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Families of Orbits in the Vicinity of the Collinear Libration Points

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Abstract

In recent years, three-dimensional periodic and quasi-periodic orbits near the collinear libration points in the Sun-Earth/Moon three-body problem have been the focus of great interest for space mission design. Thus, more effort in the astrodynamics community has been directed toward analysis and computation of families of such orbits. But families of periodic orbits in the three-body problem have been the subject of much study for many decades. The first such orbit to be employed for a spacecraft trajectory (ISEE-3) was a member of a particular type of simply symmetric three-dimensional family, i.e., a halo orbit. Thus, attention shifted to further analysis of these periodic halo families. Some of the significant work in the development of periodic solutions in the three-body problem has been reviewed and a number of the highlights from the analysis and eventual numerical computation of halo families is presented here. The halo families of periodic orbits extend from each of the libration points to the nearest primary; they appear to exist for all values of the mass ratio. Thus, this further understanding may serve to support future spacecraft mission planning as well.

Introduction

In 1964, Robert Farquhar enrolled at Stanford University where Professor John Breakwell would eventually agree to become his thesis advisor [1]. Near the same time, a challenging problem had captured the interest of Ralph Pringle at Lockheed. In a small study for NASA Marshall, Pringle, who previously collaborated with Breakwell, was investigating spacecraft oscillating about the Earth-Moon L_2 libration point for a communications satellite that could service the backside of the Moon—work that was never published. The suggested solution involved a satellite that would oscillate back and forth in the plane of motion of the Earth and Moon. Such a solution is unsatisfactory because the satellite periodically passes behind the Moon and is not visible from Earth. Farquhar suggested that the problem be addressed by using an out-of-plane solution. The in-plane and out-of-plane frequencies of the motion are close but not exact; the satellite is still out of sight for some time. However, Farquhar's idea incorporated a forced orbit using periodic out-of-plane

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pulsing [2]. A few months later, Breakwell suggested an alternative continuous controller to force the frequencies to be equal, thus ensuring a periodic out-of-plane solution that is always visible from Earth. This control was slightly cheaper but not very practical [2]. Hence, the basic concept for a "halo orbit" was born. This view of the origin of halo orbits and halo families for space applications is one that is now most familiar to those in the astrodynamics community. And, indeed, this independent insight into bounded motion in the vicinity of the collinear libration points has opened new directions for spacecraft mission design. Breakwell and Farquhar each pursued the ideas further. In 1973, Farquhar and Kamel published higher order approximations to predict the existence of large natural periodic solutions in the Sun-Earth/Moon four-body problem; these would later lead to the numerically produced halo orbits that are now familiar to many trajectory analysts and mission planners. The naturally occurring solutions are characterized by a specified relationship between the in-plane and out-of-plane amplitudes. Because they are inherently unstable, stationkeeping maneuvers are required. Numerous other aspects of the problem were investigated, many reflected in Farquhar's Ph.D. thesis. Farquhar, as well as Breakwell, each continued to examine nominal trajectory design as well as potential stationkeeping strategies for spacecraft that might use such unstable orbits. Much time and effort was devoted to the translation-control problem for a libration-point satellite. Farquhar also offered a number of ideas for possible libration point missions and it is not surprising that many of the libration point missions proposed for the 1990's built on these early suggestions. Farquhar completed his thesis in 1968 [4].

Other Early Contributors

Of course, Breakwell and Farquhar were not the first to seek periodic solutions in the three-body problem. The three-body problem has been the focus of much mathematical and scientific interest, and, at least since Poincaré [5] published his fundamental work in 1892, much of the serious attention has focused on the search for periodic solutions. (In fact, Poincaré considered periodic orbits the only access to understanding the behavior in the difficult three-body problem.) Early investigations quickly narrowed the study to solutions in the planar problem and a number of families of periodic orbits were identified. By 1920, however, Moulton [6] considered the three-dimensional problem and, significantly for this discussion, he was interested in "oscillating satellites" in the vicinity of the collinear libration points. His ultimate goal would be to compute a periodic solution for such a satellite. "If it is given such an initial displacement that it revolves in the vicinity of the point of equilibrium in an orbit closed relative to the moving system, it is called an *oscillating satellite;* for, as seen from the Earth, it oscillates in the neighborhood of the equilibrium point in an apparently closed orbit."[2]

Moulton did prove that there are three types of finite periodic solutions that are generated from the infinitesimal solutions at the collinear points. Consider the linearized motion near the collinear libration point and the variational equations, relative to the equilibrium point, of the form

$$\dot{\boldsymbol{\xi}} = A\boldsymbol{\xi} \tag{1}$$

where $\boldsymbol{\xi} = (\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta})$ and A is a constant matrix. The solution for the linear variational equations can be written as

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[2] Moulton, 1920, p. 152.

$$\xi = \xi_O \cos \omega_{xy} t + \frac{\eta_O}{\alpha} \sin \omega_{xy} t \tag{2a}$$

$$\eta = \eta_O \cos \omega_{xy} t - \alpha \xi_O \sin \omega_{xy} t \tag{2b}$$

$$\zeta = c_1 \cos \omega_z t + c_2 \sin \omega_z t \tag{2c}$$

where ω_{xy} is the in-plane angular velocity for any of the three collinear libration points $L_{1,2,3}$ and ω_z is the angular velocity in the out-of-plane direction. The solution in equation (2) can be periodic in three cases:

1. $\xi = \eta = 0$ and ζ is of the form in (c).
2. $\zeta = 0$ and ξ, η are of the form (a), (b) respectively.
3. ξ, η, ζ are of the forms (a), (b), (c) respectively, and the two angular velocities ω_{xy}, ω_z are commensurable.

Moulton [6] then proved the existence of finite periodic solutions in three dimensions that are generated from these three types of infinitesimal periodic oscillations (a), (b), and (c). He also presented some numerical examples of the first two types, i.e., he calculated a small section of the families associated with two of the three existing types. Moulton's sketch of finite periodic solutions of the first, nearly vertical, type appears in Fig. 1. Such calculations relied heavily on analytical approximations based on assumptions concerning the structure of generating solutions near the equilibrium solutions. But computational tools were limited at the time and Moulton speculated that it would be "practically impossible" to determine three-dimensional orbits of the third type numerically. But, planar analysis continued. Beginning around 1922, Elis Strömgren and his colleagues at the Copenhagen Observatory identified categories for periodic orbits in the three-body problem [7]. Of fourteen different classes, those in classes *a*, *b*, and *c* are defined as retrograde periodic orbits around L_3, L_2, and L_1, respectively.

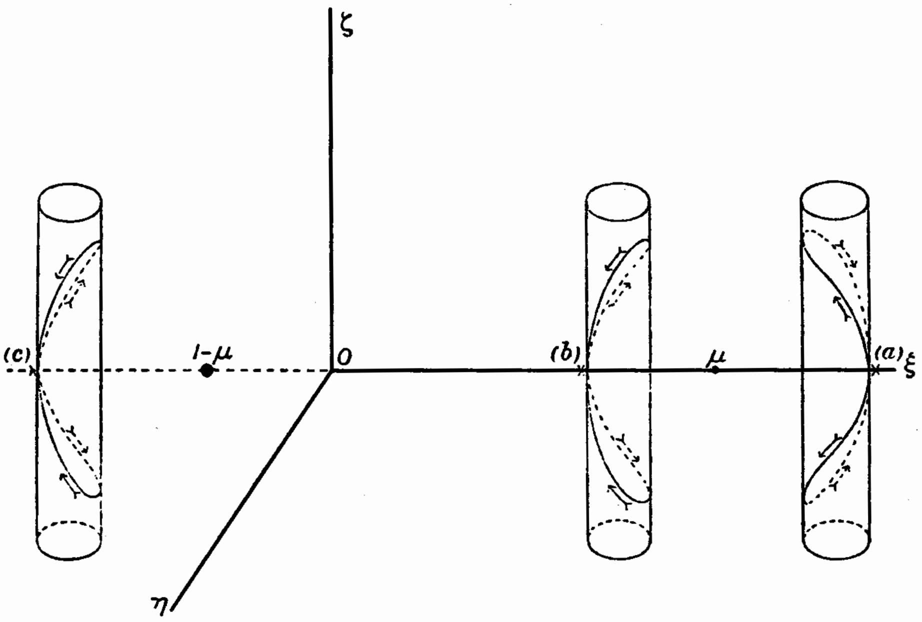


FIG. 1. Moulton's Representation of Orbits of the First Type [6].

As computational capabilities improved and more numerical procedures were employed, a wider range of three-dimensional periodic orbits could be calculated. As early as 1963, Goudas [8] computed nineteen families of three-dimensional periodic orbits in the circular restricted three-body problem. Of the nine families highlighted in the paper, all are doubly symmetric. Goudas successfully extended Moulton's families but none appears to be the families of halo orbits now familiar to many. However, Goudas discussed hodographs as an aid in the procedure to numerically compute such families. He also investigated the stability of his families of orbits and concluded that the periodic orbits in the vicinity of the collinear libration points are "very unstable." In 1967, Bray and Goudas [9] again considered three-dimensional orbits in cases of finite but not small deviations in the third dimension where linearization of this component is no longer realistic. They formulated the problem and sought analytical approximations whenever possible, but used purely numerical techniques when necessary. With the introduction of computers, Bray and Goudas were some of the first to exploit extensive computational capabilities in this problem.

Of course, Hénon [10] has long been considered an expert on this problem and, in 1973, he published results of a study of plane periodic orbits, but focused on out-of-plane perturbations and their impact on stability. His "vertical stability index" a_v was evaluated for any specified members of a planar family, including those associated with the collinear libration points, i.e., families first identified by Moulton. A planar orbit is designated a vertical critical orbit when its vertical stability index has a magnitude of one. This orbit then signals a bifurcation and represents the intersection of a family of plane periodic orbits with a family of three-dimensional periodic orbits. Hénon identified vertical critical orbits in Strömgren's plane retrograde families a, b, and c of periodic orbits around L_3, L_2, and L_1 for mass ratio $\mu = .5$; Kazantzis [11] later explored them for the Sun-Jupiter ratio $\mu = 0.00095$. Kazantzis classified his three-dimensional orbits as:

'type A'—symmetric with respect to the xz-plane;
'type B'—symmetric with respect to the $\mathbf{x}$-axis;
'type C'—both previous symmetries.

It is notable that the vertical critical orbits themselves can be far from the equilibrium point. In his work, Hénon was evaluating out-of-plane stability of the planar orbits and did not pursue the three-dimensional families, but others did. Robin and Markellos [12] examine the mechanism by which the planar families bifurcate to the three-dimensional orbits with special emphasis on the symmetry properties (axisymmetric, plane/simply symmetric, doubly symmetric). They also noted that, owing to the symmetry of the restricted problem with respect to the xy-plane, for every vertical branch there is a "mirror image" branch consisting of orbits that are the images under reflection across the xy-plane.

Various other researchers continued the search and examination of various types of families in the restricted problem; of particular interest are the few that appear to have identified members of the halo families. Results from a study to extend the vertical critical orbits of Hénon to finite three-dimensional families are partially presented by Michalodimitrakis [13] in 1978, as well as in the paper by Ichtiaroglou and Michalodimitrakis [14] in 1980. They proved the existence of the bifurcating families and demonstrated that the three-dimensional families may actually connect to more than one planar family; a few orbits were also calculated. Zagouras and

Kazantzis [15] published their study of families of three-dimensional oscillations in the vicinity of the collinear points in 1979. They were particularly interested in the Sun-Jupiter system. Zagouras and Kazantzis [15] examined infinitesimal periodic oscillations around the collinear libration points in the plane of motion of the two primaries that are continued along the families a, b, and c of plane retrograde periodic orbits around L_1, L_2, and L_3. Examining the stability index of the vertical critical orbits ($|a_v| = 1$) of the families a, b, and c as computed for the Sun-Jupiter system ($\mu = 0.00095$), it is observed that some critical members of each family are of type A symmetry, namely, symmetrical with respect to the xz-plane. Generated from plane periodic orbits, the three-dimensional families A_{1v}, C_{1v}, and B_{1v} from Zagouras and Kazantzis [15] are, in fact, the L_2, L_1, and L_3 halo families, respectively, in the Sun-Jupiter system. The authors do detect some stable orbits in the L_1 family, but the families are only partially computed and the stable orbits, now known to exist in the other two families, were not observed.

In 1967, Szebehely [7] produced a comprehensive book summarizing studies in the restricted problem to that time. Marchal [16] returned to the task in 1990 and published a book that details the more recent progress and potential future investigations in the problem. Both works contain numerous plots and discussions of periodic orbits.

Fundamental Motions Near Libration Points

In recent years, of course, computational capabilities have improved greatly. Returning to the foundations supplied by Poincaré [5], modern techniques can more extensively incorporate dynamical systems theory (DST) to support trajectory design in three-body regimes. One of the primary advantages of DST is the immediate insight gained from the geometry of the phase space in the vicinity of libration points and halo orbits. The in-plane generating families and the nearly vertical periodic orbits of the earlier studies can be computed in the context of DST.

Equilibrium solutions and periodic orbits, as well as quasi-periodic motions, are special solutions in the three-body problem. These special solutions are examples of one of the fundamental models for the phase space, i.e., invariant manifolds. A manifold is an m-dimensional analog to a two-dimensional surface in $\mathbb{R}^n$. An invariant manifold can be described as a collection of orbits that start on a surface and stay on that surface for the duration of their dynamical evolution. This definition can then characterize a variety of behaviors. In addition to the examples mentioned, there exist invariant manifolds that asymptotically approach or depart other invariant manifolds. These are called stable and unstable manifolds, respectively. Much discussion has focused lately on the computation of approximations to the stable and unstable manifolds. It is in the center manifold, however, where bounded motion exists (as well as the transitions from one type of bounded motion to another). Consider first a collinear libration point as an equilibrium solution in terms of the usual rotating coordinates in the three-body problem with origin at the barycenter. (The $\mathbf{x}$-axis of the rotating frame is coincident with the line connecting the primaries and is directed from the larger primary to the smaller one, the $\mathbf{z}$-axis is normal to the plane of motion of the primaries, and the $\mathbf{y}$-axis completes the right-handed triad.) The libration point itself possesses a one-dimensional stable manifold, a one-dimensional unstable manifold, and a four-dimensional center manifold. Within the center manifold are periodic and quasi-periodic motions. Specifically, there are two types of periodic motion that can be identified: one type of periodic solution exists

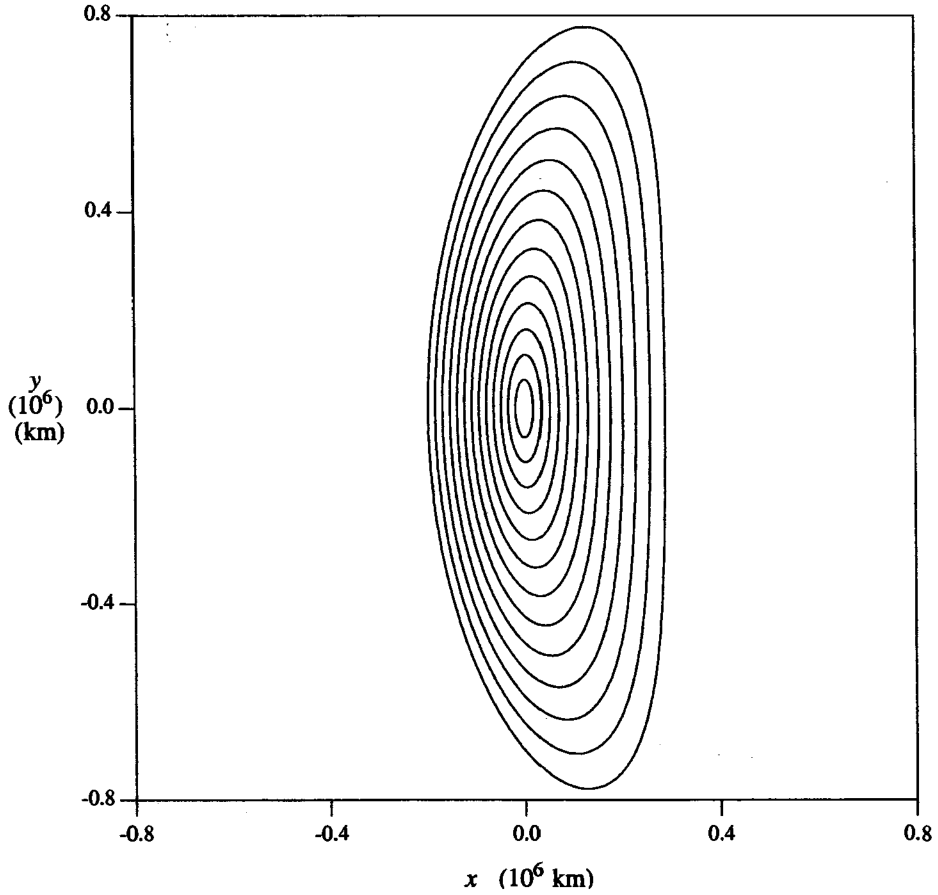


FIG. 2. Planar Lyapunov Orbits.

in the plane of motion of the primaries (i.e., has no out-of-plane component) and is sometimes called a Lyapunov orbit; the other type is dominated by the out-of-plane component and is nearly vertical [17, 18]. A number of the planar orbits are plotted in Fig. 2, relative to the L_1 point in the Sun-Earth/Moon system. A range of nearly vertical orbits appears in three planar projections in Fig. 3. In the figure, the xy projection appears in the upper left, the xz projection in the lower left, and the yz projection in the lower right. Note that this range includes orbits with a very large out-of-plane component; the large amplitude members of the family more clearly demonstrate the nonlinearity. The orbits in Fig. 3 are Moulton's orbits of the first type [6], the vertical orbits of Hénon [10], and the family L_{1v}^{e} of type C in Zagouras and Kazantzis [15]. Also residing in the center subspace are quasi-periodic solutions that are related to both the planar solutions and the vertical orbits. Such three-dimensional, quasi-periodic solutions have been typically described as Lissajous trajectories [17, 19]. A typical Lissajous curve is plotted in Fig. 4 for an arbitrary ratio of the in-plane and out-of-plane amplitudes. Each of these Lissajous trajectories lies on two-dimensional tori that are symmetric with respect to both the xz-plane and the yz-plane. An example of one of these tori near L_1 in the Sun-Earth/Moon system appears in Fig. 5. In each of the views in the figure, the blue, red, and black

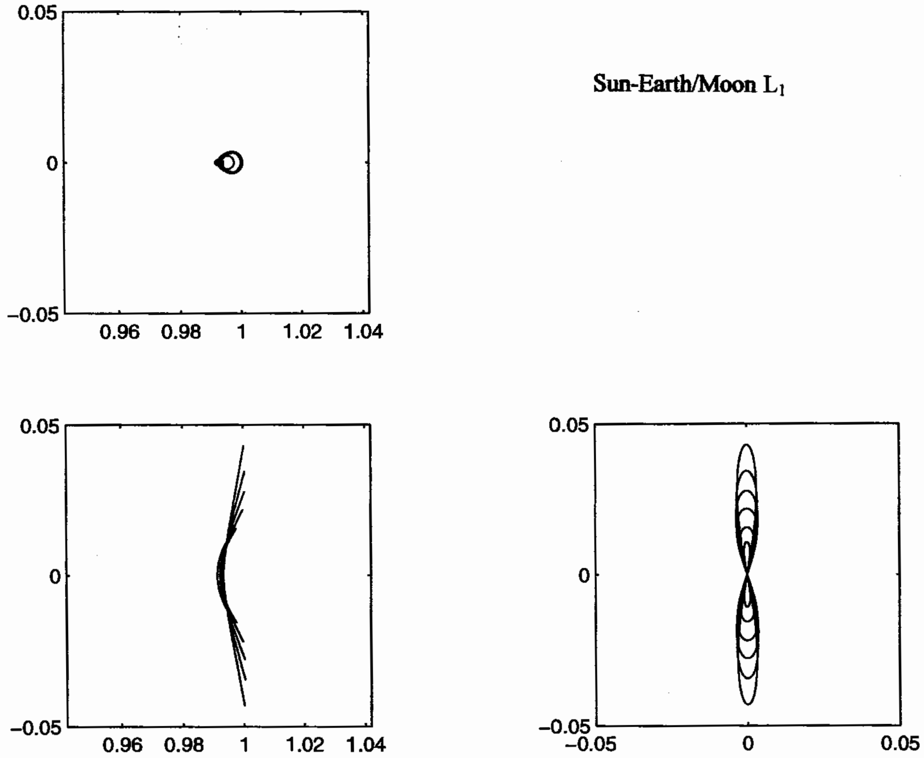


FIG. 3. Nearly Vertical Orbits.

triad is anchored to the same corner of the bounding box with the blue corresponding to the **x**-axis, the red axis parallel to the **y**-axis, and the black axis directed along the **z**-axis. (All subsequent figures of this type use the same convention.) It is important to note that the surface was actually generated using trajectories that lie on the surface.

The third type of periodic orbit is the halo orbit. The halo orbits are, in fact, the result of a pitchfork bifurcation from one of the Lyapunov orbits. As the amplitude of the planar orbits increases along the family, eventually a critical amplitude is reached where a bifurcation occurs. This point correlates with the vertical critical orbits of Hénon [10]. While the specific amplitude (i.e., orbit) cannot yet be predicted by any analytical means, it can easily be identified by monitoring the eigenvalues of the monodromy matrix corresponding to each of the Lyapunov orbits; this corresponds to Hénon's planar orbit with a stability index a_v (calculated from the eigenvalues) at a value of one. For any Lyapunov orbit with an amplitude less than the critical amplitude, there is one unstable eigenvalue, one stable eigenvalue, and four center eigenvalues. Of the center eigenvalues, two have a value of one and two lie on the unit circle, i.e., a nonzero imaginary part. As the amplitude of the Lyapunov orbit approaches that of the critical orbit, the two center eigenvalues with nonzero imaginary part approach a value of one (along the unit circle). At the critical orbit, there are four eigenvalues with a value of one. Similar to the local phase space of the libration point itself, within this four-dimensional subspace there are two types of periodic motions. One set of eigenvectors can be identified that point to nearby Lyapunov orbits. However, the other eigenvectors in this subspace point to

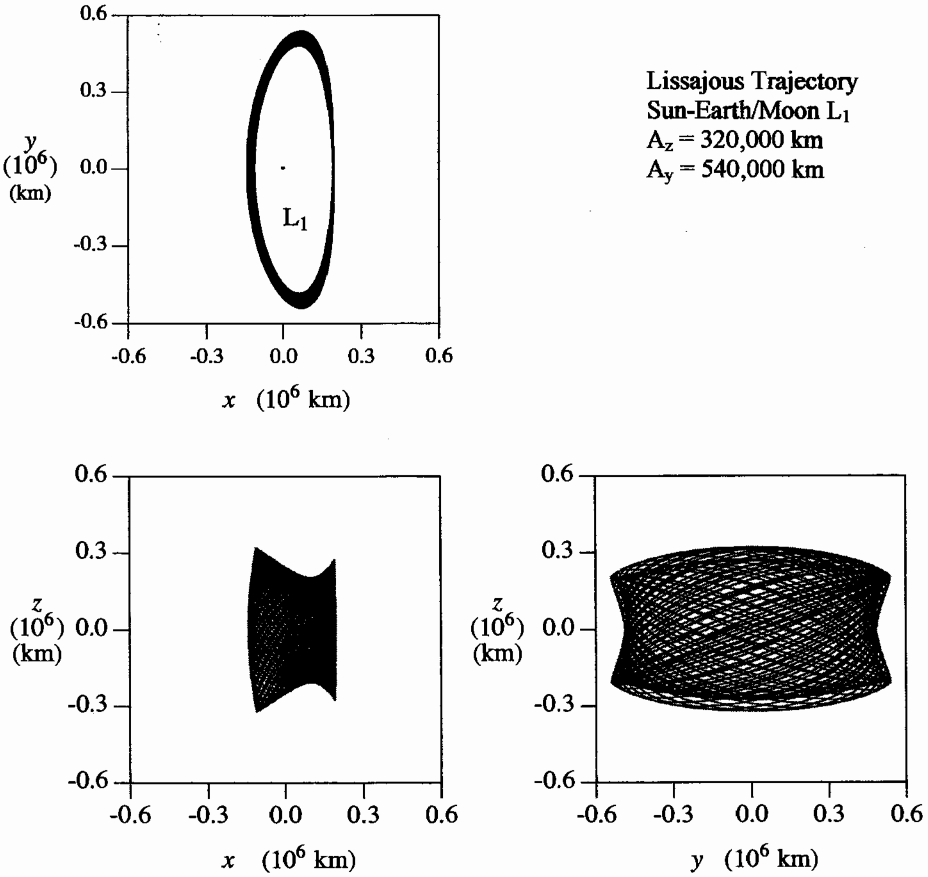


FIG. 4. Lissajous Trajectory.

periodic orbits with an out-of-plane component. These are the northern and mirror image southern halo orbits. Appearing in Fig. 6 is an example of a northern L_1 halo orbit in the Sun-Earth/Moon system. The halo orbits are also an example from Moulton's third existing type of three-dimensional finite periodic orbits.

Partly in support of mission design, the focus has shifted in recent years in attempts to further understand the flow in the vicinity of the collinear libration points and the associated periodic orbits. Significant contributions appear in Gómez, Jorba, Masdemont, and Simó [20], Simó [21], as well as Gómez, Masdemont, and Simó [22]. These authors have approached the problem by expanding the Hamiltonian in the vicinity of a collinear libration point in a power series where the degrees of freedom are then reduced to the center manifold. This results in a reduced Hamiltonian without an unstable term. Thus, long integrations produce an insightful Poincaré map, at a given value of energy, in the vicinity of a collinear point, that is, one that offers a broader view of the phase space. Using suitable coordinates, one of these maps appears in Fig. 7. (In the new coordinates, it is very close to a section through $z = 0$.) The fixed point in the central part of the figure corresponds to the nearly vertical periodic orbit (essentially related to the frequency ω_z). The two fixed points at both sides of the plot represent the two symmetric halo orbits with the specified energy. The boundary of the plot is a Lyapunov planar orbit (essentially

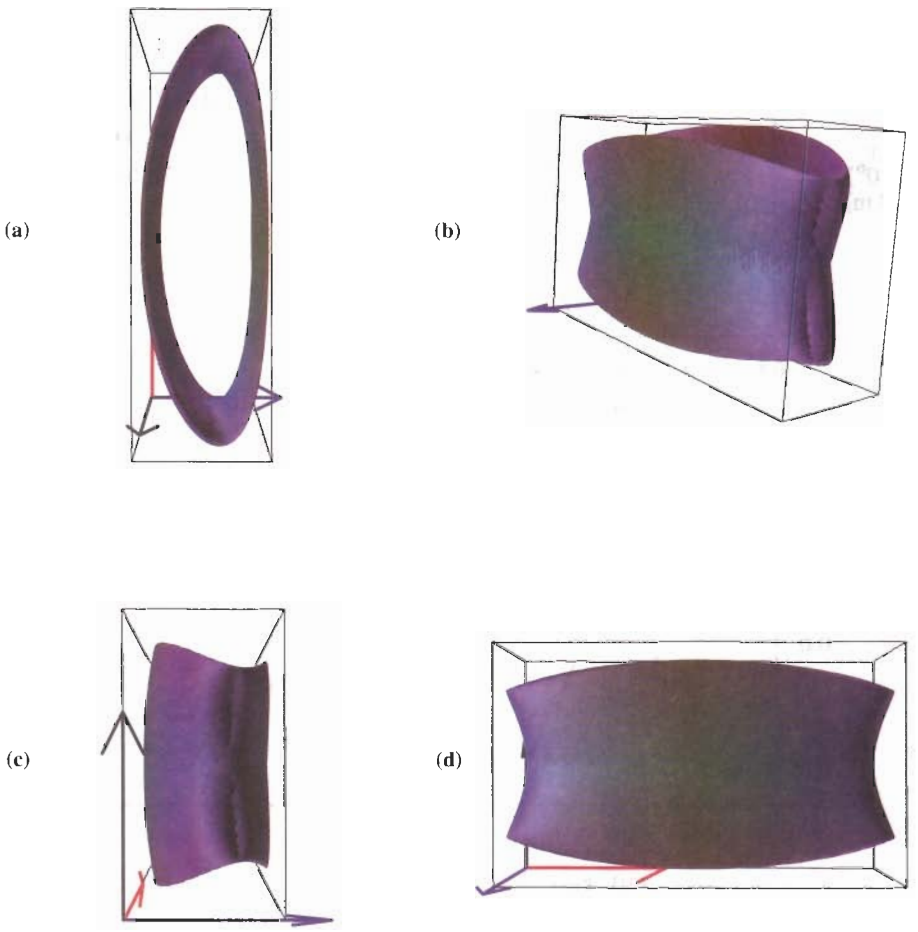


FIG. 5. Lissajous Surface Projected into Configuration Space.

related to the frequency ω_{xy}, and, as noted previously, unstable even in the center manifold). Note that enveloping both the vertical periodic orbit and the halo orbits are invariant curves that represent the intersections of two-dimensional, invariant tori, regions that usually include quasi-periodic orbits. These orbits are also known as nonlinear Lissajous orbits. The inner ones are Lissajous orbits around the almost vertical periodic orbit, and the curves surrounding the fixed points that are offset from the center are Lissajous orbits around the halo orbits; these are also denoted as quasihalo orbits. An example of one of the inner Lissajous trajectories appears in the plot in Fig. 4. The quasihalos can also be determined and these trajectories reside on surfaces, but now the surfaces envelope the halo orbits. Gómez, Masdemont, and Simó [22] derived approximations to represent this type of quasi-periodic motion; Barden and Howell [17, 18] utilized the center subspace of the monodromy matrix of a halo orbit to numerically calculate the trajectories on the surface. An example [17, 18] is presented in Fig. 8 where, in this view, the surface appears much like a torus. In fact, the surface is self-intersecting at the xz-plane crossing in this representation. However, recall that this is a two-dimensional surface in a six-dimensional phase space, being projected into three-dimensional configuration space.

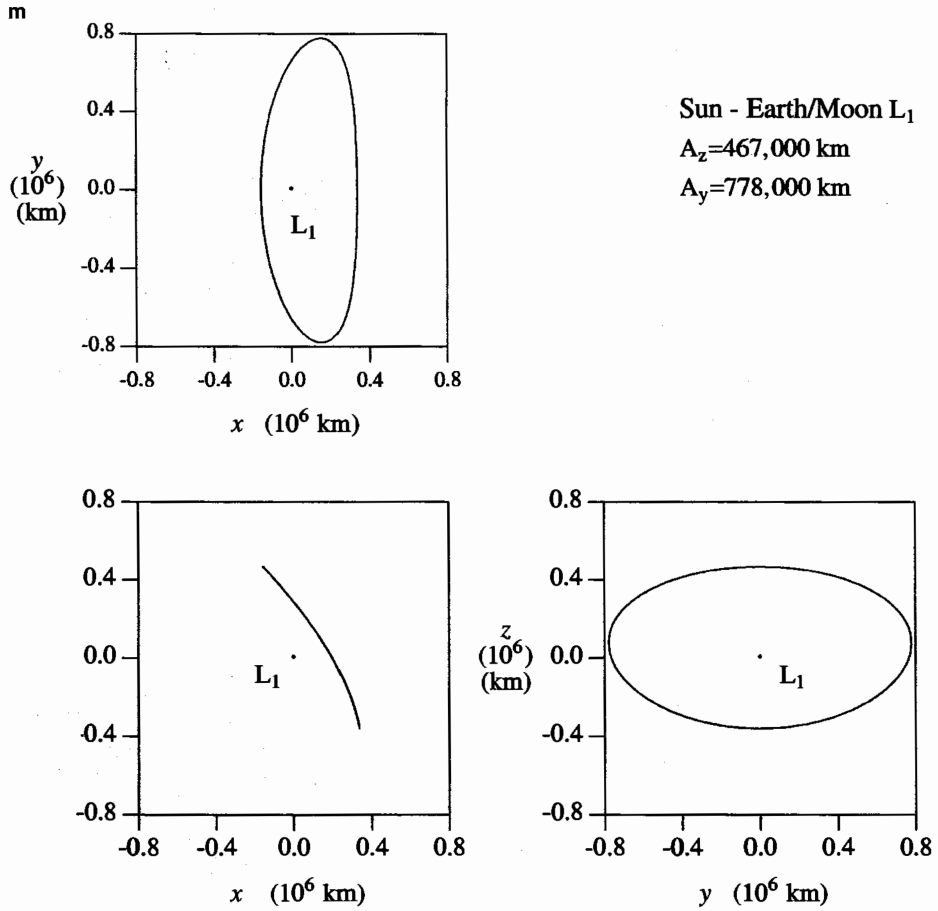


FIG. 6. Halo Orbit in the Circular Restricted Problem.

Application to Mission Design

International Sun-Earth Explorer-3

The halo orbit concept truly gained credibility for mission design with the launch of ISEE-3 (International Sun-Earth Explorer-3) on August 12, 1978, and subsequent insertion of the vehicle into a halo orbit about the Sun-Earth libration point L_1 [23]. As the first spacecraft to use a libration point orbit, it reflected a new approach to trajectory design, i.e., visualizing both problem and solution in a rotating coordinate frame. In support of the mission, NASA engineers were required to focus on the computation of an acceptable L_1 halo orbit for the sake of the mission; based on the earlier work [3], a number of analyses were completed to reach that goal [23, 24].

Families of Halo Orbits

After the success of ISEE-3, Breakwell pursued halo orbits into a theoretical context by expanding the investigation to a search for complete families of solutions. Such a search would include large amplitude halo orbits, i.e., solutions far outside the range of the analytical approximations. The very characteristics that enable the

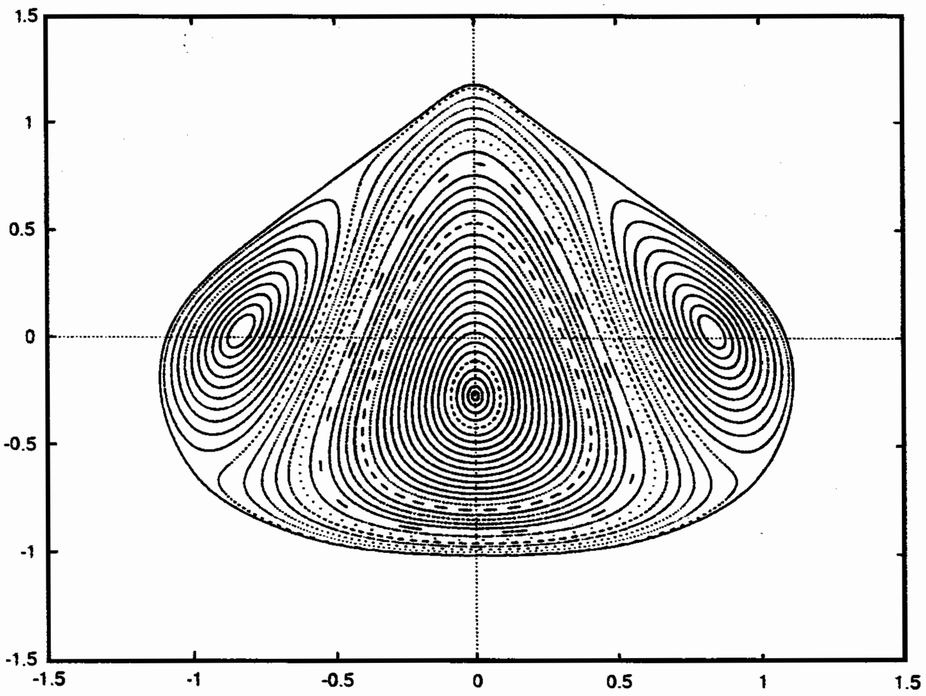


FIG. 7. Poincaré Map of the Flow Reduced to the Center Manifold; Earth-Moon L_2; Given Level of Energy [20] (with permission).

existence of periodic halo orbits made them difficult to compute. Investigation of larger orbits and families meant numerical analysis. Farquhar and Kamel [3] had already predicted and determined halo orbits near the translunar L_2 libration point in the Earth-Moon system. Breakwell and Brown [25] then extended these into an L_2 family of orbits. Also in the Earth-Moon system, they computed a family originating near the L_1 libration point between the Earth and the Moon. Of special significance, Breakwell and Brown identified a narrow band of stable halos in each family. From this significant beginning, additional halo families were computed across the complete range of mass ratios in the three-body problem [26, 27]. Their computation was complicated by the large out-of-plane amplitudes associated with any potentially stable halo orbits; also, some of the family members passed very close to one of the primaries and required regularization in three dimensions. Nevertheless, the determination of halo families in the vicinity of all three collinear points was accomplished. Figures 9, 10, and 11 include projections of the orbits from all three families. (The orbits are plotted for the mass ratio $\mu = 0.04$ to emphasize some of the characteristics.) These computed families are more complete than those of Zagouras and Kazantzis [15].

In the projections in the figures, the range of stable orbits in the L_1 and L_2 families are indicated by dashed curves. The orbits in a family are continuous, of course, and it is useful to also identify the surface associated with a particular family; the members of the family always remain on this surface. Appearing in Fig. 12 is the surface associated with the family of L_1 halo orbits in the Sun-Earth/Moon system. Note that this surface is actually constructed in six-dimensional phase

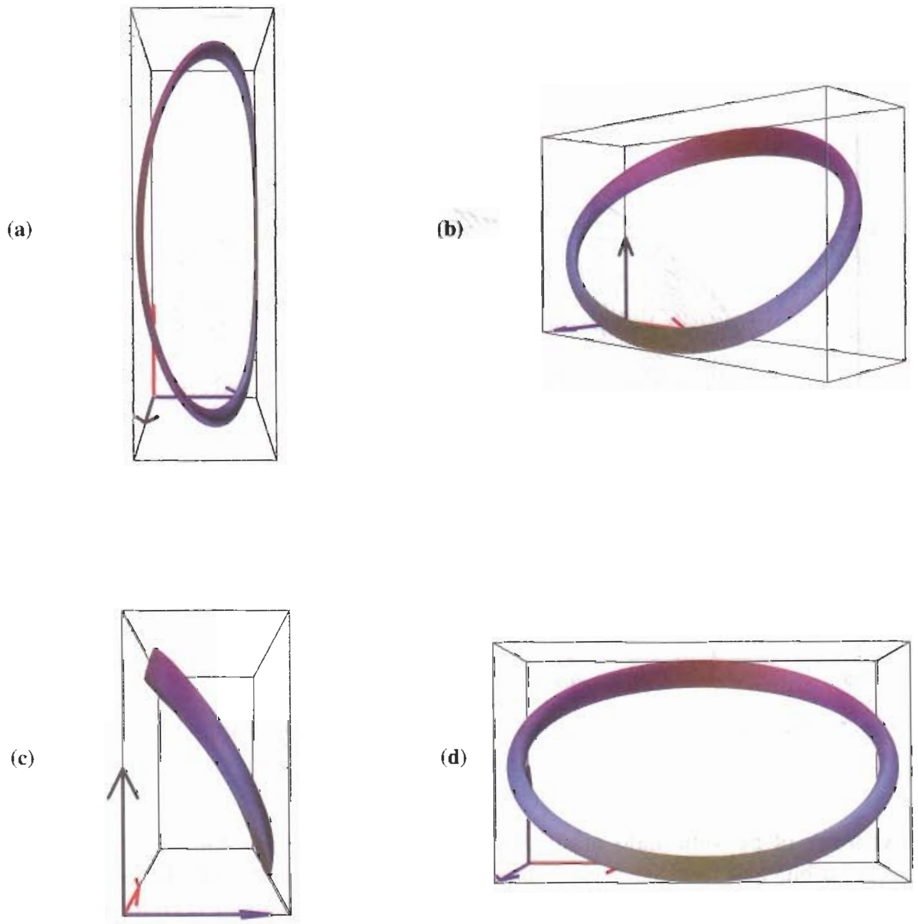
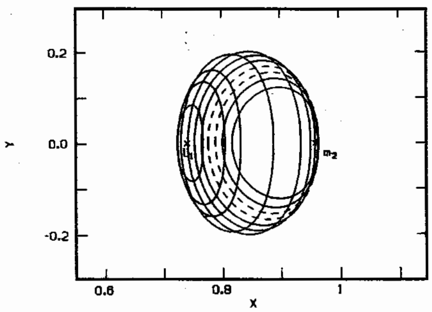


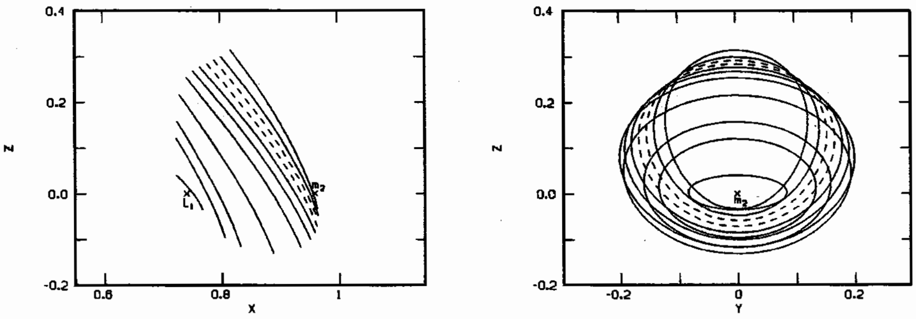
FIG. 8. Torus Enveloping Northern Halo Orbit.

space, but has been projected onto configuration space. Essentially, a surface has been created from orbits similar to those in Fig. 9 for the appropriate mass ratio. The L_1 halo orbit that was used for the ISEE-3 spacecraft, as well as for similar missions launched thus far, would correspond to the smallest halo orbit that appears in Figs. 9 or 12, even though the ISEE-3 trajectory is characterized by excursions of 120,000 km in the out-of-plane direction and approximately 666,000 km in the in-plane y direction. If all of the members of the halo families in the figures were plotted, the trajectories and surfaces would extend in both directions. As the family is extended toward the libration point, both L_1 and L_2 families would collapse to the in-plane generating orbit, i.e., the critical Lyapunov orbit previously mentioned. As the families are continued toward the smaller primary, each family tends toward rectilinear orbits perpendicular to the plane of motion of the primaries. The L_2 family will again collapse toward the xy-plane; in contrast, the rectilinear members of the L_1 family will continue to increase in size in the direction of the out-of-plane component.

It should not be surprising that the L_1 family has characteristics similar to those of an L_2 family of orbits. For computational convenience, the L_3 orbits in Fig. 11



X-Y Projection, L1 Family, μ = .04.



X-Z Projection, L1 Family, μ = .04.

Y-Z Projection, L1 Family, μ = .04.

FIG. 9. Family of L_1 Halo Orbits; $\mu = 0.04$ [26].

were computed for a system with mass ratio μ by computing the L_2 orbits for a family with mass ratio $1 - \mu$. Therefore, extensions of the family in the figures would produce characteristics similar to those discussed for the L_2 orbits. That is, the family appears to both originate and terminate with a planar orbit. The L_3 orbits are much larger, however, and the approach to the primary involves rectilinear orbits that pass even closer to the primary. In fact, the stable range for L_3 orbits is located where the orbits in the figure pass closest to the primary; this is also the point at which the out-of-plane excursion is the largest. To compare the relative size of the orbits, surfaces for L_3 families in the Sun-Earth, Sun-Jupiter, and Sun-Neptune systems appear in Fig. 13. The in-plane excursions of these orbits can reach a distance equal to the distance between the primaries, the out-of-plane excursion can be 1.5 times that value. Of course, such orbits exist not only in Sun-planet systems, but planet-moon systems as well, as demonstrated by the analysis in the Earth-Moon region.

There are an endless number of periodic orbits in the region of space governed by the force model in the three-body problem. They include orbits and families of various types of symmetry and multiplicity. To exploit these types of trajectories in a problem with no analytical solutions, it is necessary to understand the structure of the space as much as possible, and to parameterize, classify, and categorize as is feasible. It is a daunting task. Yet, a mission designer will want to identify a desirable orbit and/or arc quickly and compute it efficiently. Thus, study of the halo and

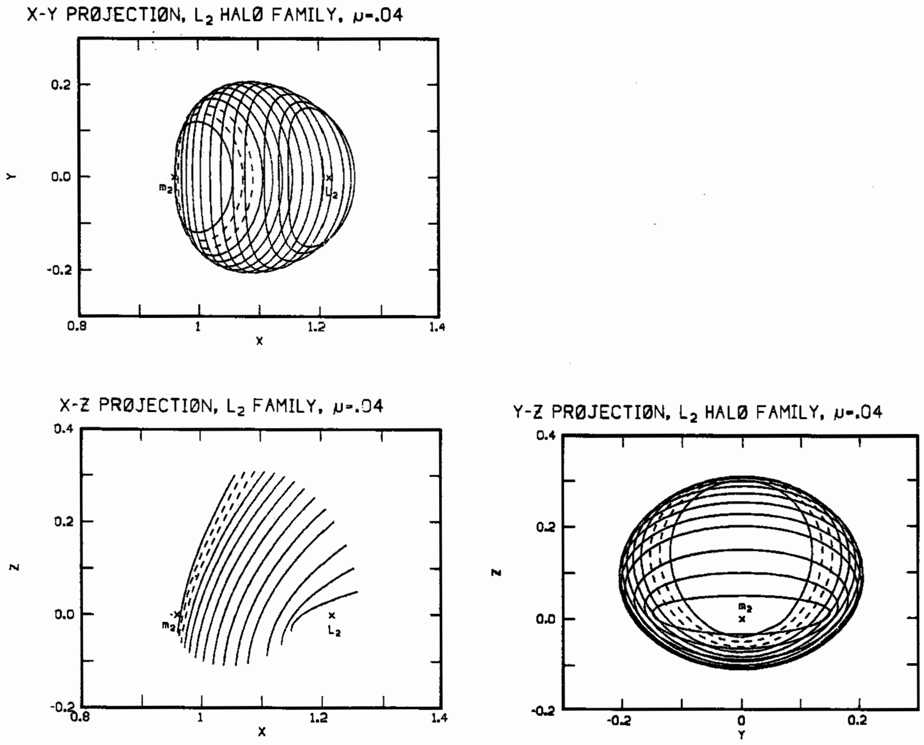


FIG. 10. Family of L_2 Halo Orbits; $\mu = 0.04$ [26].

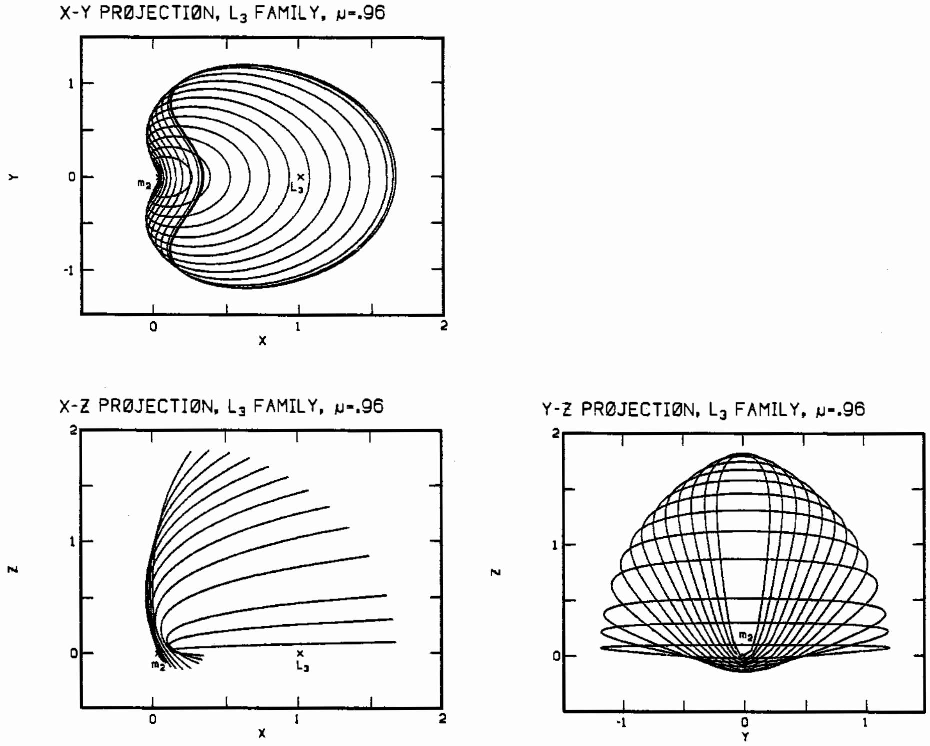


FIG. 11. Family of L_3 Halo Orbits; $\mu = 0.04$ [26].

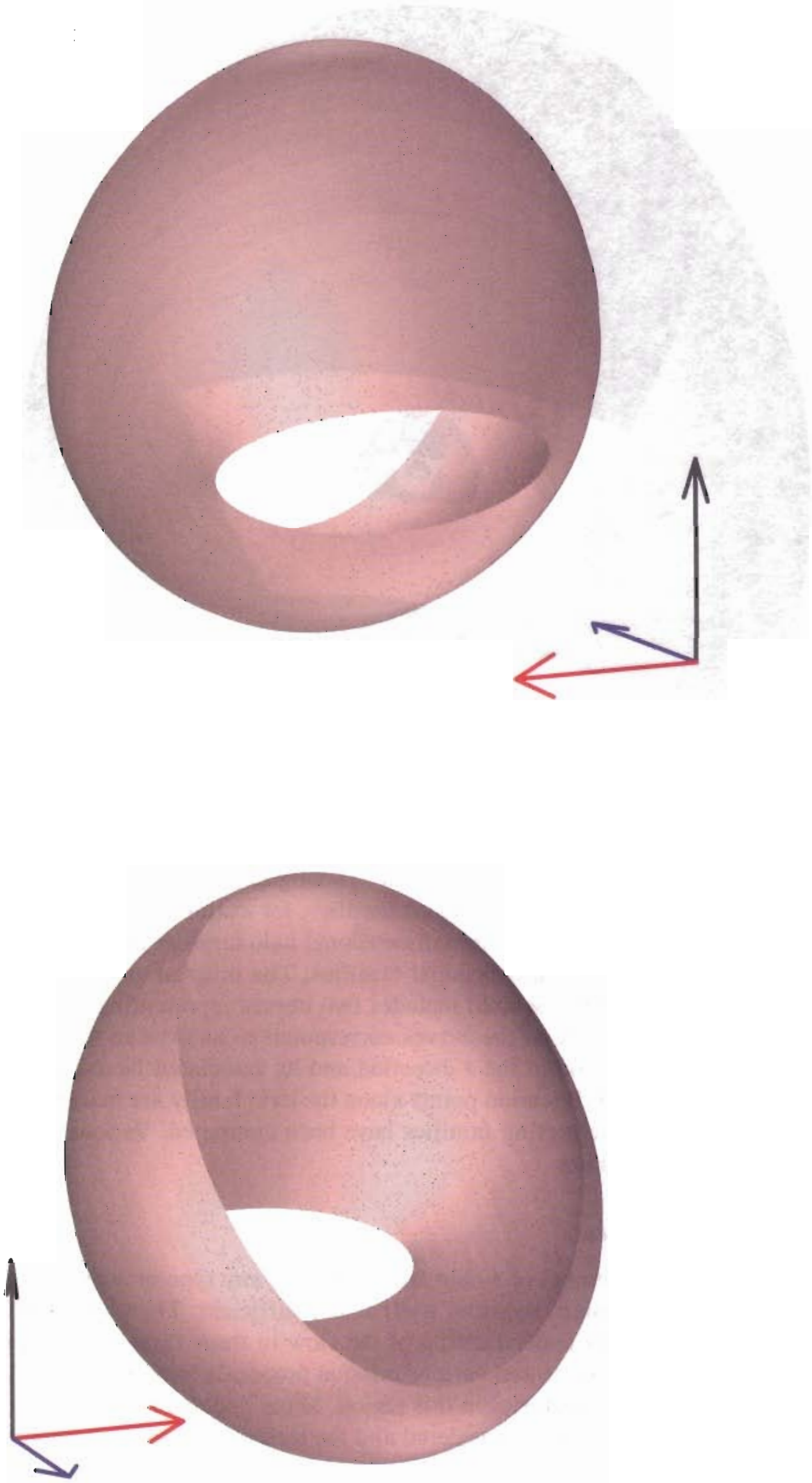


FIG. 12. Surface Generated from the Sun-Earth/Moon L_1 Halo Family.

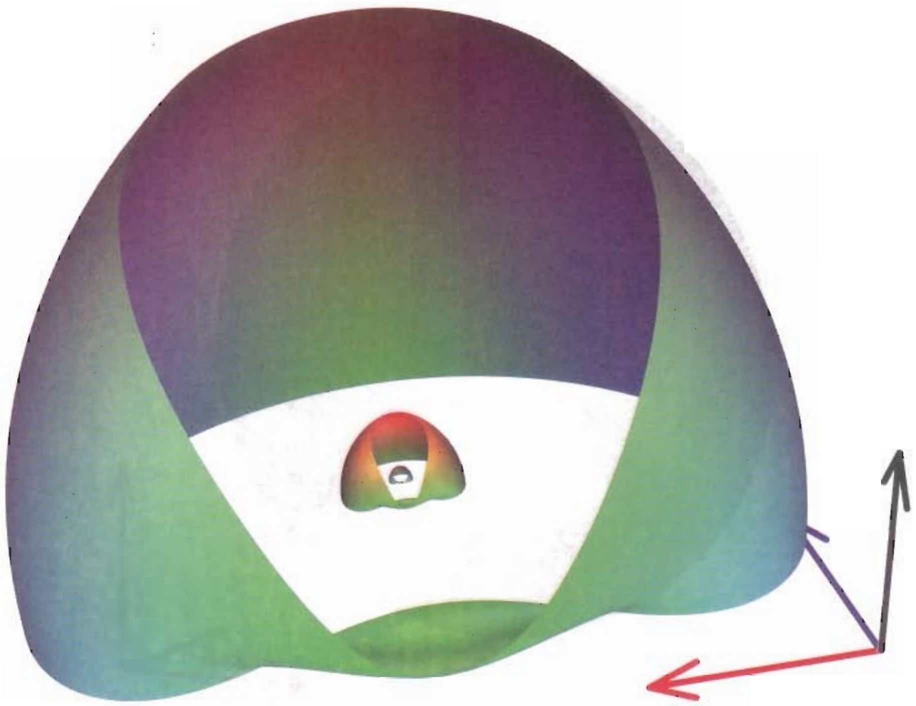


FIG. 13. Sun-Neptune, Sun-Jupiter, and Sun-Earth L_3 Halo Families.

Lissajous type trajectories and their potential applications continues. Further, the search for connections between these orbits and other periodic families, begun many decades ago, remains incomplete. Having determined the bifurcations from plane orbits to the three-dimensional families, for example, it is interesting to consider the stability across the three-dimensional halo families and determine bifurcations to yet other three-dimensional families. The bifurcation diagram in Fig. 14, produced in just such a search, includes two curves representing the same L_1 halo family [28]. Each point on the curves corresponds to an orbit in the family through its maximum excursion in the x direction and its associated Jacobi constant (or energy). Many of the bifurcation points along the halo family are marked. Members of a number of the intersecting families have been computed. Various researchers are continuing such studies.

Transfer Trajectories

To exploit any member of a halo family or even any type of quasi-periodic trajectory for mission design, the orbit itself is not sufficient. Thus, in recent years, the necessity for a better understanding of the flow in these three-body regimes is apparent. Planned missions and various mission proposals have involved halo orbits or the quasi-periodic trajectories in this region. Many papers in the literature testify to the different options being considered and the techniques used to compute the paths of motion.

The families of orbits in the vicinity of the collinear points themselves may offer the best method to design transfers. Since most of the orbits in the periodic halo

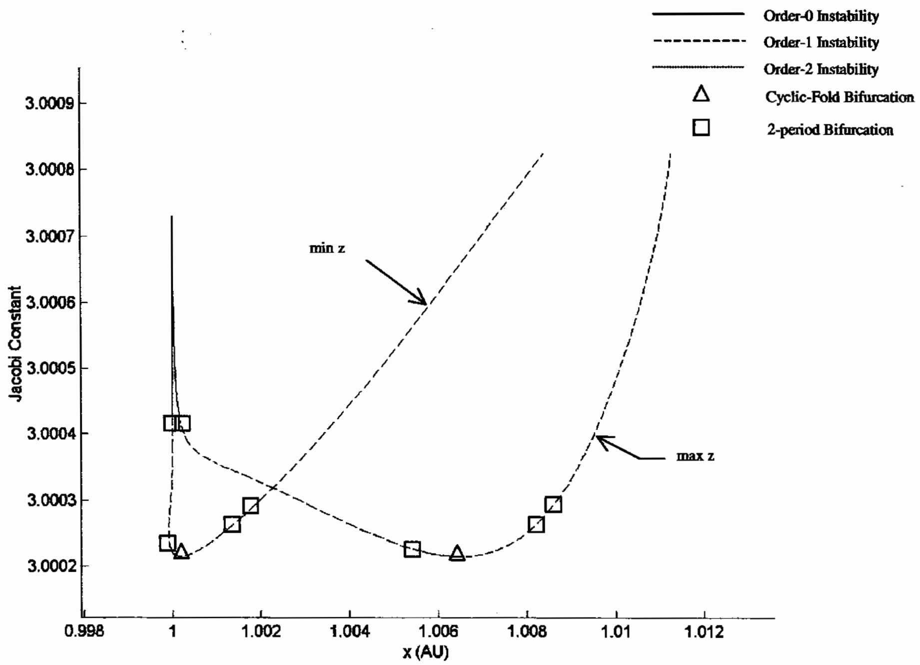


FIG. 14. Bifurcation Diagram for the Three-Dimensional L_2 Halo Family.

families are unstable, the corresponding stable and unstable manifolds govern the behavior of the paths of motion in the vicinity of the primaries and the collinear libration points. The stable and unstable manifolds become a natural choice for consideration of transfer trajectories and computation of point-to-point arcs to accomplish various objectives. A number of investigations that shift the study of the periodic families of orbits in this direction are now underway and demonstrate potential for supporting space trajectory design [29–31].

Concluding Remarks

The three-body problem has fascinated mathematicians, scientists, and engineers for over 200 years. Study of only the region in the vicinity of the collinear libration points has been rich with surprises and opportunities. In space mission design, accomplishment of many short- and long-term science and exploration goals will increasingly require innovative and complex spacecraft trajectory concepts. Thus, it is an appropriate time to further exploit this regime. Clearly, a number of successful missions, including ISEE-3 and SOHO, have now proved the viability of using the three- (and four-) body problem to develop baseline trajectory concepts. Numerous proposals from various international agencies have based missions on these successes. Additional study of these families and the flow in their vicinity should receive increasing attention.

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