_{m0} = 1$ and $\lambda_{mf}$ is determined by propagation along the transfer path.

The final step in defining the 2PBVP is construction of the transversality conditions to ensure local optimality. The first differentials of the Bolza function, Eq. (4.13), with respect to the independent time-like parameters $\tau$ and $\alpha$ supply the conditions

$$\chi_{\tau v_0}^T \frac{\partial x_I(\tau_0, \alpha_0)}{\partial \tau_0} = 0 \quad (4.24)$$
$$\chi_{\tau v_0}^T \frac{\partial x_I(\tau_0, \alpha_0)}{\partial \alpha_0} = 0 \quad (4.25)$$
$$\chi_{\tau v_f}^T \frac{\partial x_T(\tau_f, \alpha_f)}{\partial \alpha_f} = 0 \quad (4.26)$$
$$\chi_{\tau v_f}^T \frac{\partial x_T(\tau_f, \alpha_f)}{\partial \tau_f} = 0. \quad (4.27)$$

If the design for a particular transfer bypasses the use of a manifold, then one or both of Eqs. (4.25) and (4.26) are omitted from the boundary value problem. The change in the spacecraft state with respect to the parameter $\alpha$ is simply

$$\frac{\partial x}{\partial \alpha} = \begin{cases} v \\
\begin{bmatrix} f_1^T 
\end{bmatrix} \end{cases} \frac{\partial t}{\partial \alpha}. \quad (4.28)$$
For the differential that represents the spacecraft state with respect to the orbit parameter \( \tau \), a more complex expression is developed that incorporates the insertion onto the manifold as well as the STM along the manifold. From Senent [13], this expression is

\[
\frac{\partial x}{\partial \tau} = \Phi(\alpha, 0) \Upsilon(\tau) = \Phi(\alpha, 0) \left[ f(x(\tau, 0)) \pm \epsilon [I_6 - \hat{v}(\tau) \hat{v}(\tau)^T] A_6(x(\tau, 0)) \hat{v}(\tau) \right] \frac{dt}{d\tau}
\]

(4.29)

where the term \( \Upsilon(\tau) \) represents the change in the initial state on the manifold arc and the STM, \( \Phi(\alpha, 0) \), shifts this variation to the end of the manifold trajectory. The sign in \( \Upsilon(\tau) \) is specified by the sign selected in Eq. (3.23).

The fixed-time transfer between periodic orbits, with or without the use of invariant manifolds, is now formulated as a 2PBVP. The design variable vector is

\[
X_S = \begin{pmatrix} \tau_0 \\ \alpha_0 \\ \lambda_1 \\ \alpha_f \\ \tau_f \end{pmatrix}
\]

(4.30)

where \( \lambda_1 = (\lambda_{r1}, \lambda_{v1})^T \), that is, the position and velocity co-states at the beginning of thrust. The constraint equation

\[
F_S(X_S) = \begin{bmatrix} \dot{x}^f(\tau_0, \alpha_0, \lambda_1) - x_T(\tau_f, \alpha_f) \\ \lambda_T^x \frac{\partial x_f(\tau_0, \alpha_0)}{\partial \tau_0} \\ \lambda_T^x \frac{\partial x_f(\tau_0, \alpha_0)}{\partial \alpha_0} \\ \lambda_T^x \frac{\partial x_f(\tau_f, \alpha_f)}{\partial \alpha_f} \\ \lambda_T^x \frac{\partial x_f(\tau_f, \alpha_f)}{\partial \tau_f} \end{bmatrix} = 0
\]

(4.31)

includes both continuity and optimality constraints. The superscript \( t \) on the state \( x \) in the continuity constraint indicates the final state evolving from a numerical propagation of the EOMs. The subscripts on the terminal co-state vector \( \lambda_f \) indicate the needed elements, so \( \lambda_{x,f} \) is composed of the position and velocity co-states at the end of the thrust arc. The design vector \( X_S \) and the constraint vector \( F_S(X_S) \)
comprise a single shooting targeting scheme with no patch points internal to the trajectory. The 2PBVP can be solved as formulated but a guess for the initial co-state vector is not immediately available or intuitive.

### 4.3 Adjoint Control Transformation

Dixon [61] developed the Adjoint Control Transformation (ACT) to supply more intuitive variables as a basis to generate an initial guess for the co-states and to increase the convergence radius of the locally optimal solutions. To accomplish these goals, the ACT relates the thrust direction in a vehicle-centered frame to the initial co-states $\lambda_1$. The spacecraft-centered frame $V$ is defined with the right-handed triad

$$
\hat{V} = \frac{v}{|v|}, \quad \hat{n} = \frac{r \times v}{|r \times v|}, \quad \hat{b} = \hat{n} \times \hat{V} \tag{4.32}
$$

where $\hat{V}$ is a unit vector in the direction of the spacecraft velocity, $\hat{n}$ is the instantaneous orbit normal, and $\hat{b}$ completes the orthonormal set of unit vectors. Two spherical angles, $\rho$ and $\beta$, specify the thrust direction in this frame. These two angles, the unit vectors defining frame $V$, and the thrust vector $T$ are all illustrated in Fig. 4.1.

![Figure 4.1. Spacecraft-centered frame $V$ with thrust vector](image-url)
The thrust direction is now expressed in terms of the angles $\rho$ and $\beta$ and then transformed to the rotating frame $R$. In terms of the angles, the unit vector that defines the thrust direction in frame $V$ is written

$$u_V = \begin{bmatrix} \cos \rho \cos \beta \\ \sin \rho \cos \beta \\ \sin \beta \end{bmatrix}. \tag{4.33}$$

Then, the differential vector

$$\dot{u}_V = \begin{bmatrix} -\dot{\rho} \sin \rho \cos \beta - \dot{\beta} \cos \rho \sin \beta \\ \dot{\rho} \cos \rho \cos \beta - \dot{\beta} \sin \rho \sin \beta \\ \dot{\beta} \cos \beta \end{bmatrix} \tag{4.34}$$

represents the rate of change of the thrust direction as observed from the vehicle frame. The transformation of the thrust direction vector from frame $V$ to frame $R$ is accomplished via

$$u_R = Du_V \tag{4.35}$$

where the rotation DCM $D$ is defined

$$D = \begin{bmatrix} \hat{x} \cdot \hat{V} & \hat{x} \cdot \hat{b} & \hat{x} \cdot \hat{n} \\ \hat{y} \cdot \hat{V} & \hat{y} \cdot \hat{b} & \hat{y} \cdot \hat{n} \\ \hat{z} \cdot \hat{V} & \hat{z} \cdot \hat{b} & \hat{z} \cdot \hat{n} \end{bmatrix}. \tag{4.36}$$

Rotation of the thrust direction rate of change yields

$$\dot{u}_R = \dot{D}u_V + Du_V \tag{4.37}$$

where

$$\dot{D} = D \cdot \left( \frac{r \times v}{r^2} \right) \tag{4.38}$$

delivers the DCM rate of change.

The direct connection between the optimal controls and the spherical angles $\rho$ and $\beta$ remains to be established. The primer vector from Eq. (4.20) is rearranged to appear

$$\lambda_v = \lambda_v u_R \tag{4.39}$$
with the time derivative

$$\dot{\lambda}_v = \dot{\lambda}_v u_R + \lambda_v \dot{u}_R.$$  

(4.40)

Recall the velocity co-state equations of motion in Eq. (4.23). The initial position co-state is then

$$\lambda_r = -\dot{\lambda}_v u_R - \lambda_v \dot{u}_R - \frac{\partial f_1}{\partial v} \lambda_v$$  

(4.41)

where Eqs. (4.39) and (4.41) relate the initial position and velocity co-states to the thrust direction angles and the magnitude the velocity co-state vector and its corresponding rate of change, i.e.,

$$\dot{\lambda}_v = -\frac{1}{u^T v} \left[ \lambda_v \dot{u}_v + \lambda^T_v \frac{\partial f_1}{\partial v} v - \lambda^T_v f_1 \right].$$  

(4.42)

The design vector

$$X_{SACT} = \begin{bmatrix} \tau_0 \\ \alpha_0 \\ \rho_0 \\ \dot{\rho}_0 \\ \beta_0 \\ \dot{\beta}_0 \\ \lambda_{v1} \\ \dot{\lambda}_{v1} \\ \alpha_f \\ \tau_f \end{bmatrix}$$  

(4.43)

and the previous constraint vector from Eq. (4.31) now constitute another formulation of the 2PBVP that provides more physical intuition in constructing initial guesses and reduces sensitivity during the targeting process.

### 4.4 Computation of Unconstrained Optimal Transfers

Indirect optimization methods pose fuel optimal transfers as 2PBVPs with a low number of both free variables and conditions that must be satisfied but, in general, the
initial conditions must be numerically computed. The general corrections procedure from Chapter 3 is applied to numerically converge on solutions to any given 2PBVP. When no nearby low-thrust arcs are known, single shooting is employed to reduce the number of initial variables that require a starting value. Alternatively, if a low-thrust solution is known, multiple shooting increases solution accuracy within a continuation scheme.

4.4.1 Optimal Transfers from Single Shooting

A single shooting algorithm of the type discussed in Ch. 3 is implemented to determine solitary, unconstrained optimal transfers. Newton’s method suffices to solve the 10-variable and 10-constraint single shooting optimization problems in Sections 4.2 and 4.3, but the algorithm requires computation of the gradient of the constraint function $F_S(X_S)$. Incorporating the ACT, the constraint gradient requires the partial of each constraint in Eq. (4.31) with respect to each state in Eq. (4.43). However, analytically formulating each of the partials is a laborious process and the ACT formulation is used sparingly to generate single transfers. Therefore, the constraint gradient is constructed with finite differencing whenever the ACT is employed.

When the co-states are included in the set of design variables, the constraint gradient is determined using the previously defined $6 \times 6$ STM $\Phi(t, t_0)$ (Section 3.1.1) as well as a new $14 \times 14$ STM $\Psi(t, t_0)$ incorporating the seven states (position, velocity, mass) and the corresponding co-states. To create the new STM, define the combined state and co-state vector

$$\xi = \begin{bmatrix} \chi \\ \lambda \end{bmatrix}$$

(4.44)

with the associated vector derivative

$$a = \dot{\xi} = \begin{bmatrix} \dot{\chi} \\ \dot{\lambda} \end{bmatrix}$$

(4.45)
where the vectors $\boldsymbol{\chi}$ and $\boldsymbol{\lambda}$ include the states and co-states, respectively. A new linear dynamical matrix of partials $A_{14}(t)$ is evaluated as

$$A_{14}(t) = \frac{d\alpha(\xi)}{d\xi} =$$

$$\begin{bmatrix}
0_3 & I_3 & 0_{3,1} & 0_3 & 0_{3,1} & 0_{3,1} & 0_{3,1} \\
-\frac{2P_{\text{max}}\lambda v}{m^2\lambda_m} & 0_{1,3} & \frac{P_{\text{max}}}{m^2\lambda_m} & 0 & 0 & -\frac{P_{\text{max}}\lambda v}{m^2\lambda_m} \\
\frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} & -\frac{2P_{\text{max}}\lambda v}{m^2\lambda_m} & 0_{1,3} & 0 & \frac{P_{\text{max}}}{m^2\lambda_m} & 0 & -\frac{P_{\text{max}}\lambda v}{m^2\lambda_m} \\
0_{1,3} & 0_{1,3} & \frac{P_{\text{max}}\lambda v}{m^2\lambda_m} & 0_{1,3} & -\frac{P_{\text{max}}\lambda v}{m^2\lambda_m} & \frac{P_{\text{max}}\lambda v}{m^2\lambda_m} & P_{\text{max}} & 0_{1,3} \\
-W & 0_3 & 0_{3,1} & 0_3 & -\left(\frac{\partial f_1}{\partial r}\right)^T & 0_{3,1} \\
0_3 & 0_3 & 0_{3,1} & -I_3 & -\left(\frac{\partial f_1}{\partial \theta}\right)^T & 0_{3,1} \\
0_{1,3} & 0_{1,3} & -\frac{3P_{\text{max}}\lambda v^2}{m^4\lambda_m} & 0_{1,3} & -\frac{2P_{\text{max}}\lambda v}{m^2\lambda_m} & \frac{2P_{\text{max}}\lambda v}{m^2\lambda_m} & \frac{2P_{\text{max}}\lambda v}{m^2\lambda_m} & -\frac{P_{\text{max}}\lambda v}{m^2\lambda_m}
\end{bmatrix}$$

(4.46)

where the subscripts on $0$ and $I$ indicate the size of the zero or identity matrix. (The individual subscript ‘3’ indicates a $3 \times 3$ matrix; the subscript combination ‘3,1’ denotes a 3-element column vector, and ‘1,3’ a 3-element row vector.) The $3 \times 3$ sub-matrix

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\
W_{21} & W_{22} & W_{23} \\
W_{31} & W_{32} & W_{33} \end{bmatrix}$$

(4.47)
incorporates the third partials of the pseudo-potential function $U^*$ from Eq. (2.19)

$$
W_{11} = \lambda_{v_x} U^*_{x^3} + \lambda_{v_y} U^*_{x^2y} + \lambda_{v_z} U^*_{x^2z}
$$

$$
W_{12} = \lambda_{v_x} U^*_{x^2y} + \lambda_{v_y} U^*_{xy^2} + \lambda_{v_z} U^*_{xyz}
$$

$$
W_{13} = \lambda_{v_x} U^*_{x^2z} + \lambda_{v_y} U^*_{xyz} + \lambda_{v_z} U^*_{xz^2}
$$

$$
W_{21} = \lambda_{v_x} U^*_{x^2y} + \lambda_{v_y} U^*_{xy^2} + \lambda_{v_z} U^*_{xyz}
$$

$$
W_{22} = \lambda_{v_x} U^*_{xy^2} + \lambda_{v_y} U^*_{y^3} + \lambda_{v_z} U^*_{y^2z}
$$

$$
W_{23} = \lambda_{v_x} U^*_{xyz} + \lambda_{v_y} U^*_{yz^2} + \lambda_{v_z} U^*_{z^3}
$$

$$
W_{31} = \lambda_{v_x} U^*_{x^2z} + \lambda_{v_y} U^*_{xyz} + \lambda_{v_z} U^*_{xz^2}
$$

$$
W_{32} = \lambda_{v_x} U^*_{xyz} + \lambda_{v_y} U^*_{yz^2} + \lambda_{v_z} U^*_{z^3}
$$

$$
W_{33} = \lambda_{v_x} U^*_{xz^2} + \lambda_{v_y} U^*_{yz^2} + \lambda_{v_z} U^*_{z^3}
$$

(4.48)

Then, the new $14 \times 14$ STM is defined such that

$$
\delta \xi(t) = \frac{\partial \xi}{\partial \xi_0} \delta \xi_0 = \Psi(t, t_0) \delta \xi_0
$$

(4.49)

$$
\dot{\Psi}(t, t_0) = \frac{\partial a}{\partial \xi} \Psi(t, t_0) = A_{14}(t) \Psi(t, t_0)
$$

(4.50)

where the initial STM is equal to the identity matrix, $\Psi(t_0, t_0) = I$. Note that the

$14 \times 14$ STM $\Psi$ reduces to the $6 \times 6$ STM $\Phi$ when the spacecraft mass and the co-
states are removed as variables (e.g., the VSI engine is turned off or a thrust profile
is specified).

The elements of the complete constraint gradient are now constructed using the

STMs $\Phi$ and $\Psi$. Recall from Eq. (4.31) that the constraint vector $F_S(X_S)$ includes six

constraints enforcing position and velocity continuity between the end of the thrust
arc and the target stable manifold as well as four transversality constraints ensuring local optimality. The constraint gradient for $F_S(X_S)$ is then

$$
DF_S(X_S) =
\begin{bmatrix}
\Psi_{x,x} \frac{\partial x_I}{\partial \tau_0} & \Psi_{x,x} \frac{\partial x_I}{\partial \alpha_0} & \Psi_x \lambda_1 & -\frac{\partial x_T}{\partial \alpha_f} & -\frac{\partial x_T}{\partial \tau_f} \\
\lambda_1^T \frac{\partial^2 x_I}{\partial \tau_0 \partial \alpha_0} & \lambda_1^T \frac{\partial^2 x_I}{\partial \tau_0 \partial \alpha_0} & \left( \frac{\partial x_I}{\partial \tau_0} \right)^T & 0 & 0 \\
\lambda_1^T \frac{\partial^2 x_I}{\partial \tau_0 \partial \alpha_0} & \lambda_1^T \frac{\partial^2 x_I}{\partial \tau_0 \partial \alpha_0} & \left( \frac{\partial x_I}{\partial \tau_0} \right)^T & 0 & 0 \\
\left( \frac{\partial x_T}{\partial \alpha_f} \right)^T \Psi_{\lambda,x} \frac{\partial x_I}{\partial \tau_0} & \left( \frac{\partial x_T}{\partial \alpha_f} \right)^T \Psi_{\lambda,x} \frac{\partial x_I}{\partial \alpha_0} & \left( \frac{\partial x_T}{\partial \alpha_f} \right)^T \Psi_{\lambda,x} \lambda_1 & \left( \frac{\partial x_T}{\partial \alpha_f} \right)^T \Psi_{x,f} \frac{\partial^2 x_I}{\partial \tau_f \partial \alpha_f} & \left( \frac{\partial x_T}{\partial \alpha_f} \right)^T \Psi_{x,f} \frac{\partial^2 x_I}{\partial \tau_f \partial \alpha_f} \\
\left( \frac{\partial x_T}{\partial \tau_f} \right)^T \Psi_{\lambda,x} \frac{\partial x_I}{\partial \tau_0} & \left( \frac{\partial x_T}{\partial \tau_f} \right)^T \Psi_{\lambda,x} \frac{\partial x_I}{\partial \alpha_0} & \left( \frac{\partial x_T}{\partial \tau_f} \right)^T \Psi_{\lambda,x} \lambda_1 & \left( \frac{\partial x_T}{\partial \tau_f} \right)^T \Psi_{x,f} \frac{\partial^2 x_I}{\partial \alpha_f \partial \tau_f} & \left( \frac{\partial x_T}{\partial \tau_f} \right)^T \Psi_{x,f} \frac{\partial^2 x_I}{\partial \alpha_f \partial \tau_f}
\end{bmatrix}
\tag{4.51}
$$

where the subscripts on the STM $\Psi$ indicate sub-matrices with the respective rows and columns (e.g., $\Psi_{\lambda,x}$ implies the sub-matrix formed from the rows corresponding to the position and velocity co-states and the columns associated with the position and velocity). Likewise, the subscripts on the co-state vector $\lambda$ denote the elements to be extracted, such that $\lambda_{x,f}$ is composed of the position and velocity co-states at the end of the thrust arc. Recall that $\lambda_1$ is defined $\lambda_1 = (\lambda_{r_1}, \lambda_{v_1})^T$. Expressions for the second partials of the initial and target states $x_I$ and $x_T$ with respect to $\tau$ and $\alpha$ are available in all cases. The partials with respect to the $\alpha$ parameters are all in the form

$$
\frac{\partial^2 x(\tau, \alpha)}{\partial \alpha^2} = A_{14}(\alpha) \begin{bmatrix} v \\ f_1 \end{bmatrix}
\tag{4.52}
$$

while, for the second partials with respect to both $\alpha$ and $\tau$, the expression is

$$
\frac{\partial^2 x(\tau, \alpha)}{\partial \tau \partial \alpha} = \Phi(\alpha, 0) \gamma(\tau).
\tag{4.53}
$$

There does exist an expression for $\frac{\partial^2 x}{\partial \tau^2}$ but it is cumbersome and requires the second-order mapping of initial to final states $\frac{\partial \Phi(t_{i_1})}{\partial x(t_{i_1})}$. Unfortunately, this process requires the computation of second-order sensitivity partials and the integration of an additional 216 equations of motion. Thus, it is simply more computationally efficient to compute a numerical gradient for $\frac{\partial^2 x}{\partial \tau^2}$ with finite differencing.
4.4.2 Application of Multiple Shooting to Optimal Transfers

This analysis primarily employs multiple shooting to apply constraints and to implement a continuation scheme where solutions for varying thrust durations are generated given a known solution to the 2PBVP. For an unconstrained transfer using multiple shooting with $n$ segments, the state vector is defined

$$X_M = \begin{bmatrix} \tau_0 \\ \alpha_0 \\ \lambda_1 \\ TD_1 \\ \xi_2 \\ TD_2 \\ \vdots \\ \xi_n \\ TD_n \\ \alpha_f \\ \tau_f \end{bmatrix}$$

(4.54)

where $\xi_i$ is the combined state and co-state vector at the $i$-th patch point and $TD_i$ is the duration along the $i$-th segment. Segment duration is retained as a variable for greater flexibility when developing solutions, though the consequence is an underconstrained system that requires a minimum-norm iterative procedure. The constraint
vector is then expanded to include continuity constraints on the segments to ensure a smooth path

\[
F_M(X_M) = \begin{pmatrix}
\xi^t_2(\tau_0, \alpha_0, \lambda_1) - \xi_2 \\
\xi^t_3(\xi_2) - \xi_3 \\
\vdots \\
x^t_T(\xi_n) - x_T(\tau_f, \alpha_f)
\end{pmatrix}
\]

\[
\begin{aligned}
\lambda_1^T \frac{\partial x^t\xi(\tau_0, \alpha_0)}{\partial \tau_0} \\
\lambda_1^T \frac{\partial x^t\xi(\tau_0, \alpha_0)}{\partial \alpha_0} \\
\lambda^T_{x_f} \frac{\partial x_T(\tau_f, \alpha_f)}{\partial \alpha_f} \\
\lambda^T_{x_f} \frac{\partial x_T(\tau_f, \alpha_f)}{\partial \tau_f} \\
\sum_{i=1}^n TD_i - TD
\end{aligned}
\]

\[
= 0 \quad (4.55)
\]

where the superscript \( t \) indicates states that are reached after propagation from a state \( \xi_i \) for the associated segment duration. The final constraint in the vector \( F_M \) involves the thrust duration along the segments and ensures that the overall transfer includes thrust durations that sum to the total specified value \( TD \). The multiple shooting problem is solved via a boundary value problem solver or equation solver using either numerical or analytical gradients.

The design variable vector is greatly expanded in a multiple shooting scheme and the matrix of partials \( DF_M \) grows accordingly. The gradient of the constraint function, Eq. (4.55), with respect to the state vector, Eq. (4.54), is written
\[ DFM(X_M) = \begin{bmatrix}
\Psi_{1,x} & 0 & \cdots & 0 & a_1 & -I_{14} & 0 & \cdots & 0 & a_2 & -I_{14} & 0 & \cdots & 0 & a_n & -I_{14} & 0 & \cdots & 0 & \Psi_{2,x} & 0 & \cdots & 0 & \Psi_{x,n} & -I_{14} & 0 & \cdots & 0 & \Psi_{\lambda a} & -I_{14} & 0 & \cdots & 0 & \Psi_{\lambda n} & -I_{14} & 0 & \cdots & 0 & \end{bmatrix}
\]
where the variation in the vector state $a$ is evaluated at the end of each segment and the order of the subscripts on the $Ψ$ state transition matrix indicates whether columns or rows are extracted. In the subscripts, a number preceding the variables indicates that columns are used, while a variable preceding a number reflects the fact that rows are employed (e.g., $Ψ_{1,x}$ implies that the columns corresponding to the position and velocity states from the STM corresponding to the first segment are to be extracted).

The three necessary components for generating fuel optimal spacecraft trajectories are now fully constructed. The system model from Ch. 2, the corrections procedures and periodic orbits of Ch. 3, and the optimization algorithms of the current chapter culminate in a tool that generates transfers between periodic orbits that are free of path constraints. Subsequent chapters investigate the results of unconstrained transfers and address the inclusion of restrictions on the trajectory and spacecraft limitations.
5. RESULTS THAT YIELD UNCONSTRAINED TRANSFERS

Spacecraft missions incorporating trajectories in the vicinity of libration point orbits require a variety of orbit transfer types as well as maneuvers: samples of possible mission scenarios are offered here. All the transfers in these examples are free of path constraints. Thus it is only necessary to satisfy the boundary and local optimality conditions derived via the indirect methods in Chapter 4. The sample scenarios encompass a wide range of transfer options so that a general picture of optimal thrust arcs in this regime emerges and serves as a basis for future predictions.

5.1 Sample Transfers

A wide variety of transfer options are required to gain a broad perspective on the advantages as well as the limitations of low-thrust arcs in general and the VSI engine in particular. To this end, the following types of thrust arcs are examined:

- $L_1$-halo-to-$L_1$-halo orbit transfers, sans invariant manifold coasts.
- Phase shifting along an $L_1$ halo orbit, without incorporating invariant manifolds.
- $L_1$ orbit to $L_2$ orbit transfers, using invariant manifolds corresponding to the initial and target orbits. Both halo-to-halo orbit and vertical-to-halo orbit transfers are considered.
- $L_2$-planar-Lyapunov orbit to $L_4$-short-period-planar orbit transfers using only the unstable invariant manifold from the initial $L_2$ orbit.

All transfers occur in the Earth-Moon system and most of the initial and target orbits are located in the vicinity of the Moon, a region of rich natural dynamics and much
scientific interest. For all cases, the spacecraft is assumed to possess an initial mass of 500 kilograms and the maximum power of the VSI engine is 2 kilowatts. The spacecraft and Earth-Moon system parameters are listed in Table 5.1.

Table 5.1 Earth-Moon system and spacecraft parameter values.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth mass ($M_e$)</td>
<td>5.972x10^{24}</td>
<td>kg</td>
</tr>
<tr>
<td>Lunar mass ($M_m$)</td>
<td>7.346x10^{22}</td>
<td>kg</td>
</tr>
<tr>
<td>Mass parameter ($\mu$)</td>
<td>0.01215</td>
<td>NA</td>
</tr>
<tr>
<td>Earth-Moon distance ($l^*$)</td>
<td>3.84400</td>
<td>km</td>
</tr>
<tr>
<td>Characteristic Time ($t^*$)</td>
<td>3.752x10^{5}</td>
<td>sec</td>
</tr>
<tr>
<td>Characteristic Time ($t^*_d$)</td>
<td>4.342</td>
<td>days</td>
</tr>
<tr>
<td>Initial spacecraft mass ($m_0$)</td>
<td>500</td>
<td>kg</td>
</tr>
<tr>
<td>Maximum spacecraft power ($P_{\text{max}}$)</td>
<td>2.0</td>
<td>kW</td>
</tr>
</tbody>
</table>

5.2 Halo-to-Halo orbit transfers without arcs along invariant manifolds

The first application is the direct transfer from one northern $L_1$ halo orbit to another. Because the orbits are in the vicinity of the same collinear point, the use of the invariant manifolds offers little advantage, generally true when the transfer times are relatively short. Recall that the parameters $\alpha_0$ and $\alpha_f$ represent coast durations along invariant manifolds associated with the original and target periodic orbit, respectively. Therefore, without manifold coasts, $\alpha_0$ and $\alpha_f$ and their transversality conditions are omitted from the 2PBVP. The spacecraft transfer is directed from the initial inner halo of lower $z$-amplitude to the final outer halo orbit that possesses a higher $z$-amplitude. (Note that the $z$-amplitude is evaluated at the northern crossing of the $xz$-plane.) Characteristics of the two orbits are summarized in Table 5.2.
An initial transfer of thrust duration 2.4 days is computed via one of the single shooting strategies in Sections 4.2 or 4.3. Because the transversality conditions only assure local optimality, multiple solutions to the 2PBVP are available. For a thrust duration of 2.4 days, two locally optimal transfers are generated; the elements of the solution vector $X_5^*$ appear in Table 5.3. Since manifolds are not incorporated in this example, $\alpha_0$ and $\alpha_f$ are removed as design variables. Note that $\tau_0 = 0$ and $\tau_f = 0$ define the northern crossing of the $xz$-plane on both halo orbits. Tracking the evolution of the co-states along the trajectory illustrates the differing thrust histories for the two solutions. Recalling that the velocity co-states define the thrust direction, the $\hat{x}$ and $\hat{z}$ thrust components dominate the ends of the first trajectory while the $\hat{x}$ and $\hat{y}$ thrust components dominate the ends of the second transfer, as evident in Fig. 5.1. The central sections of the velocity co-state plots, where the co-state magnitudes are lower, indicate via Eq. (4.19) an interval of relatively low-thrust. In both solutions, the dominant thrust terms display opposing evolutions: as $\lambda_{vx}$ decreases, the other dominant term, i.e., $\lambda_{vz}$, increases. The net effect of these opposing trends is that, for both transfers, the initial and final thrust directions are nearly anti-parallel.

Transfers of increasing thrust duration are computed using the multiple shooting transfer formulation from Section 4.4. Generation of a starting guess for $X_S$ corresponding to each solution, identified by the thrust duration value, is accomplished by natural parameter continuation, that is, borrowing information from the previous known solution. Using thrust duration as the continuation parameter, a “family” of
Table 5.3 Optimal solutions for $L_1$ halo-to-$L_1$ halo transfers, $TD = 2.388364$ days

<table>
<thead>
<tr>
<th>$X^*_S$ Element</th>
<th>Solution 1</th>
<th>Solution 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0$, days</td>
<td>4.750296</td>
<td>7.244336</td>
</tr>
<tr>
<td>$\lambda_{rx}$</td>
<td>1.633695</td>
<td>1.312959</td>
</tr>
<tr>
<td>$\lambda_{ry}$</td>
<td>0.004675</td>
<td>-1.222621</td>
</tr>
<tr>
<td>$\lambda_{rz}$</td>
<td>-1.322825</td>
<td>-0.253665</td>
</tr>
<tr>
<td>$\lambda_{vx}$</td>
<td>0.390007</td>
<td>0.400902</td>
</tr>
<tr>
<td>$\lambda_{vy}$</td>
<td>0.029405</td>
<td>-0.318431</td>
</tr>
<tr>
<td>$\lambda_{vz}$</td>
<td>-0.466104</td>
<td>0.070604</td>
</tr>
<tr>
<td>$\tau_f$, days</td>
<td>7.212005</td>
<td>9.529088</td>
</tr>
</tbody>
</table>

Figure 5.1. Control history for $L_1$ halo-to-$L_1$ halo optimal thrust arcs, $TD = 2.388364$ days.
transfers emerges from a single known solution; this definition for a family of transfer arcs is consistent throughout this investigation. Two families, each comprised of 30 locally optimal transfers, and originating from the optimal solutions in Table 5.3, appear in Fig. 5.2. (Note the origin of the coordinate frame is $L_1$ and the motion of the spacecraft is clockwise around the orbits.) Family 1, continued from Solution 1, originates in the lower right (positive $y$, negative $z$) along the initial (blue) halo, while Family 2, corresponding to Solution 2, emanates from the lower left (negative $y$ and $z$) in the initial halo. Notice that, as these transfers are numerically generated, they possess initial and final states that are relatively close within the family. The final mass of the spacecraft as a function of the transfer time is plotted in Fig. 5.3. For the shortest thrust duration, the transfer from Family 1, with $m_f = 475\text{kg}$, consumes about 5 more kg of fuel than the corresponding transfer from Family 2. For longer times of flight, the difference in propellant consumption is less dramatic, but Family

![Figure 5.2. Two families of locally optimal transfers from $L_1$ halo-to-$L_1$ halo.](image-url)
1 always uses more mass to achieve a transfer of the same duration. Note that all transfer families included in this analysis demonstrate a similar asymptotic behavior when plotted against thrust duration, regardless of orbit type or the use of manifolds.

Since Family 2 yields better performance for a given thrust duration, this family is used to investigate the behavior along the transfer arc for increasingly long times of flight. Thus, transfers in Family 2 are expanded to widen the range of thrust durations. A more complete representation of the family, including transfers of thrust durations from 2.4 to 21.5 days, appears in Fig. 5.4. Three specific sample transfer arcs, corresponding to continuous thrust arcs of duration equal to 2.4, 11.9, and 21.5 days of thrust, are highlighted. An interesting feature of this transfer family is that all the end states are confined to a narrow region of the target halo orbit, i.e., the upper left (negative y, positive z). This region is not sharply defined in terms of the xy- and xz-planes, but is instead some distance from these two planes.

The transfers highlighted in Fig. 5.4 are isolated in Fig. 5.5, along with their respective $I_{sp}$ time histories. Note the shortest duration transfer forms a more-or-less
Figure 5.4. Optimal transfers of thrust duration from 2.4 to 21.5 days.

Thrust duration: 2.4 days  Thrust duration: 11.9 days  Thrust duration: 21.5 days

Figure 5.5. Three specific orbit transfers and their $I_{sp}$ profiles.
direct arc from the initial orbit to the target orbit; the two longer trajectory arcs complete one or (nearly) two revolutions before arrival. Also, the $I_{sp}$ profiles change significantly with increasing thrust duration, from one peak and no local minima for the shortest thrust duration to multiple local maxima and minima for the longest thrust duration profile. Indeed, these $I_{sp}$ profiles begin to approximate thrust-coast-thrust architectures or a series of impulsive maneuvers separated by coast arcs. For further investigation, the plots from Fig. 5.6 demonstrate the evolving nature of the highest, average, and lowest values of $I_{sp}$ as a function of thrust duration. The departure and arrival locations along each halo orbit are also dependent on thrust duration. Thus, the initial and target orbit parameters $\tau_0$ and $\tau_f$ are also plotted in Fig. 5.6. The discontinuities in the curves representing the lowest and highest values of $I_{sp}$ coincide with the decrease in $\tau_f$ and rapid change in $\tau_0$, almost as if these two parameters act to “reset” the exponential growth in highest $I_{sp}$ and the logarithmic growth in lowest $I_{sp}$. As a consequence, the average $I_{sp}$ value increases nearly linearly. During these notable thrust duration intervals, the arrival point moves backward along the target halo orbit. The rapid changes in the departure point occur in the upper left (negative $y$, positive $z$) and the lower right (positive $y$, negative $z$) of the initial halo orbit, as displayed in Fig. 5.4.
Figure 5.6. Evolution of $I_{sp}$ and orbit parameters with respect to thrust duration. Note: the change in $\tau_0$ is multiplied by a factor of 10 to better demonstrate its behavior.
5.3 Phase shifting along an \( L_1 \) halo orbit using a VSI engine

Rather than a transfer from a departure orbit to a different arrival orbit, some missions require a simple phase shift along the current orbit of the spacecraft. On a periodic orbit, these phase shifts are defined as either an angle or a time-like parameter: for simplicity, in this investigation the phase alterations are specified by the time shift \( P_S \). Then, given the most general set of design variables in Eq. (4.30), the 2PBVP reduces to a set of seven design parameters, that is,

\[
X_{PS} = \begin{cases} 
\tau_0 \\
\lambda_1 
\end{cases}
\]  
(5.1)

and seven constraints

\[
F_{PS}(X_{PS}) = \begin{cases} 
x_t(\tau_0, \lambda_1) - x_T(\tau_0 + TD + P_S) \\
\lambda_1^T \frac{\partial x_I(\tau_0)}{\partial \tau_0} 
\end{cases} = 0.
\]  
(5.2)

The target vector \( x_T(\tau_R) \) with \( \tau_R = \tau_0 + TD + P_S \) is the return state on the original orbit defined by the orbit parameter \( \tau_0 \), thrust duration \( TD \), and the phase shift \( P_S \). For any specific arc to yield a phase shift, the thrust duration and phase shift are fixed, thus, the complete derivative of \( x_T \) is \( \frac{\partial x_T}{\partial \tau_0} \) evaluated at \( x_T(\tau_R) \). The constraint gradient for a phase shift is then

\[
DF_{PS}(X_{PS}) = \begin{bmatrix} 
\Psi_{x,x} \frac{\partial x_I}{\partial \tau_0} - \frac{\partial x_T}{\partial \tau_0} & \Psi_{x,\lambda_1} \\
\lambda_1^T \frac{\partial^2 x_I}{\partial \tau_0^2} & \left( \frac{\partial x_I}{\partial \tau_0} \right)^T 
\end{bmatrix}
\]  
(5.3)

where \( DF_{PS}(X_{PS}) \) is a square, \( 7 \times 7 \) matrix. Assuming that the matrix in Eq. (5.3) is invertible, Newton’s method produces a unique solution.

Phase shifts within an orbit can also be accomplished using impulsive maneuvers; such impulsive maneuvers are also computed to compare the performance of the VSI engine to more conventional propulsion systems. For this analysis, the first impulsive maneuver to initiate the phase shift occurs at the initial position along the VSI thrust arc. Then, the final position on the VSI thrust arc is targeted using the shooting
methods detailed in Chapter 3. For this application of the shooting algorithm, the free variable vector is

\[ X_{Imp} = \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} \]  

with the constraint vector

\[ F_{Imp}(X_{Imp}) = \begin{bmatrix} x_{Imp,f} - x_{VSI,f} \\ y_{Imp,f} - y_{VSI,f} \\ z_{Imp,f} - z_{VSI,f} \end{bmatrix}. \]  

Since the target state is fixed as the end position along the VSI phase shift, the constraint gradient contains only elements of the STM in the CR3BP

\[ DF_{Imp}(X_{Imp}) = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix}. \]  

and Newton’s method is applied to determine the velocity that initiates the shifting arc. This process results in a coast arc that connects the end positions of the thrust arc but where the velocities at the terminal points do not match the velocities at the same location on the periodic orbit. These two velocity discontinuities are evaluated as

\[ \Delta v = v_{orb} - v_{Imp} \]  

where \( v_{orb} \) is the velocity on the orbit and \( v_{Imp} \) is the velocity on the impulsive transfer arc. The magnitude \( \Delta v \) of each impulsive \( \Delta v \) vector reflects the cost to implement these maneuvers. However, the velocity costs must be translated into fuel consumption for a comparison between the two approaches to produce a phase shift: impulsive maneuvers versus low-thrust from a VSI engine. This conversion is accomplished using a form of the Ideal (or Tsiolkovsky) Rocket Equation [62], that is

\[ m_{prop,Imp} = m_f \left( e^{\frac{\Delta v}{Isp,Impg_0}} - 1 \right) \]  

(5.8)
where $m_{prop,imp}$ is the propellant consumed by the impulsive engine, and $I_{sp,imp}$ is the specific impulse of the impulsive engine. The gravitational constant $g_0$ is defined in Section 4.2, and the mass $m_f$ is equal to the mass of the spacecraft at the end of the VSI thrust arc. Though the scheme to deliver an impulsive phase shift as outlined here is not optimal, a baseline comparison between a VSI engine and a more conventional high-impluse engine is now possible.

The smaller halo orbit from Section 5.2 is used to investigate phase shifting. Recall that this northern $L_1$ halo is defined by a $z$-amplitude of 17298 km and a period of 11.967 days. An optimal thrust arc of duration $TD = 2.3884$ days for a phase shift of $P_S = 2.1712$ days appears in Fig. 5.7. The halo orbit is plotted in red, the impulsive coast arc in brown, and the black curve is the path of the spacecraft along the VSI thrust arc. The green arrows along the thrust arc signify the thrust direction and relative magnitude. The initial point of the two phase-shifting arcs are marked by a blue circle while the red triangle denotes the end state. The black star denotes $\bm{x}(\tau_C)$ where $\tau_C = \tau_0 + TD$, the state of the spacecraft if the phase shift is not implemented. As demonstrated in Fig. 5.7(b), both phase shift arcs are represented with a curvature that is roughly consistent with that of the $xz$ projection of the halo orbit. The thrust from the VSI engine is generally directed radially inward toward $L_1$. Additionally, the phase shifting arc, as determined from impulsive maneuvers, follows a physical path of shorter distance. The parameter $\tau_0$ and the initial co-states for this trajectory are located in Table 5.4.

A more complete understanding of the trade-offs between the cost of the VSI engines and the corresponding impulsive maneuvers is available from an examination of solutions for different thrust durations and phase shifts in this same problem. To this end, Tables 5.5 and 5.6 summarize the results for variations in the phase shift parameter $P_S$ and the thrust duration $TD$, respectively. To represent the impulsive maneuvers in terms of the mass usage corresponding to a high-thrust engine, assume the specific impulse is $I_{sp,imp} = 450$ seconds; the fuel consumed by the VSI engine is defined $m_{prop,VSI} = m_0 - m_f$. As is apparent in the tables, there are several
Projection onto $yz$-plane; motion clockwise around orbit.

Projection onto $xz$-plane

Figure 5.7. Phase shifts for $P_S = 2.1712$ days, $TD = 2.3884$ days

Table 5.4 Optimal solution for $L_1$ halo phase shift, $P_S = 2.1712$ days, $TD = 2.3884$ days

<table>
<thead>
<tr>
<th>$X_{PS}$ Element</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0$, days</td>
<td>6.679786</td>
</tr>
<tr>
<td>$\lambda_{rx}$</td>
<td>-1.036781</td>
</tr>
<tr>
<td>$\lambda_{ry}$</td>
<td>-0.081309</td>
</tr>
<tr>
<td>$\lambda_{rz}$</td>
<td>1.797388</td>
</tr>
<tr>
<td>$\lambda_{vx}$</td>
<td>-0.367652</td>
</tr>
<tr>
<td>$\lambda_{vy}$</td>
<td>0.120284</td>
</tr>
<tr>
<td>$\lambda_{vz}$</td>
<td>0.595317</td>
</tr>
</tbody>
</table>

instances when even non-optimal impulsive maneuvers out-perform the optimal VSI arc, at least from the perspective of fuel mass consumed. However, these cases are also characterized by an average VSI specific impulse relatively close to the specific
impulse assumed for the conventional high-thrust engine. In reality, a VSI engine is operated at much higher values of specific impulse than a conventional thrust system.

The proper selection of a propulsion system, that is, comparing transfers and phase shifts between various engine types may offer novel concepts for mission planning. As an example, consider the phase shift $P_S = -2.1712$ days with $TD = 2.3884$ days, as plotted in Fig. 5.8. Both propulsion systems shift the phase by the proper amount, but the paths by the respective spacecraft are markedly different. Even though referencing Table 5.5 reveals that the impulsive manuevers are already more fuel efficient, the looping structure of the VSI phase shift indicates that it is likely
that additional propellant can be saved with a more sophisticated impulsive scheme. Further investigation is warranted.

Figure 5.8. Phase shifts for $P_S = -2.1712$ days, $TD = 2.3884$ days
5.4 Orbit transfers using two invariant manifold coasts

Trajectories in the vicinity of the Moon are not restricted to arcs near only one libration point, especially when extended- or end-of-life scenarios are considered. For trajectories that shift between libration points, a sensible design conserves fuel by exploiting the existing stable and unstable invariant manifolds associated with the periodic orbits that are defined as the initial and final orbits. For the $L_1$ and $L_2$ libration points, stable and unstable manifolds are apparent from the phase space associated with the initial and target orbits, so the parameters $\alpha_0$ and $\alpha_f$ are included in the 2PBVP. Two specific examples are investigated: (i) transferring from an $L_1$ halo orbit to an $L_2$ halo orbit; and, (ii) transferring from an $L_1$ vertical orbit to an $L_2$ halo orbit.

5.4.1 Example 1: Halo-to-halo orbit transfers

For the halo-to-halo transfer, the initial halo is an orbit from the northern family associated with $L_1$ and the target halo is either a northern or southern halo near $L_2$ and with a Jacobi constant consistent with the initial halo, all in the Earth-Moon CR3BP. The information defining the departure and arrival halo orbits is summarized in Table 5.7. Note that the $z$-amplitude of the southern halo orbit is defined at the southern crossing of the $xz$-plane; the $z$-amplitudes of the northern halo orbits are consistent with Section 5.2.

<table>
<thead>
<tr>
<th>Type</th>
<th>Initial $L_1$ Orbit</th>
<th>Target $L_2$ Orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Northern Halo</td>
<td>Northern Halo</td>
</tr>
<tr>
<td>$z$-amplitude</td>
<td>34596</td>
<td>36997</td>
</tr>
<tr>
<td>Period (P)</td>
<td>12.080</td>
<td>14.480</td>
</tr>
<tr>
<td>Jacobi Constant</td>
<td>3.1149</td>
<td>3.1149</td>
</tr>
</tbody>
</table>
Two possible sets of transfers from the northern $L_1$ halo to both the northern and southern $L_2$ halos are plotted in Fig. 5.9. The red arcs (partially obscured in the image) are coasts along the unstable manifold as the path departs from the $L_1$ orbit. Green reflects coasts along the stable manifold toward of the $L_2$ arrival orbits, and gold represents the thrust arcs. The respective colored arrows indicate the direction of motion of the spacecraft. These two families are composed of transfers with thrust durations from approximately 0.65 to 13 days. The arrivals onto the stable manifolds associated with the $L_2$ northern halo are relatively spread out and it is apparent that the thrust arcs radically change the spacecraft motion while transitioning between the two manifolds; that is, the link between the $L_1$ unstable manifold and the $L_2$ stable manifold. In contrast, the transfer arcs to the southern $L_2$ halo appear generally consistent with the natural dynamics in the CR3BP, that is, thrust arcs appear to closely follow the paths of the invariant manifolds. The manifold parameters and initial co-states for the $TD = 0.65$ days transfers are listed in Table 5.8 and the control histories corresponding to these two trajectories are plotted in Fig. 5.10. Comparison of the $\lambda_{vy}$ co-states demonstrates that the $\hat{y}$ thrust components for the two thrusting arcs are opposite in sense. Thus, the $L_1$-northern-halo-to-$L_2$-southern-halo transfer arc generally thrusts towards the Moon, while the $L_1$ northern halo-to-$L_2$ northern halo powered arc incorporates a thrust direction away from the Moon. As is apparent in Fig. 5.11, increasing the thrust duration yields northern-to-southern transfers that are not characterized by a notable increase in overall transfer time nor is there a significant difference in the nature of the flight path. The black dot denotes the position at the beginning of thrust while the black triangle marks the terminal position of the thrust arc.

The relationship between thrust duration and final mass, apparent in Fig. 5.3, is similar in profile for all the transfers generated in this analysis. However, with the inclusion of the coast times along the invariant manifolds, the overall transfer time becomes an interesting new variable for comparison relative to final mass: this relationship is plotted in Fig. 5.12 for both target orbits in this example. Clearly,
Figure 5.9. Optimal transfers from a northern Earth-Moon $L_1$ halo to both northern and southern halos using invariant manifolds.

Table 5.8 Optimal solutions for $L_1$ halo-to-$L_2$ halo transfers, $TD = 0.651372$ days

<table>
<thead>
<tr>
<th>$X_S^*$ Element</th>
<th>North to North</th>
<th>North to South</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0$, days</td>
<td>8.911241</td>
<td>5.987270</td>
</tr>
<tr>
<td>$\alpha_0$, days</td>
<td>12.401702</td>
<td>12.335960</td>
</tr>
<tr>
<td>$\lambda_{rx}$</td>
<td>1.633695</td>
<td>-0.301078</td>
</tr>
<tr>
<td>$\lambda_{ry}$</td>
<td>-0.387813</td>
<td>2.066642</td>
</tr>
<tr>
<td>$\lambda_{rz}$</td>
<td>1.382599</td>
<td>-0.641259</td>
</tr>
<tr>
<td>$\lambda_{vx}$</td>
<td>0.226281</td>
<td>0.049607</td>
</tr>
<tr>
<td>$\lambda_{vy}$</td>
<td>-1.189463</td>
<td>-0.180066</td>
</tr>
<tr>
<td>$\lambda_{vz}$</td>
<td>-0.639796</td>
<td>-3.583915</td>
</tr>
<tr>
<td>$\alpha_f$, days</td>
<td>-16.527235</td>
<td>-15.563078</td>
</tr>
<tr>
<td>$\tau_f$, days</td>
<td>6.407347</td>
<td>8.322113</td>
</tr>
</tbody>
</table>
Figure 5.10. Control History for the $L_1$ halo-to-$L_2$ halo optimal manifold transfers, $TD = 0.651372$ days.

Figure 5.11. Specific northern $L_1$ halo to southern $L_2$ halo transfers.

(a) Thrust duration: 0.65 days   Thrust duration: 6.38 days   Thrust duration: 12.12 days
Transfer time: 28.55 days   Transfer time: 28.34 days   Transfer time: 29.23 days
the transfers to the southern $L_2$ halo are more fuel efficient and of shorter overall
duration than the transfers to the northern halo orbit. In this example, the transfer
that “switches the hemisphere” between the initial and target halos (i.e., a northern-
to-southern transfer) is more efficient in exploiting the natural system dynamics and,
thus, conserves both mass and flight time. Additional analysis is warranted. Furthermore, it is apparent that, for both families, there are some instances where a longer
thrust duration, in addition to using less mass, also achieves the goal with a shorter
overall transfer time.

![Figure 5.12. Behavior of final mass with respect to the total transfer
time between $L_1$ and $L_2$ halos.](image)

### 5.4.2 Example 2: Vertical-to-halo orbit transfers

In the second example that incorporates both stable and unstable invariant mani-
folds, the spacecraft transfers from an $L_1$ vertical orbit (Grebow [48]) to the same
two target $L_2$ halos that were examined in Example 1. The vertical orbit possesses a
different Jacobi constant than the target halos, so the use of the VSI engine also must
change the energy level of the natural spacecraft motion. The characteristics describ-
ing the vertical departure orbit as well as the two halo arrival orbits are summarized in Table 5.9.

Table 5.9 Vertical and halo $L_1$ and $L_2$ orbit characteristics.

<table>
<thead>
<tr>
<th>Type</th>
<th>Initial $L_1$ Orbit</th>
<th>Target $L_2$ Orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$-amplitude</td>
<td>97362</td>
<td>36997</td>
</tr>
<tr>
<td>Period (P)</td>
<td>18.238</td>
<td>14.480</td>
</tr>
<tr>
<td>Jacobi Constant</td>
<td>2.9793</td>
<td>3.1149</td>
</tr>
</tbody>
</table>

Two families of transfers incorporating invariant manifold coasts and originating from the $L_1$ vertical orbit with arrival at either of the $L_2$ halos appear in Fig. 5.13; the total thrust duration ranges from 0.65 to 4.7 days. Immediately noticeable is the symmetry associated with the two families: the families appear nearly mirrored across the $xy$-plane, as is most apparent in Fig. 5.13(b). This symmetry implies that the two families should also possess many other similar characteristics, which is indeed true. From a design perspective, this symmetry can be exploited by a focus on the analysis of transfers to one of the target halos and then translating the results to transfer arcs that deliver the spacecraft to the other halo orbit. As is inferred from Fig. 5.14, this translation is a relatively simple procedure: switch the signs on the $z$ position and $z$ velocity co-states and shift the initial orbit parameter $\tau_0$ by one-half period. Although the two families are constructed using a continuation process generated from initial solutions of differing thrust duration, the design vectors in the generating transfers in Table 5.10 also display the necessary sign switches to the $z$ co-states and the initial orbit parameter $\tau_0$. 
Figure 5.13. Two views of transfers from $L_1$ vertical to $L_2$ northern and southern halos.
Figure 5.14. Initial values of $L_1$ vertical to $L_2$ northern (black) and southern (blue) halo transfers.

Table 5.10 Generating solutions for the families of optimal transfers from $L_1$ vertical-to-$L_2$ halo orbits

<table>
<thead>
<tr>
<th>$X^*_5$ Element</th>
<th>$L_2$ North, $TD = 0.868496$ days</th>
<th>$L_2$ South, $TD = 0.434248$ days</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0$, days</td>
<td>12.369213</td>
<td>3.336346</td>
</tr>
<tr>
<td>$\alpha_0$, days</td>
<td>19.723969</td>
<td>19.807087</td>
</tr>
<tr>
<td>$\lambda_{rx}$</td>
<td>1.327334</td>
<td>2.979157</td>
</tr>
<tr>
<td>$\lambda_{ry}$</td>
<td>2.830894</td>
<td>5.918445</td>
</tr>
<tr>
<td>$\lambda_{rz}$</td>
<td>-1.724492</td>
<td>4.683892</td>
</tr>
<tr>
<td>$\lambda_{vx}$</td>
<td>0.741994</td>
<td>1.100041</td>
</tr>
<tr>
<td>$\lambda_{vy}$</td>
<td>-1.077153</td>
<td>-2.638947</td>
</tr>
<tr>
<td>$\lambda_{vz}$</td>
<td>-1.481400</td>
<td>2.697948</td>
</tr>
<tr>
<td>$\alpha_f$, days</td>
<td>-13.936787</td>
<td>-14.206993</td>
</tr>
<tr>
<td>$\tau_f$, days</td>
<td>3.502478</td>
<td>3.602611</td>
</tr>
</tbody>
</table>
5.5 Orbit transfers incorporating one invariant manifold coast

The triangular libration points $L_4$ and $L_5$ are attractive for observation and communications spacecraft because of the long-term stability of these equilibrium solutions. Therefore, transfers to a periodic orbit in the vicinity of $L_4$ are investigated. The initial orbit belongs to the $L_2$ planar Lyapunov family and the target orbit is a member of the $L_4$ short-period planar orbit family. The orbits are specifically selected with Jacobi constants that are equal. These departure and arrival orbits are determined in Irrgang [63], each with a Jacobi constant value $C = 2.9784$; information on the orbits is summarized in Table 5.11. For the short-period orbit associated with $L_4$, the initial state is denoted by the parameter $\tau_f = 0$, a location which is defined by a point along the orbit at the crossing of a line parallel to the axis $\hat{x}$ passing through $L_4$, where the crossing occurs in the direction of increasing $y$ such that $\dot{y} > 0$. Unfortunately, the target $L_4$ orbit possesses only a center manifold, so the parameter $\alpha_f$ and its accompanying transversality condition must, of necessity, be removed from the 2PBVP. However, a portion of the unstable invariant manifold associated with the initial or departure $L_2$ orbit passes through the vicinity of $L_4$, so an extensive coast arc is still realizable.

Table 5.11 Planar periodic orbit characteristics.

<table>
<thead>
<tr>
<th>Type</th>
<th>Initial Orbit</th>
<th>Final Orbit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_2$ planar Lyapunov</td>
<td>21.824 days</td>
<td>28.551 days</td>
</tr>
<tr>
<td>$L_4$ short period</td>
<td>2.9784</td>
<td>2.9784</td>
</tr>
</tbody>
</table>

A family of transfer arcs between the $L_2$ Lyapunov and the $L_4$ short-period orbit is plotted in Fig. 5.15. Note the close lunar passage of the unstable invariant manifold (red in the figure). The transfers in this family range in thrust duration from 0.0868 to 16.3277 days, in total transfer time from 37.8304 to 47.4616 days, and the final spacecraft mass is always between 454.2583 and 499.7230 kg. For the generating
transfer that subsequently produces this family, the initial condition of the optimal solution \( X^*_S \) to the 2PBVP is listed in Table 5.12. Note that \( \lambda_{rz} = \lambda_{uz} = 0 \) for all the trajectories in this family, indicating that all thrusting occurs in the \( xy \)-plane since this is a planar problem. In this instance, increasing the thrust duration increases both the total transfer time and the final mass of the spacecraft. Thus, a design application to satisfy some set of mission constraints likely requires a trade-off between transfer time and fuel efficiency.

Once again, the performance of the VSI spacecraft is employed to offer some comparison to conventional impulsive maneuvers. From the ideal rocket equation (5.8), an “equivalent” \( \Delta v \) for the low-thrust arc is computed using the mass consumed and the average specific impulse that is assumed for the thrust duration. The “equivalent” \( \Delta v \) behavior across the computed family is plotted in Fig. 5.16, where the \( \Delta v \) generally increases as the thrust duration is extended, except for a very small \( TD \) value. This increase in \( \Delta v \) is consistent when compared to the cost of similar, but non-optimal, impulsive transfers, as apparent in Table 5.13. Therefore, a search
Table 5.12 Optimal solution for $L_2$ planar Lyapunov-to-$L_4$ short period orbit, $TD = 0.0868$ days

<table>
<thead>
<tr>
<th>$X_S^*$ Element</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0$, days</td>
<td>7.775058</td>
</tr>
<tr>
<td>$\alpha_0$, days</td>
<td>37.743581</td>
</tr>
<tr>
<td>$\lambda_{rx}$</td>
<td>1.552856</td>
</tr>
<tr>
<td>$\lambda_{ry}$</td>
<td>4.989447</td>
</tr>
<tr>
<td>$\lambda_{rz}$</td>
<td>0.000000</td>
</tr>
<tr>
<td>$\lambda_{vx}$</td>
<td>-2.534419</td>
</tr>
<tr>
<td>$\lambda_{vy}$</td>
<td>0.787231</td>
</tr>
<tr>
<td>$\lambda_{vz}$</td>
<td>0.000000</td>
</tr>
<tr>
<td>$\tau_f$, days</td>
<td>2.711170</td>
</tr>
</tbody>
</table>
for an optimal impulsive maneuver transfer begins where the $L_2$ unstable manifold approaches the $L_4$ orbit.

Figure 5.16. Equivalent $\Delta v$ values corresponding to the various transfers along the $L_2$-to-$L_4$ orbit transfer family

Table 5.13 Low-thrust and impulsive $\Delta v$ costs for $L_2$-to-$L_4$ periodic orbit transfers.

<table>
<thead>
<tr>
<th>$TD$, days</th>
<th>$\Delta v_{VSL}$, m/s</th>
<th>$\Delta v_{Imp}$, m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0868</td>
<td>77.8078</td>
<td>77.7501</td>
</tr>
<tr>
<td>0.8685</td>
<td>77.7629</td>
<td>77.9524</td>
</tr>
<tr>
<td>5.1241</td>
<td>78.2124</td>
<td>84.9786</td>
</tr>
<tr>
<td>9.4666</td>
<td>79.1175</td>
<td>103.5790</td>
</tr>
<tr>
<td>16.3277</td>
<td>80.2172</td>
<td>149.1312</td>
</tr>
</tbody>
</table>
6. PATH CONSTRAINTS AND DIRECT OPTIMIZATION

To satisfy mission objectives or address spacecraft limitations, path constraints may be placed upon transfer paths. However, for the inclusion of path constraints in an indirect optimization scheme, the 2PBVP must be fully reformulated for each new constraint and any constraint combinations. To avoid reformulating the problem for every new constraint, a hybrid optimization strategy combines the low dimensionality of indirect methods with the flexibility of a direct approach. To develop the hybrid process, the direct optimization method of sequential quadratic programming is first introduced. Application of a path constraint to a known unconstrained transfer is then demonstrated.

6.1 Introducing Sequential Quadratic Programming

Direct optimization methods are explicitly formulated to extremize a cost function. This approach is advantageous in contrast to the reformulation as a boundary value problem that characterizes indirect schemes. Constrained direct optimization is therefore generally formulated as

\[
\min J(x) \quad (6.1)
\]

subject to the constraints

\[
g(x) \leq 0 \quad (6.2a)
\]

\[
h(x) = 0 \quad (6.2b)
\]

\[
x^L \leq x \leq x^U \quad (6.2c)
\]

where \(g(x)\) is an \(l\)-vector of inequality constraints, \(h(x)\) is an \(p\)-vector of equality constraints, and the state vector \(x\) is bound between lower and upper limits on each
state. A maximization problem is easily converted to a minimization problem by switching the sign (±) of the cost function and all of the inequality constraints are written as in Eq. (6.2a) by switching signs on the constraints as needed. Consistent with indirect optimization, the constraints are adjoined to the cost function to create the Lagrangian function Λ, i.e.,

\[
\Lambda(x) = J(x) + \sum_{j=1}^{l} \nu_j \{\max[0, g_j(x)]\} + \sum_{k=1}^{p} \nu_{l+k} |h_k(x)|
\] (6.3)

where, as in Ch. 4, the variables ν are Lagrange multipliers. Then, the well-known Karush-Kuhn-Tucker (KKT) conditions [24,25] for an optimal solution \(x^*\) are:

1. The solution \(x^*\) is feasible (all constraints in Eq. (6.2) are satisfied).

2. The relation

\[
\nu_j g_j(x^*) = 0
\] (6.4)

is separately satisfied for every \(j\) and where all \(\nu_j \geq 0\).

3. The gradient of the cost function and adjoined constraints vanishes, that is,

\[
\nabla J(x^*) + \sum_{j=1}^{l} \nu_j \nabla g_j(x^*) + \sum_{k=1}^{p} \nu_{l+k} \nabla h_k(x^*) = 0
\] (6.5)

for all \(\nu_j \geq 0\) and unrestricted \(\nu_{l+k}\).

These conditions ensure local optimality of the cost function either within or on the boundaries defined by the constraints.

Sequential Quadratic Programming (SQP), one type of non-linear programming (NLP) algorithm, optimizes the cost function by solving a series of sub-problems where the objective function is approximated as quadratic and the constraints are approximated as linear. The SQP method iterates the update equation

\[
x^q = x^{q-1} + \zeta s^q
\] (6.6)
where $\zeta$ dictates step size. The iteration process continues until a solution $x^q$ satisfies the KKT conditions for constrained optimality. The first sub-problem is the determination of the search direction $s^q$ by minimizing the quadratic function $Q$, that is

$$Q(s^q) = J(x^q) + \nabla J(x^q)^T s^q + \frac{1}{2} (s^q)^T B s^q,$$  \hspace{1cm} (6.7)

subject to the linear constraints

$$\nabla g_j(x^q)^T s^q + \sigma_j g_j(x^q) \leq 0 \quad j = 1, \ldots, l$$ \hspace{1cm} (6.8a)

$$\nabla h_k(x^q)^T s^q + \bar{\sigma} h_k(x^q) = 0 \quad k = 1, \ldots, p.$$ \hspace{1cm} (6.8b)

The matrix $B$ approximates the Hessian matrix of the Lagrangian, $\nabla^2 \Lambda$, and the parameters $\sigma_j$ and $\bar{\sigma}$ ensure that the sub-problem possesses a feasible solution. Because of its form, the search direction sub-problem is amenable to solution via one of the various quadratic programming techniques available in Nocedal [22]. The optimal step size $\zeta$ in Eq. (6.6) is the focus of the second sub-problem and is determined such that the Lagrangian function $\Lambda(x^q)$ is minimized. Because this sub-problem is one-dimensional, a simple line search method is employed to compute $\zeta$ [64]. Though this introduction to SQP is brief, a more thorough discussion is available in Eldersveld [23].

There are a variety of SQP implementations, but two popular algorithms are MATLAB’s fmincon and the SNOPT packages developed by Gill, Murray, and Saunders [27]. This investigation uses the SNOPT algorithm because of its ease in implementation, quick computation times, and the ability to mix analytical and numerical derivatives. In addition, the SNOPT package solves the search direction and step size sub-problems internally, and the approximation matrix $B$ is computed with no input from the user. One feature of SNOPT, however, is that all inequality constraints, such as those in Eq. (6.2a), must be converted to equality constraints through the inclusion of slack variables. So, to ensure $g_j(x) \leq 0$, the inequality constraints are rewritten

$$g_j(x) + \kappa_j^2 = 0$$  \hspace{1cm} (6.9)
where the slack variable $\kappa_j$ is squared to ensure it is always positive. If a constraint $g_j(x) \geq 0$ is required, then the sign preceding $\kappa_j$ in Eq. (6.9) is switched.

### 6.2 Transfers with Path Constraints on $I_{sp}$

Now one of the chief benefits of the multiple shooting formulation is displayed: easy addition of path constraints to the patch points along the thrust arc. As an example, an inequality constraint is placed on allowable $I_{sp}$ to simulate limitations on engine performance:

$$I_{sp,\text{limit,lower}} \leq I_{sp} \leq I_{sp,\text{limit,upper}}.$$  

(6.10)

The first step is the modification of the constraint function in Eq. (4.55) to include the new constraints on the patch points:

$$F_C(X_C) = \begin{cases} 
I_{sp,1}(\lambda_1) - I_{sp,\text{limit}} \pm \kappa_1^2 \\
\xi_2^t(\tau_0, \alpha_0, \lambda_1) - \xi_2 \\
I_{sp,2}(\tau_0, \alpha_0, \lambda_1) - I_{sp,\text{limit}} \pm \kappa_2^2 \\
\xi_3^t(\xi_2) - \xi_3 \\
\vdots \\
x_T^t(\xi_n) - x_T(\tau_0, \alpha_0) \\
I_{sp,T}(\xi_n) - I_{sp,\text{limit}} \pm \kappa_T^2 
\end{cases} = 0$$  

(6.11)

where the variables $\kappa_i$ are as defined in Section 6.1 and the appropriate sign ($\pm$) is selected for each node. The $I_{sp}$ constraint is applied to each of the interior patch points as well as the initial point (denoted by the subscript 1). In addition, the final constraint, with the subscript $T$, is applied at the final state along the thrust arc. This formulation assumes that only one of the $I_{sp}$ limits is being applied at each patch point, requiring \textit{a priori} knowledge of a rough approximation of the $I_{sp}$ magnitudes at each patch point and, thus, the nodes with an $I_{sp}$ constraint that is most likely to be violated. However, both constraints are easily applied at each patch point, if so desired. Also note that the transversality conditions are removed since SNOPT is
used to directly optimize $m_f$. The time-of-flight constraint is also removed because
the trajectory segments are now defined with a fixed duration. The fixed duration
along each segment ensures that the patch points do not simply move to a location
along the thrust arc where the $I_{sp}$ constraints are already satisfied. The state vector
is then rewritten

$$X_C = \begin{bmatrix}
\tau_0 \\
\alpha_0 \\
\lambda_1 \\
\kappa_1 \\
\xi_2 \\
\vdots \\
\xi_n \\
\kappa_n \\
\kappa_T \\
\alpha_f \\
\tau_f \\
\end{bmatrix}, \quad (6.12)$$
The elements of the constraint gradient are then represented as

\[ DF_C(X_C) = \begin{bmatrix}
0 & 0 & \frac{\partial I_{sp}}{\partial \lambda_v} & \pm 2\kappa_1 & 0 & \ldots & 0 \\
\Psi_1, x & \frac{\partial \varphi_1}{\partial r_0} & \Psi_1, x & \frac{\partial \varphi_1}{\partial \varphi_0} & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \frac{\partial I_{sp}}{\partial (m, \lambda)} & \pm 2\kappa_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \Psi_2 & 0 & -I_{14} & 0 & \ldots & 0 \\
0 & \ldots & \ldots & 0 & -I_{14} & 0 & 0 & 0 & 0 \\
0 & \ldots & \ldots & 0 & \frac{\partial I_{sp}}{\partial (m, \lambda)} & \pm 2\kappa_n & 0 & 0 & 0 \\
0 & \ldots & \ldots & 0 & \Psi_{x, n} & 0 & 0 & -\frac{\partial \varphi_n}{\partial \alpha_f} & -\frac{\partial \varphi_n}{\partial \gamma_f} \\
0 & \ldots & \ldots & 0 & \frac{\partial I_{sp}}{\partial (m, \lambda)} & \Psi_{n, m, \lambda} & 0 & \pm 2\kappa_T & 0 & 0
\end{bmatrix} \]

(6.13)

where this gradient matrix is relatively sparse. Additionally, no finite difference derivatives are required to construct the gradient matrix. Therefore, the states, constraints, objective function, and all gradients are inputs to the SNOPT routines. This combination of indirectly optimized thrust laws with directly optimized path constraints is termed a “hybrid” optimization method.

The specific trajectory to be constrained is the 9.77-day transfer from the first sample mission application, that is, a transfer from one $L_1$ halo orbit to another. The upper limit on $I_{sp}$ is specified as 5100 sec and 4500 sec is defined as the lower limit so that the unconstrained optimal transfer violates both constraints. Knowledge of the unconstrained $I_{sp}$ history is employed to position the patch points in preparation for the constrained optimization scheme. The thrust arc segments are then defined so that the patch points are located at times corresponding to the local extrema in $I_{sp}$ from the unconstrained trajectory. Applying the constraint on $I_{sp}$ only slightly shifts the physical path of the spacecraft and, indeed, the final mass is nearly equal, that is, the value is reduced only from 498.40 kg to 498.37 kg. Yet, the behavior of the $I_{sp}$ for the transfer is noticeably modified as demonstrated in Fig. 6.1. While the constrained
trajectory fits within the limits on $I_{sp}$, it is interesting to note that the lowest value of $I_{sp}$ has shifted in both magnitude and the time during engine operation.

![Graph showing time history of $I_{sp}$ for unconstrained (dashed) and constrained (blue) trajectory, lower $I_{sp}$ limit 4500 sec, upper $I_{sp}$ limit 5100 sec.](image)

Figure 6.1. Time history of $I_{sp}$ for unconstrained (dashed) and constrained (blue) trajectory, lower $I_{sp}$ limit 4500 sec, upper $I_{sp}$ limit 5100 sec.

Constraints on the highest and lowest allowable values of specific impulse are easily applied concurrently, so long as these constraints only slightly change the pre-existing upper and lower values. When applying more restrictive constraints simultaneously to both the upper and lower boundaries on $I_{sp}$, the hybrid optimizer usually fails to determine a solution that completely satisfies the constraints. However, the hybrid algorithm converges quite readily if a restrictive constraint is applied only to the maximum value of $I_{sp}$. For example, the specific impulse history for an $I_{sp}$ upper limit of 4500 sec, and no lower limit, appears in Fig. 6.2. In comparison to the previous example, the unconstrained transfer with a final mass of 498.40 kg evolves into the new constrained path that is produced with a final mass equal to 497.79 kg. The $I_{sp}$ histories are markedly different, however. Two local low extremes and a local high extreme are apparent in the middle of the unconstrained transfer path but are replaced by one local low once the constraint is applied. A local high in the $I_{sp}$ value
does appear near the beginning of the constrained transfer arc. This local high value violates the constraint on specific impulse, however. This constraint violation can be overcome in several ways, including:

- Tightening the constraints to be more restrictive than is absolutely necessary.
- Employing many additional node points distributed along the thrust arc.
- Repeating the hybrid optimization process several times, with each successive application using the previous constrained solution as the initial solution guess.

Each method offers certain advantages, so operational and design considerations will determine the solution appropriate solution method.

![Figure 6.2. Time history of $I_{sp}$ for unconstrained (dashed) and constrained (blue) trajectory, upper $I_{sp}$ limit 4500 sec.](image)

Though only one example constraint is applied, a mission incorporating libration point orbits will need to satisfy many constraints throughout the operational life of the spacecraft. In addition to investigating the application of individual constraint types, several constraints must be applied simultaneously to truly determine the capabilities of the hybrid optimization algorithm.
7. SUMMARY AND CONCLUSIONS

An examination of fuel-optimal transfers using a variable specific impulse engine is the focus of this investigation. Transfers between periodic orbits in the vicinity of the libration points of the Earth-Moon circular restricted three-body problem represent the specific application. To generate transfers with no path constraints, an indirect optimization scheme is implemented, allowing the incorporation of coast arcs along the invariant manifolds associated with the periodic orbits. Solutions are obtained via shooting methods because of the low dimensionality of the indirect method and the relatively short thrust intervals. Furthermore, the indirect optimization scheme is combined with direct optimization methods to create a hybrid algorithm that maintains the low-dimensionality of the indirect scheme but readily allows the application of path constraints. A variety of transfers between libration point orbits are generated and analyzed to gain insight into the underlying dynamics and the thrust options for mission design.

7.1 Transfers between Libration Point Orbits

The orbit transfers that are generated in this investigation offer an overview of several important considerations for mission design. If short-duration transfers between periodic orbits in the vicinity of the same libration point are desired, coast arcs along the invariant manifolds can be excluded without a significant increase in fuel mass consumption. In addition, for the particular $L_1$ halo orbit combination, the average specific impulse value of the VSI engine demonstrates linear growth with respect to increasing thrust duration. In contrast, the minimum and maximum values of $I_{sp}$ demonstrate logarithmic and exponential growth, respectively, for an increasing length of the thrust interval. Additionally, phase shifts along one periodic orbit are
implemented with the VSI engine. The thrust arcs for a 2.4 day thrust duration and 
-2.17 day phase shift offers novel insights into the design of transfers achieved with 
more conventional impulsive maneuvers.

With the inclusion of coasts along the invariant manifolds associated with periodic 
orbits, a richer set of transfers are available and a much deeper understanding of the 
underlying dynamics is gained. The transfers from an $L_1$ halo orbit to an $L_2$ halo orbit 
demonstrate the fuel advantage available by exploiting the natural system dynamics. 
This family of transfers also demonstrates that, when coasts along invariant manifolds 
are blended with thrust arcs, it is possible to generate a transfer with both lower fuel 
consumption and shorter transfer time. Furthermore, with clever exploitation of the 
symmetry in the CR3BP, certain types of transfers can be modified to produce mirror 
arcs. This feature is exhibited by transfers between a vertical orbit near $L_1$ and halo 
orbits in the $L_2$ region. Additional insights into the performance of the VSI engine 
and predictions for impulsive maneuvers are gained by a comparison of the fuel cost 
for a low-thrust transfer to an equivalent energy cost, that is, the $\Delta v$ costs for the 
same transfer.

7.2 Optimization Methodologies

The two major types of optimization strategies, direct and indirect, are combined 
to produce a hybrid optimization algorithm that rapidly generates solutions while 
simultaneously allowing the inclusion of path constraints. The calculus of variations 
is applied to convert an unconstrained optimization problem to a two-point boundary 
value problem with a low number of variables and conditions. This low dimensionality 
allows for rapid computation to produce a large number of optimal solutions. If 
path constraints are required to satisfy mission objectives, the indirect optimization 
scheme is blended with a direct optimization routine that addresses the objective 
and the constraints directly. The capability of this hybrid optimization approach is 
demonstrated by applying limits to the values of specific impulse to be delivered by
the VSI engine. Depending on the difference between the enforced limits and the unconstrained performance of the VSI engine, radically different thrust profiles can be generated.

### 7.3 Recommendations for Future Investigation

The study of trajectory design employing a VSI engine offers a wealth of opportunities for additional investigation and insights into parallel research topics. Some suggestions for future work include:

- Continue investigation of unconstrained transfers in the CR3BP, especially into various other types of periodic orbits and in other dynamical regions of interest, such as the vicinities of $L_3$ and $L_5$. Extension to higher fidelity models, whether through perturbation techniques or ephemeris models, is of high interest. In particular, the symmetry effects of CR3BP and their implications for higher fidelity models is intriguing.

- All transfers in this analysis are only assured of local optimality. Therefore, a more thorough investigation into the available trade space and the global optimization issues, along with the attendant increase in computational cost, is warranted.

- A wide variety of path constraints are required for mission design, but only one is analyzed in this investigation. Therefore, constraints such as line-of-sight and shadowing should be also implemented. Additionally, constraints should be combined so that the performance of the hybrid optimization scheme can be more fully examined.

- Investigation of the VSI engine thrust profiles has offered much insight for impulsive maneuver placement. The low-thrust transfers offer novel and non-intuitive possibilities for transfers to actually be accomplished using conventional high-
impulse engines in non-Keplerian regimes. Thus, further comparison between optimal low-thrust and impulsive maneuver trajectories is warranted.

- A comparison of low-thrust transfers that are achieved using constant specific impulse engines in contrast to VSI engines is beneficial. A method to convert results from one engine type to another could greatly expand solution options and save valuable time in mission design.

- All transfers in this investigation are open-loop and no guidance or error-correction issues have been addressed. For mission design, a closed-loop control scheme to return to or alter the nominal thrust arc is required. Thus, a robust control system should be developed and tested.

Future investigation can, in general, follow two non-mutually exclusive paths: develop further mission design capabilities for yet-to-be-realized propulsion systems or translate results and design capabilities to currently available hardware. Both branches offer tantalizing possibilities for mission design.
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