Representations of higher-dimensional Poincaré maps with applications to spacecraft trajectory design

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Abstract

The Circular Restricted Three-Body Problem (CR3BP) offers a useful framework for preliminary trajectory design in multi-body regimes. While the CR3BP offers a simplified model with useful symmetries, trajectory design in this dynamical regime is often nontrivial. It is essential to gain insight into the available solution space to generate trajectories that meet a variety of constraints. The Poincaré map is a powerful tool that, in combination with a constraint on the energy level, allows a reduction in dimension such that, for the planar problem, the system is reduced to two-dimensions and the phase space is fully represented by the projection onto a plane. In the spatial problem, however, Poincaré maps must represent at least four states and are therefore challenging to visualize. In this investigation, a method to represent the information contained in higher-dimensional Poincaré maps using a two-dimensional image for visualization is explored and is applied for trajectory design. Four-dimensional map representations are demonstrated to compute transfers, including heteroclinic and homoclinic connections, between periodic and quasi-periodic orbits in the spatial problem. Alternative maps, such as the periapse Poincaré map, require that at least five Cartesian states be represented. Map representations are generated to visualize the full state associated with crossings of the periapse map, and are employed to locate families of periodic orbits about the Moon, as well as transfers to these orbits via transit trajectories.

1. Introduction

The Circular Restricted Three-Body Problem (CR3BP) serves as a useful framework for preliminary trajectory design in a multi-body force environment. Libration point orbits are well known solutions that have been employed in numerous missions. Missions to libration point orbits in the Sun–Earth system include observatories in the vicinity of $L_1$ (ISEE-3 [33], SOHO [35], ACE [34], WIND [34], Genesis [34]), and near $L_2$ (WMAP [36] and the Herschel and Planck Space Observatories [13]). ARTEMIS was the first libration point mission in the Earth–Moon system; two spacecraft were maintained in large quasi-periodic orbits about the $L_1$ and $L_2$ points before entering long-term lunar orbits [16]. While the existence of periodic and quasi-periodic solutions near the libration points in the CR3BP is generally well understood, the design of trajectories that incorporate these orbits is nontrivial. To generate trajectories that meet a variety of constraints, tools that offer insight into the available solution space are essential. The Poincaré map is a powerful tool that is useful to evaluate the solution space and to compute trajectories with specified characteristics. In combination with a constraint on the energy level, Poincaré maps allow a reduction in dimension such that, for the planar problem, the system is reduced to two dimensions and the state space is fully represented via a planar projection. In the spatial problem, Poincaré maps must represent at least four state variables. In this investigation, methods to represent the available...

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information in higher-dimensional Poincaré maps are explored and are applied for trajectory design.

Previous researchers have employed a variety of techniques to facilitate trajectory design in the spatial problem. To examine behavior in the vicinity of the collinear libration points, Jorda and Masedemont [28], as well as Gómez et al. [19], obtain higher-order normal form expansions of the Hamiltonian in the vicinity of the equilibrium points to decouple the stable and unstable motion. Removing unstable behavior via a reduction to the center manifold yields a system with two degrees of freedom; thus, maps associated with periodic and quasi-periodic orbits in the spatial problem for a particular energy level are two-dimensional. Gómez et al. [18] compute the stable/unstable manifold asymptotic to the center manifold associated with the collinear points for a particular energy level and employ a Poincaré map to reduce the dimension of the problem. By selecting two additional parameters, the four-dimensional maps are reduced to two-dimensions. Thus, the state space is fully represented by projecting the map onto a plane, and the problem of locating trajectories with prescribed characteristics is more tractable. Paskowitz and Scheeres [38] employ periastron maps to classify trajectory behavior in the spatial problem; these authors represent higher-dimensional maps by plotting vectors such that the basepoint represents the three-dimensional position state and the magnitude and direction of the vector indicate the velocity at periapsis.

The problem of representing multivariate data sets graphically is treated extensively in the field of data visualization. Strategies to represent complicated and interconnected information support the exploration of higher-dimensional data sets by exploiting the human ability to perceive and recognize patterns. While algorithms may be employed for data analysis, the development of such algorithms often requires a priori knowledge concerning the solutions of interest. Exploiting the capabilities of human perception allows the potential to reveal new or unexpected solutions. From a trajectory design perspective, a visual representation of the data allows the designer to both develop intuition about the solution space and to recall trends and conclusions. Visual representations for Poincaré maps additionally prove useful to facilitate an interactive trajectory design approach, based on user interaction with visual tools. The field of data visualization offers many strategies to aid in visually presenting multivariate data sets. Parallel coordinates [27], mosaic plots [23], scatterplot matrices, and trellis displays [4] are examples of modifications to the conventional two-axis plot that enable visualization of multivariate data. An alternative approach offered from data visualization is the use of glyphs, i.e., graphical objects whose physical characteristics are determined by a data set. A variety of glyph definitions have been previously demonstrated, and are discussed by Ward [46].

Glyphs prove useful for the development of a visual representation for higher-dimensional Poincaré maps. These map representations provide a wealth of insight into the solution space, and facilitate the location of a variety of new solutions. In this investigation, the use of glyphs is demonstrated to visualize four or more state variables associated with the crossings of a Poincaré map, and the resulting map representations are demonstrated for several mission design scenarios, including the search for maneuver-free transfers between libration point orbits in the spatial problem in both the Earth–Moon and Sun–Earth systems.

Poincaré maps are also useful to locate solutions such as transit and non-transit trajectories [8,18,30] by propagating a batch of initial conditions and examining subsequent returns to the map. The periapsis Poincaré map has been employed by several researchers to investigate the dynamics in the vicinity of the smaller primary in both the planar and spatial problems [9,10,21,22,26,38,45]. In the planar problem, the projection of the periapsis map onto configuration space provides a complete and intuitive representation of the state space. Such maps offer insight into the dynamics facilitating temporary-capture and escape in the vicinity of the smaller primary. To fully represent the periapsis map in terms of Cartesian coordinates in the spatial problem, at least five states must be represented. Sample periapsis map representations are generated to display the full six-dimensional Cartesian state space, and these maps are employed to examine regions of transit, as well as regions of long-term capture. Periodic orbits about the Moon are located by visual inspection of the structures associated with nearby capture orbits on the map, and are used to initialize continuation schemes to compute families of periodic orbits. By observing map crossings associated with transit trajectories in the vicinity of the periodic orbits, a sample transfer from the Earth to one of these lunar capture orbits is computed.

2. Fundamental motions in the CR3BP

In the CR3BP, the motion of a spacecraft, assumed massless, is examined as it moves in the vicinity of two primary bodies, $P_1$ and $P_2$. A rotating frame, centered at the system barycenter, $B$, is defined such that the rotating $\mathbf{x}$–axis is directed from the larger primary ($P_1$) to the smaller ($P_2$), the $\mathbf{z}$–axis is parallel to the direction of the angular velocity of the primary system, and the $\mathbf{y}$–axis completes the dextral, orthonormal triad. The system is nondimensionalized using the following characteristic quantities: total mass, $m^* = m_1 + m_2$; the distance between the primaries, $e^*$; and, characteristic time, $t^* = \sqrt{e^*/Gm^*}$. The nondimensional distances to the primaries are $r_1 = \mu$, $r_2 = 1 - \mu$, where $\mu = m_2/m^*$. The position vectors, defined in terms of rotating coordinates relative to $B$, $P_1$, and $P_2$, are $\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$, $\mathbf{r}_{13} = (x - \mu)x + \mathbf{y} + \mathbf{z}$, and $\mathbf{r}_{23} = (x - 1 + \mu)x + \mathbf{y} + \mathbf{z}$. The first-order vector equation of motion for $P_3$ is written with respect to the rotating frame in dimensionless variables as

$$\mathbf{\dot{x}} = \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{f} = [\mathbf{x}, \mathbf{y}, \mathbf{z}, 2\mathbf{y} + \Omega_x, -2\dot{x} + \Omega_y, \Omega_z]^T.$$
The pseudo-potential, $\Omega$, is defined as
\begin{equation}
\Omega(x, y, z) = \frac{1 - \mu}{r^2_{13}} + \frac{\mu}{r^2_{23}} + \frac{1}{2} \left( x^2 + y^2 \right),
\end{equation}
and the quantities $\Omega_x$, $\Omega_y$, $\Omega_z$ represent partial derivatives of $\Omega$ with respect to rotating variables. Five libration points exist, and are depicted in Fig. 1. The integral of the motion, denoted the Jacobi constant, is represented as
\begin{equation}
C = 2\Omega(x, y, z) - v^2,
\end{equation}
where $v = \sqrt{x^2 + y^2 + z^2}$.

Within the framework of the CR3BP, several fundamental solutions exist that prove useful in mission design. Families of periodic and quasi-periodic solutions exist within the vicinity of the collinear libration points and have been studied by several researchers [5,6,11,12,14,17,25,29,43]. Some examples include the planar and vertical Lyapunov orbits, as well as the family of halo orbits; sample orbits about the $L_1$ and $L_2$ points in the Earth–Moon system are plotted in Fig. 2. For unstable periodic orbits in the CR3BP, invariant manifold structures exist that provide transport into and away from the orbits. A trajectory along the stable/unstable manifold is computed by introducing a perturbation in the state along a periodic orbit (period $T$) in the direction of the stable/unstable eigenvector. Assume that $\lambda_s < 1$ and $\lambda_u = 1/\lambda_s$ are the stable and unstable eigenvalues from the monodromy matrix, $\Phi(t_0 + T, t_0)$, corresponding to a fixed point $x^*$ associated with time $t_0$ along an unstable periodic orbit. Let $w^+$ and $w^-$ be their associated eigenvectors, computed by solving the equations $\Phi(t_0 + T, t_0)w^+ = e^{\lambda_s T}w^+$, $\Phi(t_0 + T, t_0)w^- = e^{\lambda_u T}w^-$. Define $w^+$ and $w^-$ as the two directions associated with each eigenvector. A particular trajectory along the local half-manifold, $W^{w^+}_{loc}$ or $W^{w^-}_{loc}$, is approximated by introducing a perturbation relative to the fixed point, $x^*$, along the periodic orbit in the direction $w^+$ or $w^-$, respectively. Likewise, a perturbation relative to $x^*$ in the directions $w^+$ or $w^-$, respectively, produces a trajectory along the local half-manifold $W^{w^+}_{loc}$ or $W^{w^-}_{loc}$. The step along the direction of the eigenvector is denoted $d$, and the initial state along the local stable manifold trajectories are $x^{w^+} = x^* + d \cdot w^+$, $x^{w^-} = x^* - d \cdot w^-$, where $w^+$ and $w^-$ are normalized so that the vector containing the position components of the eigenvector is of unit length; this normalization provides a physical meaning for the value of $d$ as a distance. The value of $d$ is critical because it determines the accuracy with which the local manifolds are approximated. For the purposes of this investigation, $d=20$ km is selected for the Earth–Moon system and yields errors on the order of $4 \rightarrow 250$ km and $0.2 \rightarrow 1.4$ m/s between $x^*(t_0 + T)$ and $x^*(2T)$. In the Sun–Earth system, $d=300$ km yields errors less than 700 km and 0.4 m/s for all examples.) An individual trajectory on the local stable manifold is globalized by propagating the states $x^{w^+}$ and $x^{w^-}$ in reverse-time in the nonlinear model. This process yields the numerical approximation for the global manifold $W^{w^+}_s$ and $W^{w^-}_s$. The same procedure is employed to approximate the global unstable manifolds, $W^{w^+}_u$ and $W^{w^-}_u$. The stable/unstable manifold surfaces $W^{w^+}_s$, $W^{w^-}_s$, $W^{w^+}_u$, and $W^{w^-}_u$, composed of global manifold trajectories $W^{w^+}_s$, $W^{w^-}_s$, $W^{w^+}_u$, and $W^{w^-}_u$, associated with each point along the periodic orbit $(x^*(t), t_0 \leq t \leq t_0 + T)$, reflect asymptotic flow into and away from the periodic orbit.

3. Poincaré maps

The Poincaré map is a valuable tool that offers insight into the complicated dynamics in the CR3BP. Defining a surface of section, $\Sigma$, a map is generated by propagating initial conditions and displaying crossings of the resulting trajectories with $\Sigma$. The use of a Poincaré section, in addition to a constraint on the value of Jacobi constant, reduces the dimension of the system by two. In the planar problem, the state space is, therefore, entirely represented by the projection onto a plane. As an example, consider the Poincaré map appearing in Fig. 3 that depicts a projection of the crossings of the unstable manifold associated with a planar $L_1$ Lyapunov orbit and the stable manifold asymptotic to an $L_2$ Lyapunov with the surface $\Sigma$ defined by $x=0$ in Moon-centered coordinates for $C=3.15$ in the Earth–Moon system; these intersections with $\Sigma$ form contours in the $y$--$y$ phase space. The map is one-sided, that is, only crossings in one direction ($x > 0$ in this example) are considered. For each point on the map, $x=0$, $y$ and $y$ are plotted in the map, and $\hat{x}$ is computed from the specified value of the Jacobi constant. Thus, an intersection of the two contours in the $y$--$y$ plane.
Fig. 3. A Poincaré map is demonstrated for the planar problem, and two heteroclinic connections are determined from the intersection of the contours on the map (Moon-centered view).

Fig. 4. Possible representations for crossings of higher-dimensional Poincaré maps.

used to represent the velocity states $\dot{y}$ and $\dot{z}$. The value $k$ is a scaling constant that is employed to improve visual clarity. The remaining state, $\dot{x}$, can be computed from the Jacobi constant. While the glyph in Fig. 4(a) is selected to represent four variables in this investigation, any other glyph definition could also be employed. This representation may be modified for use with alternative types of maps, for example, the periapse map.

Periapse maps are generated by considering intersections with the surface $\Sigma = \{x|g = 0, g > 0\}$, where $g = \sqrt{(x-x_P)^2+y^2+z^2}$ represents the magnitude of the position vector of the satellite, $P_3$, relative to either $P_1$ or $P_2$. Here, $x_P$ represents the $x$ location of the primary body $P_i$, i.e., $x_{P_1} = -\mu$, and $x_{P_2} = 1 - \mu$. The state is represented in spherical coordinates by five variables: the radius, $\rho$, in-plane angle $\theta$, out-of-plane angle $\phi$, and the angular rates $\dot{\theta}, \dot{\phi}$. One variable is eliminated through a constraint on the value of Jacobi constant, and the state is fully defined through four variables. Thus, a periapse map depicting $(\theta, \phi, \dot{\theta}, \dot{\phi})$ represents the full state associated with a periapse for a specified value of $C$. A map displayed in spherical coordinates may not offer an intuitive representation for the state space, however. Alternatively, the glyphs defined to represent four states may be extended to represent all six Cartesian states, as depicted in Fig. 4(b). In this map, information from the six state variables is displayed using a glyph comprised of a pair of connected vectors (subsequently denoted as a vector ‘chain’). The base point is defined as the periapse location in the $x–y$ plane. Then, the planar projection of the periapse position.
appears as the black point (\(\hat{A}\)) at location \((x,y)\) in the \(x-y\) plane. \(\hat{A}\) serves as the base point for the second coordinate frame, defined as the \(x-y\) plane, and the circle (\(\hat{B}\)) represents the in-plane velocity states \((x,y)\). A third coordinate plane is defined, with the point \(\hat{B}\) as its origin, and represents the out-of-plane components \(z\) and \(\hat{z}\). Thus, the square (\(\hat{C}\)) reflects the coordinates \((z,\hat{z})\) and the full state is represented by the ‘chain’ \(\hat{A}\hat{B}\hat{C}\hat{C}\). The constants \(k_1\) and \(k_2\) are used to scale the velocity and position states (only the \(z\) position is scaled in this example), respectively. This, a visualization for the full state associated with crossings of the periapeic map is realized in a two-dimensional image. While many alternate glyphs could be defined to represent four or six variables, the glyphs in Fig. 4 are employed for the examples in this investigation. Note that these glyphs are similar to the ray glyphs demonstrated by Friendly [15].

4. Heteroclinic and homoclinic connections between periodic and quasi-periodic orbits

One application of Poincaré maps for trajectory design is the location of maneuver-free heteroclinic or homoclinic connections between periodic or quasi-periodic orbits. Koon et al. [30], Gómez et al. [19], Barrabés et al. [2,3], and Parker et al. [37] demonstrate the use of a variety of Poincaré maps to locate heteroclinic connections between planar orbits in the CR3BP. Gómez et al. [18,19] produce heteroclinic and homoclinic connections associated with quasi-periodic orbits in the spatial problem. In this investigation, representations for higher-dimensional Poincaré maps employing glyphs are demonstrated for the location of free transfers between periodic and quasi-periodic orbits in the spatial problem.

4.1. Targeting heteroclinic and homoclinic connections

Poincaré maps are a useful tool to locate an initial guess that can be employed to initiate the search for a heteroclinic connection, however, a differential corrections process must be applied to compute a feasible transfer. Barrabés et al. [2] present a method to correct heteroclinic and homoclinic connections and to continue these orbits, thus, computing families of free transfers. In this investigation, two differential corrections procedures are employed to compute free transfers between (a) periodic or (b) quasi-periodic orbits, where the orbits exist in either the planar or spatial problems. Note that the converged transfers are heteroclinic in a numerical sense, that is, the trajectories are corrected to be continuous to within a tolerance of \(\sim 1\) cm and \(\sim 1 \times 10^{-8}\) m/s in the Sun–Earth examples and \(\sim 0.1\) mm and \(\sim 1 \times 10^{-9}\) m/s for Earth–Moon transfers. Within the corrections algorithms, numerical integration of the non-dimensional equations of motion is accomplished using a Runge–Kutta Prince–Dormand (8, 9) method with absolute and relative tolerances of \(10^{-14}\). The differential corrections schemes employed are described in the following sections.

4.1.1. Free transfers between periodic orbits

To locate heteroclinic connections between two periodic orbits, consider the schematic in Fig. 5. Several points along an initial guess for a transfer trajectory from ‘orbit 1’ to ‘orbit 2’ are identified and numbered. Note that the numbers associated with each point do not necessarily correspond to their sequence in time. Let the point numbered 1 represent the initial state \(X_1(t_{10})\) along periodic orbit 1, where \(T_1 = t_{10} - t_0\) is the orbital period. Point 2 represents the location \(X_2(t_{10} + r_1)\) along the orbit after a coast time of \(r_1\); departure onto the unstable manifold occurs at this location by stepping along the unstable eigenvector direction \(\mathbf{W}^U\). Point 3 represents this step and is computed as \(X_3(t_{10} + r_1) = \mathbf{d}_1 \cdot \mathbf{W}^U\), where \(\mathbf{d}_1\) may be positive or negative. The initial state, \(X_0(t_{10})\), along the unstable manifold arc is represented as point 4, and point 5 corresponds to the final state, \(X_6(t_{10})\), after a propagation time of \(t_{10} - t_0\). Similarly, points 6 and 7 depict the initial and final states, \(X_7(t_{10})\) and \(X_8(t_{10})\), along the stable manifold associated with orbit 2. The states associated with points 5 and 6 are selected from the map to generate the initial guess for the transfer (note that the unstable manifold is actually propagated in reverse-time from point 5 to point 4). Orbit 2 is defined by the initial state \(X_2(t_{20})\) at point 8, and \(T_2 = t_{20} - t_0\) is the orbit period. Finally, point 9 represents the location \(X_2(t_{20} + r_2)\) along the orbit after a coast time of \(r_2\), at which point the step onto the stable manifold, \(X_{10} = X_2(t_{20} + r_2) + \mathbf{d}_2 \cdot \mathbf{W}^S\), is depicted by point 10; note that \(\mathbf{d}_2\) may be positive or negative. In the corrections process, the quantities contained in \(\mathbf{X}\) are allowed to vary in order to satisfy the constraint \(\mathbf{F} = 0\), where

\[
\mathbf{X} = \begin{bmatrix}
X_1(t_{10})
\mathbf{T}_1
X_2(t_{20})
T_2
r_1
T_0
X_0(t_{20})
T_0
X_1(t_{10})
Y_1(t_{10})
Z_1(t_{10})
X_2(t_{10})
Y_2(t_{10})
Z_2(t_{10})
X_3(t_{10})
Y_3(t_{10})
Z_3(t_{10})
X_4(t_{10})
Y_4(t_{10})
Z_4(t_{10})
X_5(t_{10})
Y_5(t_{10})
Z_5(t_{10})
X_6(t_{10})
Y_6(t_{10})
Z_6(t_{10})
X_7(t_{10})
Y_7(t_{10})
Z_7(t_{10})
X_8(t_{10})
Y_8(t_{10})
Z_8(t_{10})
\end{bmatrix},
\mathbf{F} = \begin{bmatrix}
x_1(t_{10}) - x_1(t_{10})
y_1(t_{10}) - y_1(t_{10})
z_1(t_{10}) - z_1(t_{10})
x_2(t_{10}) - x_2(t_{10})
y_2(t_{10}) - y_2(t_{10})
z_2(t_{10}) - z_2(t_{10})
x_3(t_{10}) - x_3(t_{10})
y_3(t_{10}) - y_3(t_{10})
z_3(t_{10}) - z_3(t_{10})
x_4(t_{10}) - x_4(t_{10})
y_4(t_{10}) - y_4(t_{10})
z_4(t_{10}) - z_4(t_{10})
x_5(t_{10}) - x_5(t_{10})
y_5(t_{10}) - y_5(t_{10})
z_5(t_{10}) - z_5(t_{10})
x_6(t_{10}) - x_6(t_{10})
y_6(t_{10}) - y_6(t_{10})
z_6(t_{10}) - z_6(t_{10})
\end{bmatrix}.
\]

Here, the terms \(x_i(t_{20}) - x_i(t_{10})\), \(y_i(t_{20}) - y_i(t_{10})\), \(z_i(t_{20}) - z_i(t_{10})\), \(x_i(t_{20}) - x_i(t_{10})\), \(z_i(t_{20}) - z_i(t_{10})\), \(i = 1, 2\), are included to enforce that orbits 1 and 2 be periodic. Because the value of \(C\) is constant along the orbit, the remaining coordinate \(\mathbf{\dot{y}}\) is, necessarily, equal in magnitude for the initial and final states along a periodic orbit; the term \(\mathbf{\dot{y}}(t_{10}) - \mathbf{y}(t_{10})\) is included to additionally enforce that the direction of the velocity \(\mathbf{\dot{y}}\) is consistent between the initial and final states along the orbit. Note that the initial state along each periodic orbit is defined to be an \(x\)-axis crossing such that \(\mathbf{\dot{y}} \neq 0\). An additional constraint on the phase angle of the initial state may be included to retain the state location along
the x-axis, however, such a constraint was not included for the corrections in this investigation. The term \( \chi_d(t_{i0}) - \chi_d(t_{uf}) \) is included to enforce full-state continuity between the unstable and stable manifolds. Finally, the terms \( \chi_d(t_{i0}) - \chi_d(t_{if}) \) and \( \chi_d(t_{if}) - \chi_d(t_{i0}) \) are employed to constrain the trajectories \( \chi_d(t) \) and \( \chi_d(t) \) to the unstable and stable manifold surfaces associated with orbits 1 and 2, respectively. Solutions that satisfy the equation \( T = \Omega \) represent a fully continuous transfer arc between two periodic orbits, to within the specified tolerance. A discontinuity exists between the manifold state \( \chi_{m1} \) and orbit state \( \chi_i(t_{i0} + \tau_i) \) \((i = 1 \text{ or } 2)\), however, the value of \( \tau_i \) is selected such that propagating \( \chi_{m1} \) for \( 2 \cdot \tau_i \) in reverse-time for \( i = 1 \) or forward-time for \( i = 2 \) yields a trajectory that remains in the vicinity of the periodic orbit. Due to the sensitivity of these trajectories to perturbations in the initial conditions, additional patch points are included along the periodic orbits and the manifold arcs. Because the periodic orbits are not propagated as one continuous arc, the identity \( \Phi(t_f, t_{i0}) = \Phi(t_{if}, t_{in}) \cdot \Phi(t_{in}, t_{in-1}) \cdots \Phi(t_{i1}, t_{i0}) \), where \( t_{i0} < t_{i1} < \cdots < t_{in} < t_{y} \) and \( i = 1, 2 \), is employed to approximate the monodromy matrix associated with orbits 1 and 2 during the differential corrections process. By selecting orbit 2 equal to orbit 1, homoclinic connections associated with periodic orbits are located, while selecting unique orbits for orbits 1 and 2 enables the computation of heteroclinic connections.

### 4.1.2. Free transfers between quasi-periodic orbits

In some cases, the targeting algorithm to locate a transfer between periodic orbits does not converge (i.e., a free transfer between periodic orbits does not exist in the vicinity of the initial guess). Then, the constraint that orbits 1 and 2 be periodic is removed and a connection between quasi-periodic orbits is sought. In this case, multiple revolutions around orbits 1 and 2 are incorporated into the initial guess, in addition to the unstable and stable manifold arcs from periodic orbits 1 and 2, respectively. Nodes are distributed along the orbits and manifold arcs, and the full state at each node is allowed to vary, in addition to the time-of-flight along each segment. Continuity in position and velocity is enforced between subsequent segments to yield a fully continuous transfer between quasi-periodic orbits in the vicinity of the original periodic orbits.

### 4.1.3. Low-cost transfers between periodic orbits

An alternative to removing the periodicity constraint, when the targeting algorithm to locate a maneuver-free transfer between periodic orbits does not converge, is to allow maneuvers along the path. To allow a maneuver between the unstable manifold of orbit 1 and the stable manifold of orbit 2, the constraint \( \chi_d(t_{i0}) - \chi_d(t_{uf}) \) in \( T \) should be replaced with the constraints \( \chi_i(t_{i0}) - \chi_i(t_{uf}), y_i(t_{i0}) - y_i(t_{uf}), \) \( z_i(t_{i0}) - z_i(t_{uf}) \), so that only position continuity is enforced between the manifold arcs. The cost function \( J = ((x_i(t_{i0}) - x_i(t_{uf}))^2 + (y_i(t_{i0}) - y_i(t_{uf}))^2 + (z_i(t_{i0}) - z_i(t_{uf}))^2)^{1/2} \) defines the magnitude of the \( \Delta v \) required to connect the unstable and stable manifolds. Applying optimization to minimize the function \( J \) yields a low-cost transfer between periodic orbits.

#### 4.2. Employing Poincaré maps to locate free or low-cost transfers

In the following example, the use of higher-dimensional Poincaré maps to locate free and low-cost transfers is demonstrated to compute a transfer between northern and southern halo orbits in the vicinity of \( L_1 \) and \( L_2 \), respectively, in the Sun–Earth system. (Low-thrust transfers between northern and southern halo orbits are demonstrated by Stuart et al. [42], to be more fuel efficient than transfers between the northern families). The halo orbit families in the CR3BP are composed of orbits, many of which have an unobscured line-of-sight to each of the primary locations, that are symmetric across the \( x-z \) plane. For this example, halo orbits in the vicinity of \( L_1 \) and \( L_2 \) are computed for the Jacobi constant value \( C = 3.00074 \), and the unstable and stable manifolds, respectively, are computed and propagated until their first crossing of the plane defined by \( \Sigma = \{x = 1 - \mu \} \) in nondimensional, barycentric coordinates. The manifolds, surface of section, and manifold crossings are plotted in Fig. 6(a). In the figure, red represents the unstable manifold associated with the \( L_1 \) northern halo orbit, and blue denotes the stable manifold associated with the \( L_2 \) southern halo orbit. To search for a free or low-cost transfer between the \( L_1 \) and \( L_2 \) orbits, it is necessary to locate points on the map that compare closely in both position and velocity. To view the velocity information for each point on the map, vectors representing the \( y \) and \( z \) components of the manifold velocity are ‘attached’ to each crossing position in Fig. 6(b). Points that lie very near the Earth correspond to high speeds; for visual clarity, the map in Fig. 6(b) includes only those crossings that lie beyond 3 Earth radii. The red and blue contours, formed by the vector basepoints on the map, intersect at two locations: one very near the Earth and one further away. A closer view of the intersection that lies further from the Earth reveals that the vectors at these locations are of similar magnitude and orientation, indicating that \( y \) and \( z \) compare closely between the stable and unstable manifold crossings. Thus, a heteroclinic connection may exist nearby.

To locate the map appearing in Fig. 6(b), several maps were first computed for various energy levels to identify an energy level corresponding to a low-cost transfer. For example, close views of the contour intersection region (such as the region appearing in the zoom box in Fig. 6(b) for \( C = 3.00074 \)) are demonstrated in Fig. 7: two maps corresponding to lower energy levels (higher values of \( C \))
than Fig. 6(b), i.e., $C = 3.0008$, and $C = 3.00077$ appear in Fig. 7(a) and (b); one map associated with a higher energy level, $C = 3.00071$, is demonstrated in Fig. 7(c). From the maps in Fig. 7(a) and (b), it is evident that the red and blue vectors nearest the contour intersection are converging to a similar orientation as the Jacobi constant value is reduced. The angle between the positive $y$-axis and the red vector is less than the angle between the $y$-axis and the blue vector, but the difference in these angles is decreasing as $C$ is reduced. For $C = 3.00074$, the red and blue vectors near the contour intersection in Fig. 6(b) appear nearly parallel. Further reducing $C$ to a value of 3.00071, as demonstrated in Fig. 7(c), the angle between the positive $y$-axis and the red vector is greater than the angle between the $y$-axis and the blue vector, indicating that values of $C$ around 3.00074 may be nearby a local minimum in the required $\Delta v$ to transfer between the $L_1$ halo orbit unstable and $L_2$ halo orbit stable manifolds. Thus, the map in Fig. 6(b) is employed to locate an initial guess for a free transfer between the $L_1$ and $L_2$ halo orbits.

To generate an initial guess for a heteroclinic connection, a stable and unstable manifold pair is selected from the map, and appears circled in black in Fig. 6(b). The corrections algorithm, outlined previously, for heteroclinic connections between periodic orbits is employed, however the corrector chatters at $l/F \sim 7 \times 10^{-4}$ (corresponding to errors on the order of $\sim 10^5$ km and $\sim 10$ m/s), suggesting that there may not exist a free transfer between periodic orbits in the vicinity of the initial guess. For this reason, the constraint that the halo orbits be precisely periodic is removed and a connection between quasi-periodic orbits is sought. To search for a connection, 15 revolutions along each halo orbit are incorporated into the initial guess and the targeting algorithm for free transfers between quasi-periodic orbits is employed. Each revolution along the halo orbit employed for the initial guess is discretized into 4 arcs. The unstable and stable manifold trajectories are discretized into 3 arcs, and two additional segments are included to bridge the discontinuities between the states along each halo orbit at times $t_0$ and the manifold departure/insertion locations $\bar{x}_m$ defined at time $t_0 + \tau_i$. The final solution corresponds to a total of 130 nodes along the transfer. The resulting maneuver-free connection between $L_1$ and $L_2$ quasi-halo orbits, closely bound to the periodic orbits from the initial guess, appears in Fig. 8. Note that a similar solution is demonstrated by Howell et al. [24]. The red and blue arcs represent the transfer leg, the initial guess for which corresponds to the unstable manifold of the $L_1$ halo and the stable manifold of the $L_2$ halo. Including 15 revolutions along the $L_1$ and $L_2$ quasi-halo orbits, the total time-of-flight is 16.15 years, while the red/blue transfer leg requires 1.10 years. All 15 revolutions are included in the plot in Fig. 8, however, are nearly indistinguishable as they are closely bound to the central periodic orbits from the initial guess. Alternatively, a low-cost transfer is sought between periodic halo orbits, as described in Section 4.1.3. Minimizing the velocity discontinuity between the unstable and stable manifold arcs, the solution depicted in Fig. 9 is computed. The transfer requires a maneuver of 2.233 m/s to connect the

Fig. 6. Map associated with manifolds corresponding to northern $L_1$ and southern $L_2$ halo orbits for $C = 3.00074$ (Earth-centered view). (a) Manifolds and surface of section and (b) map representation. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

Fig. 7. Close view of maps for various energy levels. (a) $C = 3.0008$, (b) $C = 3.00077$ and (c) $C = 3.00071$. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)
manifold arcs, and a total time-of-flight of 1.10 years. The periodic orbits are allowed to vary independently of one another within the optimization routine, thus, the Jacobi constant values associated with the $L_1$ halo and $L_2$ halo are different, $C = 3.0007425$ and $C = 3.0007387$, respectively. Of course, by the symmetry properties of the CR3BP, connections directed from the $L_2$ to the $L_1$ orbit are obtained through the transformation $y \rightarrow -y$, $x \rightarrow -x$, $z \rightarrow -z$.

Poincaré maps such as the one appearing in Fig. 6(b) are fast to compute ($\sim 0.2$ s for this map), and prove quite useful in an interactive trajectory design setting. The computation time to simulate 1000 trajectory arcs for 63 time units ($\approx 100$ years) in the Sun–Earth system is $\sim 2$–$3$ s, using MATLAB\textsuperscript{®} and calling subroutines in C to perform numerical integration. Likewise, the time to compute representative maps corresponding to a simulation of 100 time units ($\approx 1.2$ years) in the Earth–Moon system is $\sim 2$–$3$ s.

4.3. Heteroclinic connections between northern and southern halo family members in the Earth–Moon system

In the following examples, maps displaying invariant manifolds associated with halo orbits are demonstrated in the search for heteroclinic connections between members of the northern and southern halo orbit families associated with a single libration point, either $L_1$ or $L_2$. To compute such connections in the vicinity $L_1$, both the northern and southern $L_1$ halo orbits are computed for a specified Jacobi constant value. The unstable manifold (red) associated with the northern halo orbit and the stable manifold (blue) generated from the southern halo are propagated, and crossings of the one-sided surface of section defined by $x = 1 - \mu$, $\dot{x} > 0$, in nondimensional, barycentric coordinates, are recorded. Multiple maps are examined to locate an energy level for which the existence of a low-cost transfer is indicated by the crossings of the map. The selected Jacobi constant value is $C = C_{L_1} = 0.1 \approx 3.072$, and the resulting map appears in Fig. 10. From this map, initial guesses for two potential heteroclinic connections are identified within the zoomed views numbered 1 and 2 in Fig. 10, where the base points corresponding to the manifolds that were initially selected to generate the initial guesses are circled in black. The previously defined corrections algorithm for heteroclinic connections between periodic orbits is employed, and the free transfers that appear in Figs. 11 and 12 are computed. The final Jacobi constant values for these transfers are $\sim C = 3.07230$ and $\sim C = 3.07683$. Continuity along the manifold arcs and periodic orbits is enforced with a tolerance of $1 \times 10^{-12}$, corresponding to $\sim 10^{-4}$ m in position and $\sim 10^{-12}$ m/s in velocity. The step size off the orbit and onto each manifold is $d = 20$ km; the initial/final manifold states along the unstable/stable manifold are propagated in reverse-/forward-time (toward the halo.

Fig. 8. Free transfer between northern $L_1$ and southern $L_2$ quasi-halo orbits in the Sun–Earth system for $C = 3.0007403$; the entire path requires 16.15 years, with 1.10 years spent on the red/blue transfer leg (Earth-centered view). (a) Planar projection of transfer leg and (b) spatial view. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

Fig. 9. A low-cost transfer between a northern $L_1$ halo ($C = 3.0007425$) and a southern $L_2$ halo ($C = 3.0007387$) in the Sun–Earth system; a maneuver of 2.233 m/s is required to connect the red and blue arcs, and the transfer requires 1.10 years (Earth-centered view). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

Fig. 10. Map associated with northern and southern $L_1$ halo orbit manifold crossings with the surface of section $\Sigma = \{x = 1 - \mu, \dot{x} > 0\}$ for $C = 3.072$ in the Earth–Moon system (Moon-centered view).
orbits) for twice the orbit period to demonstrate that the
manifolds remain in the vicinity of the halo orbits for at
least two revolutions.

To search for heteroclinic connections between north-
eral and southern $L_2$ halo orbits near the same level of
Jacobi constant, $C = C_{L_2} - 0.1$, the map in Fig. 13, depicting
crossings of $\Sigma = \{x = 1 - \mu, \dot{x} > 0\}$ for the unstable
manifold (red) departing from the northern $L_2$ halo and the
stable manifold (blue) asymptotic to the southern halo
orbit, is computed. A potential heteroclinic connection is
identified, and the manifold crossings circled in the
zoomed view in Fig. 13 supply the initial guess for the
targeting algorithm. Employing the differential corrections
process, including periodicity constraints for the halo
orbits, yields the heteroclinic connection plotted in
Fig. 14. Again, the initial/final manifold states along the
unstable/stable manifold are propagated in reverse-/for-
ward-time for twice the orbit period, and remain in the
vicinity of the periodic orbit. Note that transfers from the
southern to the northern halo orbits are obtained by
reflecting the transfers in Figs. 11, 12, and 14 about the
$x-y$ plane.

Clear symmetries exist for the heteroclinic connections
demonstrated in Figs. 11, 12 and 14, however, these
transfers between northern and southern halos do not
appear to exist throughout the entire halo family;
perturbing the initial guess before applying the differential
corrections process yields the precise heteroclinic
connection located from the unperturbed initial guess
(both converged solutions correspond to the same value

\[ C = 3.07230, \text{ with a travel time along the stable/unstable manifold of 39.98 days (Moon-centered view).} \]

\[ C = 3.07683, \text{ with a travel time along the stable/unstable manifold of 43.75 days (Moon-centered view).} \]

\[ \Sigma = \{x = 1 - \mu, \dot{x} > 0\} \text{ for } C = 3.072 \text{ in the Earth–Moon system (Moon-centered view).} \]
of Jacobi constant), indicating that this type of heteroclinic connection likely does not exist for other halo orbits near the initial guess. This is supported by inspection of the maps for energy levels nearby those associated with the heteroclinic connections. For example, the connection appearing in Fig. 14 corresponds to the energy level $C = 3.06856$. Maps associated with the halo orbit manifold crossings for the energy levels $C = 3.06856 - 0.005$ and $C = 3.06856 + 0.005$ appear in Fig. 15(a) and (b). Here, the locations $(y, z)$ are plotted as solid points for each crossing of the map. The intersections between the red and blue contours identify the $(y, z)$ locations for which the stable and unstable manifold surfaces intersect in configuration space. Vectors representing the velocities $\dot{y}$ and $\dot{z}$ are included for selected manifold crossings that occur to the left and to the right of the intersection between the red and blue contours. For energy levels at which a heteroclinic connection exists, the length and orientation of the red and blue vectors should align, coincident with the intersection between the red and blue contours. Examining the map in Fig. 15(a), it is evident that the angle between the positive $y$-axis and the red vector is smaller than the angle between the $y$-axis and the blue vector, for crossings both on the left and right of the contour intersection. While the vectors associated with the contour intersection are not included on the plot, the observed behavior on the map in Fig. 15(a) suggests that they do not align. Similar behavior is noted for the map in Fig. 15(b), where the angle between the $y$-axis and the red vector is greater than the angle between the $y$-axis and the blue vector, both for crossings on the left and right of the contour intersection, again suggesting that a heteroclinic connection does not exist for this energy level.

### 4.4. Homoclinic connections associated with vertical orbits in the Earth–Moon and Sun–Earth systems

The vertical orbit families, emanating from each of the libration points, are composed of orbits that are symmetric across both the $x - y$ and $x - z$ planes, with many members that are highly inclined relative to $P_1$ and $P_2$. Representations for Poincaré maps associated with vertical orbit manifold crossings prove useful in the search for homoclinic connections associated with vertical orbits. Some sample maps appear for a variety of energy levels in Fig. 16. These maps are one-sided, and correspond to the surface of section $\Sigma = \{x | y = 0\}$. The vertical orbit manifolds are propagated into the interior region for the maps.
in Fig. 16(a), (b), and (d), and into the exterior region appear for the map in Fig. 16(c). In each example, an initial guess for a homoclinic connection is located and the corresponding manifold crossings are circled on the maps. The corrections algorithm to locate free transfers between periodic orbits is employed and yields the homoclinic connections appearing in Figs. 17–20. The corrected trajectories are continuous to a tolerance of $10^{-12}$, corresponding to $10^{-12}$ m in position and $10^{-12}$ km/s in velocity. The step off of the periodic orbit and onto each manifold is $d=20$ km; the initial/final manifold states along the unstable/stable manifold are propagated in reverse-/forward-time for twice the orbit period, demonstrating that the manifolds closely shadow the periodic orbit for at least two revolutions in each example. Due to the symmetries that exist in the CR3BP, for each point on the maps in Figs. 17–20 there exists a counterpart that is reflected across the $x$–$y$ plane so that $z \rightarrow -z$ and $\dot{z} \rightarrow -\dot{z}$. Thus, for each homoclinic connection depicted, there exists a second connection computed by reflecting these trajectories across the $x$–$y$ plane. Note that several of the homoclinic connections resemble resonant orbits. In particular, the transfer appearing in Fig. 16(b) possesses behavior similar to orbits within the family of 4:3 resonant orbits computed by Vaquero [44].

Homoclinic connections associated with libration point orbits in the Sun–Earth system offer a means for transport around the Earth’s orbit, with many solutions passing by the $L_4$ and $L_5$ libration points. To locate homoclinic transfers in the Sun–Earth system, maps of vertical orbit manifold crossings for two values of Jacobi constant appear in Fig. 21. The map in Fig. 21(a) corresponds to an energy level between the values of $C$ for the $L_2$ and $L_3$ points, indicating that zero-velocity curves exist and bound the motion in the $x$–$y$ plane. Selecting the manifolds associated with the crossings circled within the zoom view in Fig. 21(a) to supply the initial guess for the transfer, a homoclinic connection is located, using the differential corrections algorithm including periodicity constraints for the vertical orbits, and is displayed in Fig. 22. Because of the large in-plane amplitudes in the Sun–Earth system, the out-of-plane amplitudes are obscured, although the manifold in this connection oscillates between $\pm 2.28 \times 10^6$ km with a period of about 1 year. To achieve a transfer with larger $z$-amplitudes, a higher energy level, i.e., a lower value of Jacobi constant,
is considered. The map in Fig. 21(b) represents intersections of the stable/unstable manifold associated with an $L_1$ vertical orbit for $C = 0.95 \cdot C_L = 2.85$; note that the $z$-amplitudes on the map have increased by more than an order of magnitude compared with the manifold crossings in Fig. 21(a). A pair of stable and unstable manifold crossings is identified that nearly overlap on the map in Fig. 21(b), and is circled in black; the associated trajectories represent the initial guess for a homoclinic connection. Application of the differential corrections algorithm including periodicity constraints for the vertical orbits leads to divergence, however. Thus, the periodicity constraints are removed and 5 revolutions along the vertical orbit are incorporated into the initial guess, both at the start and end of the transfer, to search for a connection between quasi-periodic orbits in the vicinity of the vertical orbit. The resulting maneuver-free connection between quasi-periodic orbits (tightly bound to, and visually indistinguishable from the vertical orbits employed for the initial guess) appears in Fig. 23. The orbit of the Sun–Earth Trojan asteroid 2010 TK$_7$ from October 1, 2012 to 2050 is additionally plotted in green to provide a reference for the $z$-amplitude of the vertical orbit and the associated transfer. The time-of-flight along the asteroid path is roughly equal to the time required to travel along the segment of the transfer path that is plotted in red.
4.5. Computing periodic orbits from homoclinic connections

In the previous examples, representations for higher-dimensional Poincaré maps are demonstrated for the location of homoclinic connections associated with vertical orbits in the Earth–Moon and Sun–Earth systems. Each of these connections, located using maps defined for the surface of section $\Sigma = \{x = 0\}$, possesses a crossing of $\Sigma$ for which $\dot{x} = 0$. The homoclinic connection appearing in Fig. 17, further, possesses a perpendicular crossing of $\Sigma$, i.e., $\dot{x} = \dot{z} = 0$, and is symmetric about the $x$–$z$ plane. In theory, as $t \to \infty$, each of the homoclinic connections approaches the vertical orbit and inherits the perpendicular crossings associated with the periodic orbit; thus, the homoclinic connections presented could be considered periodic with infinite period. It is possible to compute nearby periodic orbits with finite period. Some sample orbits are located and appear in Fig. 24. Orbits 1a and 1b are computed using trajectory arcs from the solution in Fig. 17 to assemble the initial guess; the initial guess for orbit 1a includes a full revolution about the vertical orbit, while the vertical orbit is removed from the initial guess for orbit 1b. Similarly, orbits 2a and 2b are located using trajectory arcs from the homoclinic connection in Fig. 18 to assemble the initial guess. Again, while the initial guess for orbit 1a includes a full revolution about the vertical orbit, the vertical orbit is not included within the initial guess for orbit 2b. The computed orbits appear in Fig. 24. Orbits 1a and 1b are computed using orbits with finite period. Some sample orbits are located using maps defined for the stable/unstable manifold is $\mathbb{C}^2$. For example, sample members from the family containing orbit 1b are computed via pseudo-arclength continuation, and are plotted in Fig. 25. (For details on pseudo-arclength continuation strategies, see [12,44].) The motion of these orbits is nearly within the $x$–$y$ plane, until a close approach with the Moon causes a ‘backflip’ that draws the orbit out of the plane and into the nearly vertical phase of motion. The backflip for the orbit in Fig. 25(a) is on the far side of the Moon, while the backflip in Fig. 25(b) is on the near side.

5. Periapse maps in the spatial problem

The periapse surface of section, initially demonstrated by Villac and Scheeres [45], as well as Paskowitz and Scheeres [38], is useful as one of a number of tools for the characterization of trajectories in the CR3BP. The periapse map is defined by the surface of section $\Sigma = \{x = 0, \rho > 0\}$, where $\rho = \sqrt{(x-x_p)^2 + y^2 + z^2}$ is the radius between the satellite and the primary body. Periapse and apoapse maps are distinguished by the radial acceleration, i.e., $\ddot{\rho} > 0$ at periapsis and $\ddot{\rho} < 0$ at apoapsis.

Periapse maps have been previously applied to locate transit trajectories, solutions that pass between adjoining regions of the zero-velocity curves through the gateways at $L_1$, $L_2$, or $L_3$. As previously demonstrated by Conley [8], Koon et al. [30], and Gómez et al. [18], among others, the invariant manifold ‘tubes’ associated with planar Lyapunov orbits facilitate transit to and from the region surrounding the smaller primary, $P_2$, in the planar CR3BP. Thus, all planar transit trajectories lie within the manifold tubes associated with the $L_1$ and $L_2$ Lyapunov orbits for a given energy level. Villac and Scheeres [45], as well as Paskowitz and Scheeres [38], use periapse maps to classify transit behavior in the vicinity of $P_2$, and identify regions on the map corresponding to immediate escape from/entry to the $P_2$ region. Haapala and Howell [20–22] and Davis and Howell [10], compute the contours formed by periapse map crossings associated with the invariant manifolds asymptotic to $L_1$ and $L_2$ Lyapunov orbits to demonstrate...
that these contours bound the regions of immediate escape/entry formed by transit periapse passages, as expected from the work of Conley [8] and others. Periapse maps have additionally been employed to demonstrate their use for various mission design scenarios [9, 10, 21, 22, 26]. In this investigation, the application of a visual representation for higher-dimensional periapse maps to trajectory design is explored.

Fig. 22. A homoclinic connection ($C = 3.0005364$) in the Sun–Earth system is located using an initial guess from the map in Fig. 21(a); travel time along the stable/unstable manifold is 17.27 years. (a) Planar projection and (b) spatial view.

Fig. 23. A homoclinic connection ($C = 2.85005$) in the Sun–Earth system is located using an initial guess from the map in Fig. 21(b); travel time for the transfer is 75.00 years. (a) Planar projection and (b) spatial view. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

Fig. 24. Periodic orbits are computed in the Earth–Moon system. (a) Orbit 1a, $C = 2.84793$, 123.34 days. (b) Orbit 1b, $C = 2.84622$, 100.49 days. (c) Orbit 2a, $C = 2.79837$, 126.76 days. (d) Orbit 2b, $C = 2.80150$, 102.73 days.
While the projection of periapse maps into the $x/C_0$ plane provides a complete description of the state space in the planar problem, visualization of periapse maps is more complicated in the spatial problem. In the spatial problem, the map cannot be fully represented using four Cartesian state space variables, however, the four-dimensional map may be rendered in spherical coordinates. This representation may not be intuitive, but to transform the map to Cartesian coordinates requires that the map represent at least five states. In the following examples, all six states associated with crossings of the periapse map are visualized, as illustrated in Fig. 4(b).

To produce a periapse map in the spatial problem for the Earth–Moon system, a grid of positions in the vicinity of the Moon and within the boundaries formed by the zero-velocity surfaces and $x_{L_1}/C_0$ is generated. For each position location, a range of velocities are assigned such that $C = 3.15$, $\rho = 0$, and the state corresponds to prograde motion. To exploit the symmetries of the CR3BP, only points for which $z > 0$ are included in the grid of initial conditions; reflecting the trajectories, obtained by integrating the initial conditions from the grid, across the $x/C_0$ plane reveals the orbits that are generated by propagating initial conditions for which $z < 0$. These initial conditions are propagated for 2.5 years in both forward- and reverse-time and the following events are recorded: (i) each passage of periapsis, (ii) escape through the $L_1$ gateway as marked by crossing $x = x_{L_1} - 0.1$, (iii) escape through the $L_2$ gateway by crossing $x = x_{L_2} + 0.1$, where 0.1 is a nondimensional distance. Trajectories that cross either of the boundaries $x = x_{L_1} - 0.1$ or $x = x_{L_2} + 0.1$ are termed transit trajectories; those that do not cross either boundary are designated as non-transit trajectories, or long-term captures (note that these long-term ‘capture’ orbits may still escape the vicinity of the Moon beyond the 5-year interval). A projection of the periapse map associated with the resulting transit (blue) and non-transit (green) trajectories for $C = 3.15$ appears in Fig. 26(a).

A large variety of long-term capture trajectories, corresponding to the green periapses in Fig. 26(a), exist, including periodic orbits. Because the crossings of the periapse map belong to trajectories generated from randomly selected initial conditions, it is unlikely that a periodic orbit will appear directly on the map, however, stable periodic orbits may be located by observing the behavior of nearby crossings on the map. For example, consider the regions delineated by the columns plotted within the cluster of green long-term capture periapses in Fig. 26(b). Column 1 is defined by the boundaries $3.704 \times 10^5 \leq x \leq 3.718 \times 10^5$ km, $-600 \leq y \leq 500$ km, $-3 \times 10^4 \leq z \leq 3 \times 10^4$ km. Considering only periapses that lie within the boundaries of the column, and plotting each periapsis using the glyph definition appearing in Fig. 4(b), the resulting map appears in Fig. 27(a). A projection of the resulting map appears in Fig. 25(a). Members from the family of periodic orbits associated with orbit 1b in the Earth–Moon system. (a) $C=2.41026$, 88.67 days. (b) $C=2.55943$, 95.99 days.

Fig. 25. Members from the family of periodic orbits associated with orbit 1b in the Earth–Moon system. (a) $C=2.41026$, 88.67 days. (b) $C=2.55943$, 95.99 days.

Fig. 26. Periapse map, projected into configuration space, depicting periapses along transit (blue) and non-transit (green) trajectories for $C=3.15$ in the Earth–Moon system. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)
correspond to blue/red). A group of four cyan chains (including one circled in red) surrounds a central chain that is circled in green; these four chains are labelled 1–4 on the map. From visual inspection of the location and geometry of the chains, it is evident that each of these four periapses possesses values of \( y, x, \) and \( z \) that are nearly zero. It is also apparent that \( z < 0 \) for periapses 1 and 3, whereas \( z > 0 \) for periapses 2 and 4. These four periapses are each close to a mirror configuration [40], for which the velocity vector associated with the spacecraft is normal to the plane containing the primary bodies, \( P_1 \) and \( P_2 \), and the spacecraft, \( P_3 \). If this configuration occurs twice along a particular trajectory arc, then the associated orbit is periodic. The periapsis map associated with periapses inside the column labeled 2 in Fig. 26(b) appears in Fig. 27(b). Similar structures appear in this map, that is, a number of green and cyan colored periapses occur near \( y = 0 \) with apparently small \( x \) and \( z \) magnitudes, indicating that a periodic orbit may exist in the vicinity.

To seed the initial guess for a periodic orbit in the vicinity of the periapses in these maps, the periapses circled in red in Fig. 27(a) and (b) are employed. As an example, consider the red circled periapsis in Fig. 27(b). This periapsis, for which \( z < 0 \), is propagated in reverse-time until it again intersects the surface defined by \( y = 0, \ y > 0 \), yielding the red arc that appears in Fig. 28(a). The open red circle represents the location of the periapsis selected from the map in Fig. 27(b). For a periodic orbit, a second mirror configuration is expected at \( y = 0, \ z > 0 \); thus, the red arc is reflected across the \( x−y \) plane, yielding the green arc. The initial guess for the periodic orbit is composed of the red arc, which is re-integrated in forward-time, concatenated with the green arc, also integrated in forward-time. To aid convergence, the red and green arcs are sampled, as depicted in Fig. 28(a), and integrated in segments. To enforce periodicity of the orbit, a differential corrections process is employed in which the quantities contained in \( \tilde{X}_P \) are allowed to vary to satisfy the constraint \( \tilde{F}_P = 0 \):

\[
\tilde{X}_P = \begin{bmatrix}
X_1(t_{i0}) \\
X_2(t_{i0}) \\
\vdots \\
X_n(t_{i0}) \\
\end{bmatrix}
\begin{bmatrix}
t_{i0} \\
t_{i1} \\
\vdots \\
t_{in} \\
\end{bmatrix}
\begin{bmatrix}
\beta \\
\end{bmatrix}
\begin{bmatrix}
X_2(t_{i0}) \quad -X_1(t_{i1}) \\
X_3(t_{i0}) \quad -X_2(t_{i1}) \\
\vdots \\
X_n(t_{i0}) \quad -X_{n-1}(t_{i-1}) \\
\end{bmatrix}
\begin{bmatrix}
y_2(t_{i0}) \quad -y_1(t_{i1}) \\
y_3(t_{i0}) \quad -y_2(t_{i1}) \\
\vdots \\
y_n(t_{i0}) \quad -y_{n-1}(t_{i-1}) \\
\end{bmatrix}
\begin{bmatrix}
z_2(t_{i0}) \quad -z_1(t_{i1}) \\
z_3(t_{i0}) \quad -z_2(t_{i1}) \\
\vdots \\
z_n(t_{i0}) \quad -z_{n-1}(t_{i-1}) \\
\end{bmatrix}
\begin{bmatrix}
\dot{y}_2(t_{i0}) \quad -\text{sign}(y_1(t_{i1})) \cdot \beta^2 \\
\dot{y}_3(t_{i0}) \quad -\text{sign}(y_2(t_{i1})) \cdot \beta^2 \\
\vdots \\
\dot{y}_n(t_{i0}) \quad -\text{sign}(y_{n-1}(t_{i-1})) \cdot \beta^2 \\
\end{bmatrix}
\end{bmatrix}
\]

Full state continuity is enforced between adjoining segments along the orbit via the constraints \( \tilde{X}_i(t_{i0}) = \tilde{X}_{i-1}(t_{i-1}) \) for \( i = 2, \ldots, n \), where \( n \) is equal to the total number of nodes employed to discretize the red and green

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Fig. 27. Poincaré maps representing periapses along non-transit trajectories for inclinations \( 0 \leq i \leq 70^\circ \) in the Earth–Moon system. (a) Map corresponding to column 1 and (b) map corresponding to column 2. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

Fig. 28. Two periodic orbits are computed using initial guesses from periapse maps in the Earth–Moon system. (a) Correcting initial guess and (b) converged orbits. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)
arcs from Fig. 28(a). Here, $t_0$ and $t_f$ represent the initial and final times along the $i$-th trajectory arc, respectively. Continuity is additionally enforced between the initial state along the first arc and the final state along the $n$-th arc via the constraints $x_i(t_f) - x_i(t_0)$, $y_i(t_f) - y_i(t_0)$, $z_i(t_f) - z_i(t_10)$, $\dot{x}_i(t_f) - \dot{x}_i(t_10)$, $\dot{y}_i(t_f) - \dot{y}_i(t_10)$, $\dot{z}_i(t_f) - \dot{z}_i(t_10)$. Because the value of $C$ is constant along a continuous trajectory, the remaining coordinate $y$ is, necessarily, equal in magnitude for the initial and final states along a periodic orbit. The term $\dot{y}_i(t_f) - \text{sign}(y_i(t_10)) \cdot \beta$, where $\beta$ is a slack variable used to apply the inequality constraint, is included to additionally enforce that the direction of the velocity $\dot{y}$ is consistent between the initial and final states along the orbit. Note that the initial state along the periodic orbit is defined to be an $x$-axis crossing for which $\dot{y} \neq 0$. This corrections process yields the periodic orbit plotted in black in Fig. 28(a), as well as in red (thin line) in Fig. 28(b). The eccentricity relative to the Moon oscillates between 0.1 and 0.7, and the inclination relative to the $x$–$y$ plane varies between 37° and 43°. Note that orbits of this geometry are demonstrated by Michalodimitrakis [32] and Lara and Russel [31], and their role in connecting families of planar periodic orbits is examined. This same process is employed to locate a periodic orbit corresponding to an initial guess generated from periapsis 3, circled in red in Fig. 27(a), yielding the bold red orbit in Fig. 28(b). The eccentricity oscillates between 0.2 and 0.65, and the inclination oscillates between about 33° and 37°. The periapses for which $z < 0$ along these periodic orbits are added to the maps in Fig. 27 and circled in green. While the orbits in Fig. 28(b) are unstable, the maximum real part of the eigenvalues associated with each monodromy matrix is very small (approximately 5 and 3).

The orbits located in Fig. 28(b) belong to families of periodic orbits that bifurcate from planar orbit families. Employing a pseudo-arclength continuation scheme, initialized from the orbits in Fig. 28(b), families of lunar orbits are computed; sample family members are displayed in
In addition to the computation of long-term capture orbits about the Moon, periapse maps are also useful to locate transfers to such orbits. Consider the transit trajectories corresponding to the blue periapses in Fig. 26(a); the first periapses along the subset of these trajectories that enter through the $L_1$ gateway, i.e., the first periapses after entry through the $L_1$ gateway along $L_1$ transit trajectories, are displayed in Fig. 30(a). The full states associated with these periapses are, again, represented using the glyph definition from Fig. 4(b), and each periapse is colored by inclination relative to the $x$–$y$ plane such that low/high inclinations correspond to blue/red. A close view of the black box in Fig. 30(a) appears in Fig. 30(b), in addition to the $z < 0$ periapse, circled in green, from the black periodic orbit in Fig. 28(a). A transit periapse that is nearby the location of the periodic orbit perilune, and that possesses a similar velocity state, is identified through visual inspection of the map, and the basepoint of the corresponding glyph appears circled in black. Integrating the state associated with this glyph in reverse-time yields an $L_1$ transit trajectory that serves as an initial guess for a transfer from the vicinity of the Earth to the periodic orbit. A discontinuity of $\sim 700$ km and $\sim 70$ m/s exists between the transit periapse and the periapse along the periodic orbit. A differential corrections process is employed to enforce continuity in position along the path, and to restrict the velocity discontinuity to a maximum value of 5 m/s (note that this sample transfer is not optimized). The resulting transit trajectory is plotted in blue in Fig. 31. A transfer from a 200 km altitude circular Earth orbit to the transit apoapsis, relative to Earth, is included in green. The total $\Delta v$ required is approximately 3.74 km/s (633 m/s to insert onto the transit path and 5 m/s to transfer onto the periodic orbit). The combined time-of-flight along the green and blue transfer trajectories is 21.08 days, and the black periodic orbit has a period of roughly 14 days.

### 6. Conclusions

While the Circular Restricted Three-Body Problem offers a useful framework for preliminary trajectory design in a multi-body force environment, mission design in this dynamical regime is nontrivial. To facilitate trajectory design in such an environment, it is desirable to develop tools that provide insight into the available solution space. The Poincaré map is a powerful tool that has proven useful to evaluate the solution space and to compute trajectories with specified characteristics. In this investigation, a method to represent the information contained in higher-dimensional Poincaré maps using a two-dimensional image is explored and is applied to trajectory design. Four-dimensional map representations are demonstrated to display the invariant manifolds associated with both halo and vertical orbits in the spatial problem for the Earth–Moon and Sun–Earth systems. Through visual inspection of these maps, free transfers (heteroclinic and homoclinic connections) associated with a variety of periodic orbits are located. Alternative maps, such as the periapse map, require that at least five Cartesian states be represented. Sample map representations to display all six Cartesian states associated with periapse map crossings are examined, and visual examination of the maps reveals structures that suggest the existence of periodic orbits. An initial guess, located from the map, is generated and a differential corrections process is employed to locate the periodic orbits. Two symmetric lunar orbits are computed and provide a starting point to generate families of periodic orbits about the Moon via numerical continuation. The periapse map representations are additionally employed to locate transfers to these orbits via transit trajectories.

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