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FULL N-BODY PROBLEM IN THE GEOMETRIC MECHANICS FRAMEWORK AND ITS REDUCTION TO CIRCULAR RESTRICTED FULL THREE-BODY PROBLEM

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A compact formalism for rigid body motion dynamics is presented in a general reference frame based on geometric mechanics. This formalism, proposed on the special Euclidean space of the Lie group, naturally accounts for the orbit/attitude coupling due to the gravitational moments and forces for the full N-body problem. Furthermore, the expressions for energy are provided. The special case for the circular restricted full three-body problem (CRF3BP) is then considered, with equations provided in the spacecraft's body and barycentric rotating frames. Several trajectories are computed for the CRF3BP, and are compared with the traditional circular restricted three body problem.

INTRODUCTION

The Cislunar region is increasingly gaining interest by several agencies worldwide as well as private organizations as a potential location not only to perform an exhaustive lunar exploration, but also to test technologies that could potentially be useful for deep space exploration. Within such a region, distinct perturbations act upon the spacecraft which makes it challenging to preserve the nominal orbit of the mission. Additionally, these perturbing effects are exacerbated when considering that the spacecrafts are rigid bodies. Consequently, it is crucial to understand the coupling between the orbital motion and the attitude of spacecrafts in that region to guarantee that the mission occurs as expected in such a highly sensitive dynamical model. Therefore, the necessary control throughout the entire lifespan of the mission can be minimized.

The high eccentricity of the near rectilinear halo orbit (NRHO) orbit may result in non-negligible coupling between the attitude and orbital motion in that orbit, particularly near perilune where the gravity gradient torques are relatively high.[1,2] The NRHOs ensure easier access from Earth when compared with direct missions to the Lunar surface, but at the cost of some complexity involving the dynamics analysis in a highly eccentric, multi-body system.[3,4] Such methodologies must be increasingly reliable as the need for autonomy grows, noting that the Gateway will be uncrewed for the majority of its operational lifetime. Thus, a robust mathematical formalism is required to address the possibly non-negligible coupling in highly nonlinear environments in which coupled effects

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such as gravity gradient torques and solar radiation pressure exist.[5,6] The NRHO is a consequence of the circular restricted three body problem (CR3BP). Consideration within the CR3BP has been paid to attitude motion using quaternions in the literature,[7] assuming a diagonal moment of inertia tensor, where trajectory design was investigated considering orbit-attitude coupling caused by solar radiation pressure. Other treatments have employed Euler angle sequences where a singularity is possible but rare to occur.[8]

Consider N rigid bodies isolated from the rest of the universe which interact solely under their mutual gravitational effects. In contrast to the *N*BP, which only considers the translational motion of the particles or center of masses of the bodies, the full N-body problem (F*N*BP) studies both the attitude and translational dynamics of N bodies under their mutual gravitational effects. The dynamic analysis of both translational and rotational motion is a challenging problem from a mathematical point of view, specifically when the coupled dynamics are considered. The rigid body rotation about its center of mass is usually considered to be decoupled from the translational motion in the literature. Using the nonlinear manifold of the special Euclidean group ($\mathsf{SE}(3)$), however, enables the treatment of both orbital and attitude motions, simultaneously. The simultaneous modeling of translational and rotational dynamics of a rigid body using $\mathsf{SE}(3)$ is especially advantageous for certain cases in which the translational and rotational dynamics are coupled. These cases include gravity forces and gravity gradient moments, attitude-dependent forces and torques due to solar radiation pressure, and orbital motion in small or highly eccentric orbits. In addition, the group-theoretic formalism of $\mathsf{SE}(3)$ enables the use of additional mathematical tools and gives rise to deeper insight into rigid body motion. However, despite the importance of this topic, few studies have been dedicated to this research.[9–12] The effect of incorporating attitude dynamics has been investigated within the context of the CR3BP as well as periodic orbits.[13,14] Additionally, it has been demonstrated that marginally stable periodic attitude behaviors leveraging the natural dynamics of the CR3BP exist that could potentially be useful for different mission applications.[15] Different perturbations, such as solar radiation pressure and structural vibrations, have been added to the coupled orbital-attitude motion with the objective to foresee challenges that may stem from these.[16] Furthermore, it has been concluded that new options for mission design may exist when the full body problem is considered.[17]

In this paper, first, a compact, general formalism is provided in $\mathsf{SE}(3)$ for the F*N*BP. This formalism considers the coupled translational and rotational dynamics of the bodies, where the equations of motion are expressed in both the inertial and the body frames of the spacecraft. It is shown that the dynamics formalism has the same structure in different coordinate frames in the proposed framework, which makes this framework quite suitable for the general formalism of F*N*BP. Then, the proposed general formalism is revised to address the circular restricted full three-body problem (CRF3BP), and it is further reduced to re-derive the well-known equations of motion for the CR3BP. One may need to scrutinize the solution and stability of the full three-body problem (F3BP) (see Refs. [18–20] for a rigorous investigation of equilibria and stability conditions of the full two-body problem and sphere-restricted full few-body problem), but such details are omitted here and the attention is devoted to the derivation of the governing equations. Finally, several trajectories are computed for the CRF3BP within the context of this general formalism, and they are compared with those obtained using the traditional CR3BP equations of motion that consider a spacecraft to be a point mass. Numerical simulations showed that the response of the system improves via normalization of the equations.

FORMULATION OF THE RIGID BODY MOTION GROUP

Consider N rigid bodies in an inertial frame $\mathcal{N}(O, \hat{n}_1, \hat{n}_2, \hat{n}_3)$ as shown in Fig. 1, where G_i denotes the center of mass of the ith body ($i = 1, 2, \cdots, N$). Let the space of rigid body rotations be the three-dimensional special orthogonal group $\mathsf{SO}(3)$, a Lie group whose matrix representation is a 3×3 rotation matrix. Then, the configuration (i.e. position and attitude) of each body is represented by an element g_i of the Lie group $\mathsf{SE}(3)$, i.e.

$$g_i = \begin{bmatrix} R_i & r_i \\ 0_{1\times 3} & 1 \end{bmatrix} \in \mathsf{SE}(3) \tag{1}$$

where $R_i \in \mathsf{SO}(3)$ is the rotation matrix from the ith body frame $\mathcal{B}_i$, $i = 1, 2, \cdots, N$, to the inertial frame $\mathcal{N}$, and $r_i \in \mathbb{R}^3$ is the position vector of the center of mass of the ith body expressed in $\mathcal{N}$. Because of the geometric structure of the configuration manifold $\mathsf{SE}(3)$, it is not diffeomorphic to a vector space. The velocity of each body with respect to the inertial frame expressed in the body frame is

$${}^{\mathcal{B}_i}V_i = [\omega_i^T, v_i^T]^T \in \mathbb{R}^6 \tag{2}$$

where $\omega_i \in \mathbb{R}^3$ and $v_i \in \mathbb{R}^3$ denote the angular and translational velocities, respectively, with respect to the inertial frame $\mathcal{N}$ expressed in the $\mathcal{B}_i$ frame. According to Eqs. (1) and (2), the state of each body can be represented by $(g_i, {}^{\mathcal{B}_i}V_i) \in \mathsf{TSE}(3)$, where $\mathsf{TSE}(3) = \mathsf{SE}(3) \times \mathbb{R}^6$ is the tangent bundle of $\mathsf{SE}(3)$ and it is a disjoint union of the tangent spaces of $\mathsf{SE}(3)$. Therefore, $\mathsf{TSE}(3) = \bigcup_{g\in\mathsf{SE}(3)} \mathsf{T}_g\mathsf{SE}(3)$, where $\mathsf{T}_g\mathsf{SE}(3)$ is the tangent space of $\mathsf{SE}(3)$ at g (see Fig. 2). Using the geometric mechanics framework based on $\mathsf{SE}(3)$, the positions and attitudes of the N bodies can be considered simultaneously, and hence performing a dynamics analysis in $\mathsf{SE}(3)$ is convenient to use in cases when orbit/attitude coupling is present.

Let the inertial angular velocity of the ith body expressed in the inertial frame $\mathcal{N}$ be denoted by

$$\Omega_i = R_i \omega_i \tag{3}$$

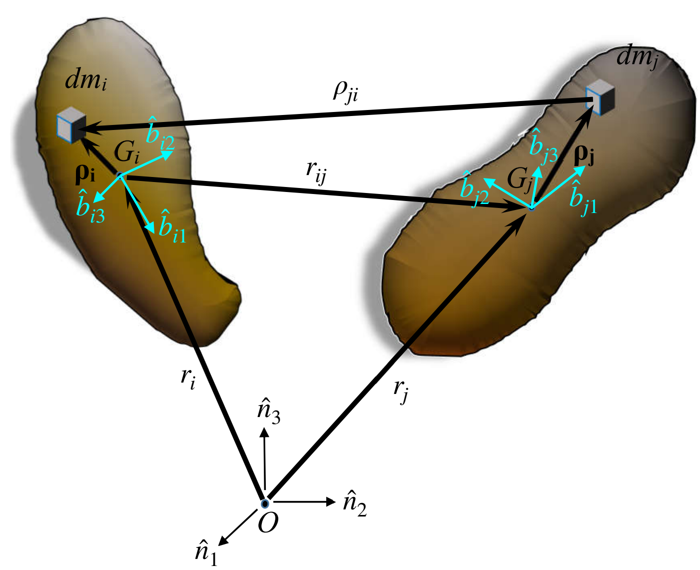

Figure 1. Any two bodies in a F*N*BP.

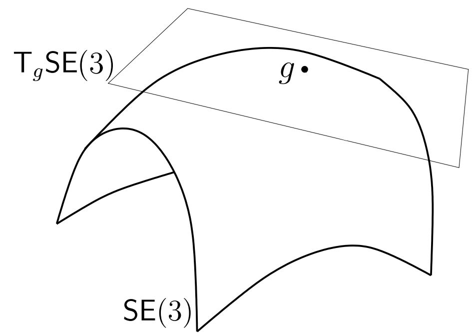

Figure 2. $\mathsf{SE}(3)$ **and its tangent space at** g**.**

Also, recall that r_i and v_i are expressed in the inertial and body frames, respectively. Hence, the inertial translational velocity of the center of mass expressed in inertial frame is

$$\dot{r}_i = R_i v_i \tag{4}$$

In order to make the study of F*N*BP convenient, we introduce the terminology of "spatial velocity" in the same form as in Ref. [21]. This terminology helps to understand the meaning of the motion of a rigid body with respect to another body, regardless of the frame we are in. For this purpose, we define the adjoint transformation associated with g, $\mathrm{Ad}_g : \mathbb{R}^6 \to \mathbb{R}^6$, as

$$\mathrm{Ad}_g = \begin{bmatrix} R & 0_{3\times 3} \\ r^\times R & R \end{bmatrix} \tag{5}$$

which is a 6×6 matrix, where the cross operator $(\cdot)^\times$ is defined for $y = [y_1, y_2, y_3]^T \in \mathbb{R}^3$ as

$$y^\times = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \tag{6}$$

such that $e_1^\times e_2 = e_1 \times e_2$ for $e_1, e_2 \in \mathbb{R}^3$. The given transformation is invertible and, using the partitioned matrix inverse method, its inverse is obtained as

$$\mathrm{Ad}_g^{-1} = \begin{bmatrix} R^T & 0_{3\times 3} \\ -R^T r^\times & R^T \end{bmatrix} \tag{7}$$

Now, using the adjoint transformation above, we define spatial velocity as

$${}^{\mathcal{N}}V_i = \mathrm{Ad}_{g_i}\, {}^{\mathcal{B}_i}V_i = \begin{bmatrix} R_i & 0 \\ r_i^\times R_i & R_i \end{bmatrix} \begin{bmatrix} \omega_i \\ v_i \end{bmatrix} = \begin{bmatrix} \Omega_i \\ r_i^\times \Omega_i + \dot{r}_i \end{bmatrix} \tag{8}$$

where the definition of Ad_g in Eq. (5) and the expressions for inertial angular and translational velocities in terms of their body-frame counterparts (Eqs. (3) and (4)) are used. The velocity ${}^{\mathcal{N}}V$ in Eq. (8) is the velocity of the inertial frame with respect to the body frame, expressed in the inertial frame (note how this interpretation compares to the definition of ${}^{\mathcal{B}_i}V$ in Eq. (2)). The terminology above relates the view of motion between the perspectives of two different frames. Using transport theorem, it can be shown that the time derivative of the body frame position that is expressed in the body frame with respect to the inertial frame and expressed in the inertial frame, i.e. $R_i \frac{d}{dt}(R_i^T r_i)$, is the same as the time derivative of the body frame position that is expressed in the inertial frame with respect to the body frame, i.e. ${}^{\mathcal{B}_i}(\frac{dr_i}{dt})$. It is important to realize that one of the frames is assumed to be the body frame and another frame is assumed to be the inertial frame. However, in general, none of the frames need to be an inertial frame. This will tremendously help to understand the F*N*BPs from different frame perspectives. Also note that the definition of the adjoint transformation here alters from that in Ref. [21], but the definition in that reference should also result in the same general conclusion here.

A set of operators and mappings required to formulate the F*N*BP in the geometric mechanics framework are defined next. The adjoint operator ad_{V_i} $(i = 1, 2, \cdots, N)$ is formulated as

$$\mathrm{ad}_{{}^{\mathcal{B}_i}V_i} = \begin{bmatrix} \omega_i^\times & 0_3 \\ v_i^\times & \omega_i^\times \end{bmatrix} \in \mathbb{R}^{6\times 6} \tag{9}$$

The co-adjoint operator is defined as

$$\mathrm{ad}^*_{{}^{\mathcal{B}_i}V_i} = \mathrm{ad}^T_{{}^{\mathcal{B}_i}V_i} = \begin{bmatrix} -\omega_i^\times & -v_i^\times \\ 0_{3\times 3} & -\omega_i^\times \end{bmatrix} \tag{10}$$

The wedge map $(.)^\vee : \mathbb{R}^6 \to \mathfrak{se}(3)$ is defined for ${}^{\mathcal{B}_i}V = [\omega^T, v^T]^T \in \mathbb{R}^6$ as

$${}^{\mathcal{B}_i}V_i^\vee = \begin{bmatrix} \omega_i^\times & v_i \\ 0_{1\times 3} & 0 \end{bmatrix} \in \mathfrak{se}(3) \tag{11}$$

Hence, $\mathbb{R}^6$ is isomorphic to the Lie algebra $\mathfrak{se}(3)$ of $\mathsf{SE}(3)$.

Newton's second law and Euler equations

Euler rotational equations of motion of the ith body are defined as

$${}^{\mathcal{B}_i}\dot{\omega}_i = -I_i^{-1}\omega_i^\times I_i\omega_i + I_i^{-1}{}^{\mathcal{B}_i}\tau_i \tag{12}$$

where $I_i \in \mathbb{R}^{3\times 3}$ is the matrix of moment of inertia of the ith body, and ${}^{\mathcal{B}_i}\tau_i$ denotes the total torque applied to that body, both expressed in its body frame. Also, according to the Newton's second law,

$$\ddot{r}_i = \frac{{}^{\mathcal{N}}f_i}{m_i} \tag{13}$$

where $m_i \in \mathbb{R}$ is the mass of the ith body, ${}^{\mathcal{N}}f_i \in \mathbb{R}^3$ denotes the total force applied to the ith body expressed in the inertial frame, and $\dot{(\cdot)}$ denotes the time derivative of $(\cdot)$ with respect to an inertial frame. Notice that the Euler equations are commonly expressed in the body frame while Newton's second law is commonly expressed in the inertial frame. If one of the bodies is a spacecraft modeled as a rigid body, formulation development in the body frame of the ith body is particularly important from the control point of view since the alignment of the actuators are usually known in the body frame of the spacecraft. If the dynamics of the celestial bodies are of interest, then the formulation should be provided in the inertial frame. Therefore, in what follows, dynamics are formulated in different reference frames, including both the body and the inertial frames, so that, depending on the application, either one of these sets can be used.

Formulation in the body frame

By taking the time derivative of the configuration g_i in Eq. (1), and leveraging the fact that $\dot{R}_i = R_i\omega_i^\times$, the kinematics of the ith body are obtained as

$$\dot{g}_i = \begin{bmatrix} R_i\omega_i^\times & R_i v_i \\ 0_{1\times 3} & 0 \end{bmatrix}$$

which, using the definition of the wedge map in Eq. (11), can be expressed in the compact form of

$$\dot{g}_i = g_i\, {}^{\mathcal{B}_i}V_i^\vee \tag{14}$$

Furthermore, by taking the time derivative of the both sides of Eq. (4), and using the fact that $\dot{R}_i = R_i\omega_i^\times$, the translational acceleration of the ith body expressed in its body frame is obtained from

$$\begin{aligned} \dot{v}_i &= -\omega_i^\times v_i + R_i^T \ddot{r}_i \\ &= -\omega_i^\times v_i + \frac{{}^{\mathcal{B}_i}f_i}{m_i} \end{aligned} \tag{15}$$

which obeys the transport theorem, being ${}^{\mathcal{B}_i}f_i = R^{T\mathcal{N}}f_i \in \mathbb{R}^3$ the total force applied to the ith body expressed in its body frame. Using Eqs. (12) and (15), as well as the co-adjoint operator and wedge map defined in Eqs. (10) and (11), the kinetics of the ith body are expressed in the compact form of

$${}^{\mathcal{B}_i}\dot{V}_i = {}^{\mathcal{B}_i}\mathbb{I}_i^{-1}\mathrm{ad}^*_{{}^{\mathcal{B}_i}V_i}\, {}^{\mathcal{B}_i}\mathbb{I}_i\, {}^{\mathcal{B}_i}V_i + {}^{\mathcal{B}_i}\mathbb{I}_i^{-1}\, {}^{\mathcal{B}_i}u_i \tag{16}$$

where ${}^{\mathcal{B}_i}V_i = [\omega_i^T, v_i^T]^T$, $\omega_i \in \mathbb{R}^3$ and $v_i \in \mathbb{R}^3$ denote the inertial angular and translational velocities, respectively, both expressed in the body frame $\mathcal{B}_i$, and ${}^{\mathcal{B}_i}u_i$ represents the augmented vector of total torques and forces applied to the body i, i.e. ${}^{\mathcal{B}_i}u_i = [{}^{\mathcal{B}_i}\tau_i{}^T, {}^{\mathcal{B}_i}f_i{}^T]^T$. Also, in Eq. (16), ${}^{\mathcal{B}_i}\mathbb{I}_i = \mathrm{blkdiag}(I_i, m_i\mathbf{1}_3)$ is called the generalized inertia matrix in the body frame, where $\mathbf{1}_3$ is the 3×3 identity matrix and $\mathrm{blkdiag}(\cdot)$ denotes the block diagonal matrix. Therefore, the dynamics (kinematics and kinetics) of the ith body are expressed in its body frame by Eqs. (14) and (16).

Formulation in the Inertial Frame

The F*N*BP is formulated in this section relative to the inertial frame in the framework of $\mathsf{SE}(3)$. The following proposition is used:

Proposition 1 *For $x \in \mathbb{R}^3$ and an orthogonal matrix R, $(Rx)^\times = Rx^\times R^T$.*
Proof. *Assume that an orthogonal coordinate frame (x, y, z) is transformed through a rotation matrix R to another orthogonal coordinate frame (ζ, η, ξ), i.e. $\zeta = Rx$, $\eta = Ry$, and $\xi = Rz$. Since the coordinates (x, y, z) are orthogonal, $z = x^\times y$. Hence,*

$$\xi = Rz = Rx^\times y = Rx^\times R^T Ry \tag{17}$$

where R is an orthogonal matrix. Also, since the coordinates (ζ, η, ξ) are orthogonal,

$$\xi = \zeta^\times \eta = (Rx)^\times Ry \tag{18}$$

Equating the right hand sides of Eqs. (17) and (18) yields

$$((Rx)^{\times} - Rx^{\times}R^{T})Ry = 0 \tag{19}$$

which should hold in any y direction, i.e. $(Rx)^{\times} = Rx^{\times}R^{T}$, and the proof is complete.

According to Eq. (3) and Proposition 1, $\omega_i^{\times} = R_i^T \Omega_i^{\times} R_i$. Hence, the time derivative of the configuration g_i in Eq. (14) may also be written as

$$\dot{g}_i = \begin{bmatrix} \Omega_i^{\times} R_i & \dot{r}_i \\ 0_{1\times 3} & 0 \end{bmatrix}$$

which is another representation of the kinematics of the ith body, and, using the definition of the wedge map in Eq. (11), its compact form corresponds to

$$\dot{g}_i = {}^{\mathcal{N}}V_i^{\vee}\, g_i \tag{20}$$

The time derivative of the adjoint transformation Ad is obtained by taking the time derivative of its partitions in Eq. (5). Using the property of Proposition 1, it is possible to show that

$$\frac{d}{dt}\mathrm{Ad}_{g_i} = \mathrm{Ad}_{g_i}\mathrm{ad}_{{}^{\mathcal{B}_i}V_i} \tag{21}$$

Now, by taking the time derivative of both sides of Eq. (8) and noting that $\mathrm{ad}_{{}^{\mathcal{B}_i}V_i}{}^{\mathcal{B}_i}V_i = 0_{6\times 1}$,

$${}^{\mathcal{N}}\dot{V}_i = \mathrm{Ad}_{g_i}{}^{\mathcal{B}_i}\dot{V}_i \tag{22}$$

Furthermore, in the proposed framework, the similarity transformation for the generalized inertia matrix is introduced as

$${}^{\mathcal{N}}\mathbb{I}_i = \mathrm{Ad}_{g_i}^{-T}\, {}^{\mathcal{B}_i}\mathbb{I}_i\, \mathrm{Ad}_{g_i}^{-1} \quad (i = 1, 2, \cdots, N), \tag{23}$$

where ${}^{\mathcal{N}}\mathbb{I}_i$ is the generalized inertia matrix in the inertial frame and the superscript $(-T)$ denotes the inverse transpose. Note that using the definition of the adjoint transformation in Eq. (5), and the rotation transformation for inertia matrix

$${}^{\mathcal{N}}I_{i_{G_i}} = R_i\, I_i R_i^T \tag{24}$$

where ${}^{\mathcal{N}}I_{i_{G_i}}$ is the matrix of moment of inertia of the ith body about its center of mass expressed in the inertial frame, we can show that

$${}^{\mathcal{N}}\mathbb{I}_i = \begin{bmatrix} {}^{\mathcal{N}}I_{i_{G_i}} - m_i r_i^{\times 2} & m_i r_i^{\times} \\ -m_i r_i^{\times} & m_i \mathbf{1}_3 \end{bmatrix} \tag{25}$$

The result obtained above is the generalized form of the parallel axis theorem, where the top-left term ${}^{\mathcal{N}}I_{i_O} \equiv {}^{\mathcal{N}}I_{i_{G_i}} - m_i r_i^{\times 2}$ is the matrix of moment of inertia of the body about the origin of the inertial frame . The inverse of the generalized inertia matrix is

$${}^{\mathcal{N}}\mathbb{I}_i^{-1} = \mathrm{Ad}_{g_i}{}^{\mathcal{B}_i}\mathbb{I}_i^{-1}\mathrm{Ad}_{g_i}^{T} = \begin{bmatrix} {}^{\mathcal{N}}I_{i_{G_i}}^{-1} & -{}^{\mathcal{N}}I_{i_{G_i}}^{-1} r_i^{\times} \\ r_i^{\times}\,{}^{\mathcal{N}}I_{i_{G_i}}^{-1} & -r_i^{\times}\,{}^{\mathcal{N}}I_{i_{G_i}}^{-1} r^{\times} + \frac{1}{m}\mathbf{1}_3 \end{bmatrix} \tag{26}$$

Note that ${}^{\mathcal{N}}\mathbb{I}_i$ and ${}^{\mathcal{N}}\mathbb{I}_i^{-1}$ are both symmetric. We have the following propositions.

Proposition 2 *For* $a, b \in \mathbb{R}^3$, $(a^\times b)^\times = a^\times b^\times - b^\times a^\times = ba^T - ab^T$.
Proof. *The proof is straightforward using the definition of the cross mapping given in Eq. (6).*

Proposition 3 *The generalized similarity transformation in Eq. (23) may also be applied on the co-adjoint operator.*
Proof. *Applying Eq. (23) onto* $\mathrm{ad}^*_{\mathcal{B}_i V}$ *and using the property in Proposition 2 yields*

$$\mathrm{Ad}_g^{-T}\mathrm{ad}^*_{\mathcal{B}_i V}\mathrm{Ad}_g^T = \begin{bmatrix} R & r^\times R \\ 0_{3\times3} & R \end{bmatrix}\begin{bmatrix} -\omega^\times & -v^\times \\ 0_{3\times3} & -\omega^\times \end{bmatrix}\begin{bmatrix} R^T & -R^T r^\times \\ 0_{3\times3} & R^T \end{bmatrix} = \begin{bmatrix} -\Omega^\times & -(r^\times\Omega + \dot{r})^\times \\ 0_{3\times3} & -\Omega^\times \end{bmatrix} \quad (27)$$

which, according to the expression for spatial velocity in Eq. (8) and the definition of co-adjoint operator in Eq. (10), is identical to $\mathrm{ad}^*_{\mathcal{N} V}$*, and the proof is complete.*

By substituting Eq. (16) into Eq. (22) as well as using Eq. (23) and Proposition 3, the following is obtained

$${}^{\mathcal{N}}\dot{V}_i = {}^{\mathcal{N}}\mathbb{I}_i^{-1}\mathrm{ad}^*_{{}^{\mathcal{N}}V_i}\,{}^{\mathcal{N}}\mathbb{I}_i\,{}^{\mathcal{N}}V_i + {}^{\mathcal{N}}\mathbb{I}_i^{-1}\,{}^{\mathcal{N}}u_i \quad (28)$$

where ${}^{\mathcal{N}}V_i$ is defined by Eq. (8), and it is the augmented inertial velocity vector of the ith body expressed in the inertial frame. Also,

$${}^{\mathcal{N}}u_i = \mathrm{Ad}_{g_i}^{-T\,\mathcal{B}_i}u_i = \begin{bmatrix} R_i & r_i^\times R_i \\ 0_{3\times3} & R_i \end{bmatrix}\begin{bmatrix} {}^{\mathcal{B}_i}\tau_i \\ {}^{\mathcal{B}_i}f_i \end{bmatrix} = \begin{bmatrix} {}^{\mathcal{N}}\tau_i + r_i^\times {}^{\mathcal{N}}f_i \\ {}^{\mathcal{N}}f_i \end{bmatrix} \quad (29)$$

denotes the external effects applied to the ith body expressed in the inertial frame, where ${}^{\mathcal{N}}f_i$ denotes the total force applied to the ith body expressed in the inertial frame, as defined below Eq. (13), and ${}^{\mathcal{N}}\tau_i$ denotes the total torque applied to the ith body about its center of mass, expressed in the inertial frame. Therefore, the dynamics of the ith body are expressed in the inertial frame by Eqs. (20) and (28). Note in Eq. (29) that $r_i^\times\,{}^{\mathcal{N}}f_i$ is the torque about the center of the inertial frame due to the force ${}^{\mathcal{N}}f_i$ that is applied to the center of mass of the rigid body. Additionally, the top row in that equation denotes the total torque about that center. Note that the rotational component of Eq. (28) is $\dot{\Omega}_i = -{}^{\mathcal{N}}I_{i_{G_i}}^{-1}\Omega_i^\times {}^{\mathcal{N}}I_{i_{G_i}}\Omega_i + {}^{\mathcal{N}}I_{i_{G_i}}^{-1}{}^{\mathcal{N}}\tau_i$, which is equivalent to Euler rotational equations of motion in Eq. (12) pre-multiplied by R_i and after transforming the terms into the inertial frame, and the translational component of Eq. (28) is the same as Newton's second law in Eq. (13).

According to the results of the past two sections, the formalism in the proposed framework enables the augmentation of Newton's second law and Euler equations in one compact form with a structure independent of the choice of the coordinate frame. However, for highly large values of r_i, the generalized inertia matrix in the inertial frame may be nearly singular, as seen in Eq. (25). Therefore, normalization of the equations may be required to overcome such a singularity issue.

KINETIC ENERGY AND GRAVITATIONAL POTENTIAL IN F*N*BP

The representation of the kinetic energy of the ith body in the proposed framework has the same form regardless of the choice of the reference frame. Additionally, a common approximation of the most significant potential, i.e. gravitational potential, is used to obtain the gravitational effects on the ith body.

Formulation of kinetic energy in the body frame: Let the position of dm_i with respect to the center of mass of the ith body in Fig. 1, i.e. ρ_i, be expressed in the inertial frame. Also, let $\sigma_i = R^T \rho_i$ be that position expressed in the body frame of the ith body. Since the ith body is assumed to be rigid, $\|\sigma_i\| = \|\rho_i\|$ is constant, and the velocity of the differential mass expressed in the body frame of the ith body corresponds to $v_i - \sigma_i^\times \omega_i$. The kinetic energy of the ith body is obtained by integrating the kinetic energy of the differential mass dm_i over the body, i.e.

$$\begin{aligned} T_i &= \frac{1}{2}\int_{\mathcal{B}_i} (v_i - \sigma_i^\times \omega_i)^T (v_i - \sigma^\times \omega_i) dm_i \\ &= \frac{1}{2} m_i v_i^T v_i - \frac{1}{2}\omega_i^T I_i \omega_i \\ &= \frac{1}{2} {}^{\mathcal{B}_i}V_i^T {}^{\mathcal{B}_i}\mathbb{I}_i {}^{\mathcal{B}_i}V_i \end{aligned} \tag{30}$$

where the definitions of center of mass, i.e. $\int_{\mathcal{B}_i} \sigma_i^\times dm_i = 0_{3\times 3}$, and inertia matrix, i.e. $I_i = -\int_{\mathcal{B}_i} {\sigma_i^\times}^2 dm_i$ as well as ${}^{\mathcal{B}_i}V_i$ and ${}^{\mathcal{B}_i}\mathbb{I}_i$ are used to obtain the given equation.

Formulation of kinetic energy in the inertial frame: The following proposition shows that, using the proposed formalism, the expression of kinetic energy of a body is independent of the reference frame.

Proposition 4 *Kinetic energy of the ith body may be expressed in inertial frame in a form similar in structure to that in the body frame.*

Proof. *Equation (8) is used to write the velocity in the body frame in terms of velocity in the inertial frame, i.e. ${}^{\mathcal{B}_i}V_i = \mathrm{Ad}_{g_i}^{-1} {}^{\mathcal{N}}V_i$. Also, using the similarity transformation in Eq. (23), the ${}^{\mathcal{B}_i}\mathbb{I}_i$, is written in terms of the generalized inertia matrix in the inertial frame, i.e. ${}^{\mathcal{N}}\mathbb{I}_i$. After substituting these into Eq. (30) and simplifying, the latter is expressed as*

$$T_i = \frac{1}{2} {}^{\mathcal{N}}V_i^T {}^{\mathcal{N}}\mathbb{I}_i^{\mathcal{N}} V_i \tag{31}$$

which has the same structure as Eq. (30).

Alternative Proof: *The velocity of the differential mass dm_i is expressed in the inertial frame as $\dot{r}_i - \rho_i^\times \Omega_i$. Hence, the kinetic energy of the ith body may be formulated in the inertial frame as*

$$\begin{aligned} T_i &= \frac{1}{2}\int_{\mathcal{B}_i} (\dot{r}_i - \rho_i^\times \Omega_i)^T (\dot{r}_i - \rho_i^\times \Omega_i) dm_i \\ &= \frac{1}{2} m_i \dot{r}_i^T \dot{r}_i - \frac{1}{2}\Omega_i^T {}^{\mathcal{N}}I_{iG_i} \Omega_i \\ &= \frac{1}{2} {}^{\mathcal{N}}V_i^T {}^{\mathcal{N}}\mathbb{I}_i^{\mathcal{N}} V_i \end{aligned} \tag{32}$$

where the definitions of center of mass (i.e. $\int_{\mathcal{B}_i} \rho_i^\times dm_i = 0_{3\times 3}$), and the inertia matrix

$$-\int_{\mathcal{B}_i} {\rho_i^\times}^2 dm_i = -\int_{\mathcal{B}_i} \left(R_i \sigma_i^\times R_i^T\right)^2 dm_i = -R_i \left(\int_{\mathcal{B}_i} {\sigma_i^\times}^2 dm_i\right) R_i^T = R_i I_i R_i^T = {}^{\mathcal{N}}I_{iG_i}$$

are used to obtain the second line of the equation, as well as ${}^{\mathcal{N}}V_i$ and ${}^{\mathcal{N}}\mathbb{I}_i$ to obtain the third line of the equation.

Finally, note that the total kinetic energy of the bodies is obtained by adding the kinetic energies of all the bodies, i.e. $T = \sum_{i=1}^{N} T_i$.

Gravitational potential and its approximation The gravitational potential on the ith body in the FNBP is

$$U = G\sum_{1\le i<j\le N}^{N}\int_{\mathcal{B}_i}\int_{\mathcal{B}_j}\frac{dm_i dm_j}{\|r_{ij}+\rho_j-\rho_i\|} \tag{33}$$

where $r_{ij} = r_j - r_i \in \mathbb{R}^3$ is the vector connecting the centers of mass of bodies i and j, and ρ_i and ρ_j are directed from the centers of masses of the ith and jth bodies to the infinitesimal mass elements dm_i and dm_j, respectively, as shown in Fig. 1. Equation (33) applies in its general form here. However, since the equations for the gravity moments and forces are expressed in terms of the gravitational potential and its derivatives with respect to the rotation matrix and position vector, it is required to obtain the gravitational potential first. In problems where the information about the masses and inertia properties of the rigid bodies are given, but their shapes are not necessarily known, Eq. (33) cannot be used directly to obtain the gravitational potential. An expression of gravitational potential in terms of the masses and inertia properties of the bodies should be used instead to obtain the gravity moments and forces. Using Legendre polynomials, the gravity potential in Eq. (33) is approximated in terms of the inertia tensors and masses of the bodies for the F2BP,[22,23] which can be extended to FNBP as:

$$\begin{aligned} U \;=\; & G\sum_{1\le i<j\le N}\Big\{\frac{m_i m_j}{\|r_{ij}\|}+\frac{1}{2\|r_{ij}\|^3}\left[m_j\mathrm{tr}(I_i)+m_i\mathrm{tr}(I_j)-3\hat{r}_{ij}^T\left(m_j R_i I_i R_i^T + m_i R_j I_j R_j^T\right)\hat{r}_{ij}\right] \\ & + O(\|r_{ij}\|^{-5})\Big\} \end{aligned} \tag{34}$$

where $\hat{r}_{ij} = \frac{r_{ij}}{\|r_{ij}\|}$. The selection of Eq. (33) or (34) to compute the gravitational potential depends on the information given about either the shape and/or inertia properties of the bodies.

Gravitational effects on the body The gravity gradient torque and gravity force applied on the ith body are obtained in its body frame from[24]

$$^{\mathcal{B}_i}\tau_i^{g\times} = R_i^T\frac{\partial U}{\partial R_i} - \frac{\partial U}{\partial R_i^T}\,R_i, \tag{35a}$$

$$^{\mathcal{N}}f_i^g = \frac{\partial U}{\partial r_i} \tag{35b}$$

Using the rotational transformation between the body and inertial frames as well as the property in Proposition 1, the gravity gradient torque applied to the ith body is obtained

$$^{\mathcal{N}}\tau_i^{g\times} = R_i\,{}^{\mathcal{B}_i}\tau_i^{g\times}R_i^T = \frac{\partial U}{\partial R_i}R_i^T - R_i\frac{\partial U}{\partial R_i^T}, \tag{36}$$

Substituting the approximate formula for U given in Eq. (34) into Eq. (36), the latter equation is rewritten as

$$\begin{aligned} ^{\mathcal{N}}\tau_i^{g\times} \;=\; & \frac{3m_jG}{2\|r_{ij}\|^3}\left(\left(-I_iR_i^T\hat{r}_{ij}\hat{r}_{ij}^T + I_iR_i^T\hat{r}_{ij}\hat{r}_{ij}^T\right)^T R_i^T + R_i\left(I_iR_i^T\hat{r}_{ij}\hat{r}_{ij}^T + I_iR_i^T\hat{r}_{ij}\hat{r}_{ij}^T\right)\right) \\ =\; & -\frac{3m_jG}{\|r_{ij}\|^3}\left(\hat{r}_{ij}\hat{r}_{ij}^T\,{}^{\mathcal{N}}I_{i_{G_i}} - {}^{\mathcal{N}}I_{i_{G_i}}\hat{r}_{ij}\hat{r}_{ij}^T\right) \end{aligned} \tag{37}$$

The matrix property $\frac{\partial b^T A^T B A c}{\partial A} = B^T A b c^T + B A c b^T$ (for $A \equiv R_i^T$, $b = c \equiv \hat{r}_{ij}$, and $B \equiv I_i$) and the property $\frac{\partial U}{\partial R_i^T} = \left(\frac{\partial U}{\partial R_i}\right)^T$ are used to obtain Eq. (37). Furthermore, according to Proposition 2, and by setting $a \equiv r_{ij}$ and $b \equiv {}^{\mathcal{N}}I_{iG_i} r_{ij}$ in that proposition, the following property is obtained:

$$(r_{ij}^{\times} {}^{\mathcal{N}}I_{iG_i} r_{ij})^{\times} = {}^{\mathcal{N}}I_{iG_i} r_{ij} r_{ij}^T - r_{ij} r_{ij}^T \, {}^{\mathcal{N}}I_{iG_i} \tag{38}$$

Substituting the property in Eq. (38) into Eq. (37) and taking the inverse of the cross mapping, the gravity moments applied to the bodies are obtained as $(i = 1, 2, 3)$

$${}^{\mathcal{N}}\tau_i^g \; = \; 3G \textstyle\sum_{j=1, j\neq i}^{3} \frac{m_j}{\|r_{ij}\|^5} r_{ij}^{\times} \, {}^{\mathcal{N}}I_{iG_i} r_{ij} \; + \; \text{h.o.t.} \tag{39}$$

where h.o.t. denotes the higher order terms. For details of higher order gravitational terms, the reader is referred to Ref. [25]. Furthermore, defining $a_1 \equiv 1/\|r_i\|$, $a_2 \equiv 1/\|r_i\|^3$, and $a_3 \equiv \hat{r}_{ij}^T(m_j \, {}^{\mathcal{N}}I_{iG_i} + m_i \, {}^{\mathcal{N}}I_{jG_j})\hat{r}_{ij}$, the gravitational potential in Eq. (34) may be represented in terms of a_1, a_2, and a_3. Then, using Eq. (35b), the gravity force applied to the body i is obtained

$${}^{\mathcal{N}}f_i^g = G \sum_{1 \le i < j \le N} m_i m_j \frac{\partial a_1}{\partial r_i} + \frac{1}{2}\frac{\partial a_2}{\partial r_i}\left(m_j \text{tr}(I_i) + m_i \text{tr}(I_j) - 3a_3\right) - \frac{3}{2} a_2 \frac{\partial a_3}{\partial r_i} \tag{40}$$

The partial derivatives of a_1, a_2, and a_3 are obtained with respect to r_i as $\frac{\partial a_1}{\partial r_{ij}} = -\frac{r_{ij}}{\|r_{ij}\|^3}$, $\frac{\partial a_2}{\partial r_{ij}} = -3\frac{r_{ij}}{\|r_{ij}\|^5}$, and, after some manipulation, $\frac{\partial a_3}{\partial r_i} = \frac{2}{\|r_{ij}\|^2}(\hat{r}_{ij}^T(m_j \, {}^{\mathcal{N}}I_{iG_i} + m_i \, {}^{\mathcal{N}}I_{jG_j})\hat{r}_{ij}\mathbf{1}_3 - (m_j \, {}^{\mathcal{N}}I_{iG_i} + m_i \, {}^{\mathcal{N}}I_{jG_j}))r_{ij}$, noting that only the r_{ij} terms are a function of r_i. Substituting the expressions above into Eq. (40), the gravity force applied to the body i is obtained as

$$\begin{aligned} {}^{\mathcal{N}}f_i^g = -G m_i \textstyle\sum_{j=1, j\neq i}^{3} \frac{m_j}{\|r_{ij}\|^3} \Big\{ \mathbf{1}_3 + \frac{3}{m_i \|r_{ij}\|^2} \Big[{}^{\mathcal{N}}I_{iG_i} + \frac{m_i}{m_j} \, {}^{\mathcal{N}}I_{jG_j} + \frac{1}{2} \Big(\text{tr}({}^{\mathcal{N}}I_{iG_i} + \frac{m_i}{m_j} \, {}^{\mathcal{N}}I_{jG_j}) \\ -5\hat{r}_{ij}^T \Big({}^{\mathcal{N}}I_{iG_i} + \frac{m_i}{m_j} \, {}^{\mathcal{N}}I_{jG_j} \Big) \hat{r}_{ij} \Big) \mathbf{1}_3 \Big] \Big\} r_{ij} + \text{h.o.t.} \end{aligned} \tag{41}$$

Therefore, both the gravity force and torque applied to the rigid body are obtained in the inertial frame.

CRF3BP IN THE PROPOSED FRAMEWORK

Consider a rigid-body spacecraft with mass m_3 and inertia I_3 in the CRF3BP shown in Fig. 3, where the two primaries are shown by (m_1, I_1) and (m_2, I_2). It is assumed that the motion of rigid-body spacecraft does not influence the motion of the primaries. Let the barycentric rotating frame be denoted by $\mathcal{C}$ and let its origin C be the same as the origin of the inertial frame, i.e. C is the barycenter of the primaries (I_1, m_1) and (I_2, m_2) which are interacting under their mutual gravitational effects. The barycentric frame is rotating with the constant angular velocity $\omega_{\mathcal{C}}\hat{K}$ with respect to the inertial frame, where $\hat{K}$ is the third axis of both barycentric and inertial frames. The configuration of the barycentric frame is

$$g_{\mathcal{C}} = \begin{bmatrix} R_{\mathcal{C}} & 0_{3\times 1} \\ 0_{1\times 3} & 1 \end{bmatrix} \tag{42}$$

where the zero column in the upper right corner is because the center origin of $\mathcal{C}$ is at the origin of the inertial frame and $R_{\mathcal{C}}$ is the rotation matrix from the barycentric to the inertial frame given by

$$R_{\mathcal{C}} = \begin{bmatrix} \cos\omega_{\mathcal{C}}t & -\sin\omega_{\mathcal{C}}t & 0 \\ \sin\omega_{\mathcal{C}}t & \cos\omega_{\mathcal{C}}t & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{43}$$

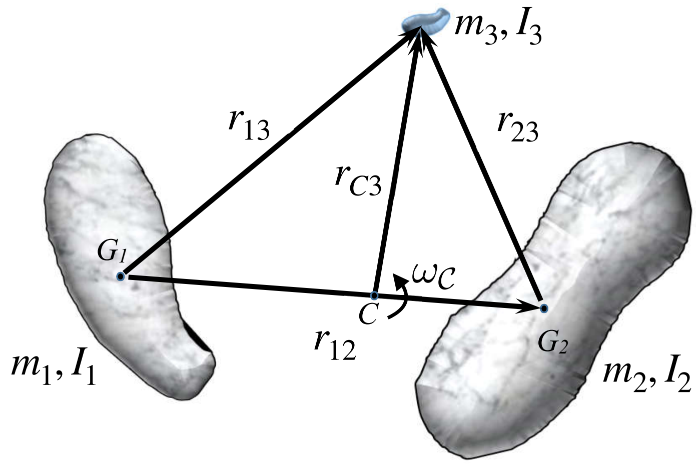

Figure 3. CRF3BP schematic

Recall that the inertial configuration of spacecraft is denoted by g as defined in Eq. (1). Then, the configuration of the spacecraft relative to the $\mathcal{C}$ frame is obtained as

$$\bar{g} = g_{\mathcal{C}}^{-1} g = \begin{bmatrix} R_{\mathcal{C}}^T R & R_{\mathcal{C}}^T r \\ 0_{1\times 3} & 1 \end{bmatrix} \tag{44}$$

where Eqs. (1) and (42) are used. The time derivative of the relative configuration above is expressed as $\dot{\bar{g}} = \frac{d}{dt} g_{\mathcal{C}}^{-1} g + g_{\mathcal{C}}^{-1} \dot{g}$ which, after application of the chain rule, implementation of Eq. (20) on g and $g_{\mathcal{C}}$, and simplifying, $\dot{\bar{g}}$ becomes

$$\dot{\bar{g}} = g_{\mathcal{C}}^{-1} (-^{\mathcal{N}} V_{\mathcal{C}} +^{\mathcal{N}} V)^{\vee} g \tag{45}$$

which, in expanded form, is expressed as

$$\dot{\bar{g}} = \begin{bmatrix} R_{\mathcal{C}}^T (\Omega - \omega_{\mathcal{C}})^{\times} R & R_{\mathcal{C}}^T (\dot{r} - \omega_{\mathcal{C}}^{\times} r) \\ 0_{1\times 3} & 0 \end{bmatrix} \tag{46}$$

The adjoint transformation in Eq. (5) applied onto $\bar{g}$ is

$$\mathrm{Ad}_{\bar{g}} = \begin{bmatrix} R_{\mathcal{C}}^T R & 0_{3\times 3} \\ R_{\mathcal{C}}^T r^{\times} R & R_{\mathcal{C}}^T R \end{bmatrix} \tag{47}$$

where the Proposition 1 is used. The transformation above is used to obtain ${}^{\mathcal{C}}V$, which denotes the velocity of the inertial frame with respect to the rigid-body spacecraft expressed in the barycentric rotating frame

$${}^{\mathcal{C}}V = \mathrm{Ad}_{\bar{g}} {}^{\mathcal{B}}V = \begin{bmatrix} {}^{\mathcal{C}}\Omega \\ {}^{\mathcal{C}}r^{\times} {}^{\mathcal{C}}\Omega +^{\mathcal{C}} \dot{r} \end{bmatrix} \tag{48}$$

After presenting the above definitions, now it is possible to define the following propositions:

Proposition 5 *The relationship between* ${}^{\mathcal{C}}\dot{V}$ *and* ${}^{\mathcal{B}}\dot{V}$ *has a structure similar to that between* ${}^{\mathcal{N}}\dot{V}$ *and* ${}^{\mathcal{B}}\dot{V}$ *in Eq. (22).*

Proof. *Taking the time derivative of both sides of Eq. (48), and noting that* $\mathrm{Ad}_{\bar{g}} = \mathrm{Ad}_{g_C}^{-1}\mathrm{Ad}_g$*, the following is obtained*

$$
{}^{\mathcal{C}}\dot{V} = \frac{d}{dt}\left(\mathrm{Ad}_{g_C}^{-1}\right)\mathrm{Ad}_g{}^{\mathcal{B}}V + \mathrm{Ad}_{g_C}^{-1}\mathrm{Ad}_g{}^{\mathcal{B}}\dot{V} \tag{49}
$$

Using the property of time derivative of the inverse of a matrix, it is shown that $\frac{d}{dt}(\mathrm{Ad}_{g_C}^{-1}) = -\left(\mathrm{Ad}_{g_C}\right)^{-1}\left(\frac{d}{dt}\mathrm{Ad}_{g_C}\right)\left(\mathrm{Ad}_{g_C}\right)^{-1}$*. Substituting this into Eq. (49) gives*

$$
\begin{aligned}
{}^{\mathcal{C}}\dot{V} &= -\mathrm{Ad}_{g_C}^{-1}\left(\frac{d}{dt}\mathrm{Ad}_{g_C}\right)\mathrm{Ad}_{g_C}^{-1}\mathrm{Ad}_g{}^{\mathcal{B}}V + \mathrm{Ad}_{g_C}^{-1}\mathrm{Ad}_g{}^{\mathcal{B}}\dot{V} \\
&= -\mathrm{Ad}_{g_C}^{-1}\left(\frac{d}{dt}\mathrm{Ad}_{g_C}\right){}^{\mathcal{C}}V + \mathrm{Ad}_{\bar{g}}{}^{\mathcal{B}}\dot{V} \\
&= \mathrm{Ad}_{\bar{g}}{}^{\mathcal{B}}\dot{V}
\end{aligned} \tag{50}
$$

which is similar in structure to Eq. (22).

Proposition 6 *The generalized similarity transformation between the* $\mathcal{C}$ *and* $\mathcal{B}$ *frames is*

$$
{}^{\mathcal{C}}\mathbb{I} = \mathrm{Ad}_{\bar{g}}^{-T}\,{}^{\mathcal{B}}\mathbb{I}\,\mathrm{Ad}_{\bar{g}}^{-1} \tag{51}
$$

where ${}^{\mathcal{C}}\mathbb{I}$ *denotes the generalized inertia matrix in the barycentric rotating frame.*
Proof. *Similar to Eq. (23), the generalized similarity transformation between* $\mathcal{C}$ *and* $\mathcal{N}$ *is retrieved, and the expression for* ${}^{\mathcal{N}}\mathbb{I}$ *in terms of* ${}^{\mathcal{C}}\mathbb{I}$ *is obtained which can be equated with its expression in terms of* ${}^{\mathcal{B}}\mathbb{I}$ *in Eq. (23). Then, using matrix inversion,* ${}^{\mathcal{C}}\mathbb{I}$ *is solved for in terms of* ${}^{\mathcal{B}}\mathbb{I}$*. Using the definition of the adjoint transformation, it is easy to show that for any* $g_1, g_2 \in \mathsf{SE}(3)$, $\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1}\mathrm{Ad}_{g_2}$*. By applying this property onto the resulting expression, Eq. (51) is obtained.*

Substituting Eq. (16) into Eq. (50), and using Eq. (51), the rigid-body spacecraft kinetics are written in the barycentric rotating frame as

$$
{}^{\mathcal{C}}\dot{V} = {}^{\mathcal{C}}\mathbb{I}^{-1}\mathrm{ad}^{*}_{{}^{\mathcal{C}}V}\,{}^{\mathcal{C}}\mathbb{I}\,{}^{\mathcal{C}}V + {}^{\mathcal{C}}\mathbb{I}^{-1}\,{}^{\mathcal{C}}u \tag{52}
$$

where

$$
{}^{\mathcal{C}}u = \mathrm{Ad}_{\bar{g}}^{-T}{}^{\mathcal{B}}u = \begin{bmatrix} R_{\mathcal{C}}^{T}R & R_{\mathcal{C}}^{T}r^{\times}R \\ 0_{3\times3} & R_{\mathcal{C}}^{T}R \end{bmatrix}\begin{bmatrix} {}^{\mathcal{B}}\tau \\ {}^{\mathcal{B}}f \end{bmatrix} = \begin{bmatrix} {}^{\mathcal{C}}\tau + {}^{\mathcal{C}}r^{\times}\,{}^{\mathcal{C}}f \\ {}^{\mathcal{C}}f \end{bmatrix} \tag{53}
$$

denotes the external effects applied to the spacecraft body expressed in the barycentric rotating frame, ${}^{\mathcal{C}}f$ denotes the total force applied to the spacecraft body, and ${}^{\mathcal{C}}\tau$ denotes the total torque applied to the spacecraft body about its center of mass, both expressed in the $\mathcal{C}$ frame. In Eq. (53), the expression for $\mathrm{Ad}_{\bar{g}}^{-T}$ is obtained by writing Eq. (7) for $\bar{g}$ and taking the transpose of the result. Also, in the equation above, ${}^{\mathcal{C}}r^{\times}\,{}^{\mathcal{C}}f$ is the torque about the center of the barycentric rotating frame due to the force ${}^{\mathcal{C}}f$ that is applied to the center of mass of the rigid body. Consequently, the top row in that equation denotes the torque about that center C. The ${}^{\mathcal{C}}\dot{V}$ term in Eq. (52) is the acceleration of the inertial frame with respect to the spacecraft expressed in the barycentric rotating frame. The left-hand-side of that equation may be written in the expanded form by taking the time derivative of Eq. (48). Substituting ${}^{\mathcal{C}}\mathbb{I}$ from Eq. (51) (considering the definition of the adjoint transformation and

${}^{\mathcal{B}}\mathbb{I}$ given below Eq. (16)) and the definition of the co-adjoint operator from Eq. (10) into Eq. (52), and after cancelling the $\frac{d}{dt}({}^{\mathcal{C}}r^{\times\mathcal{C}}\Omega)$ term from the lower partitions of both sides of the result, the expanded form of Eq. (52) is obtained as

$$\begin{aligned} {}^{\mathcal{C}}\dot{\Omega} &= -{}^{\mathcal{C}}I_G^{-1\mathcal{C}}\Omega^{\times\mathcal{C}}I_G^{\mathcal{C}}\Omega +{}^{\mathcal{C}} I_G^{-1\mathcal{C}}\tau \\ {}^{\mathcal{C}}\ddot{r} &= \frac{{}^{\mathcal{C}}f}{m} \end{aligned} \tag{54}$$

where ${}^{\mathcal{C}}I_G$ is the matrix of moment of inertia of the spacecraft about its center of mass expressed in the barycentric rotating frame. Considering the rotation of barycentric frame, the right-hand side of the equation above can be written using the transport theorem as

$$\begin{aligned} {}^{\mathcal{C}}\dot{\Omega} &= {}^{\mathcal{C}}\Omega' +{}^{\mathcal{C}} \omega_{\mathcal{C}}^{\times}\, {}^{\mathcal{C}}\Omega \\ {}^{\mathcal{C}}\ddot{r} &= {}^{\mathcal{C}}r'' + 2\, {}^{\mathcal{C}}\omega_{\mathcal{C}}^{\times} r' + {}^{\mathcal{C}}\omega_{\mathcal{C}}^{\times\,2}\, {}^{\mathcal{C}}r \end{aligned} \tag{55}$$

where $(\cdot)'$ and $(\cdot)''$ denote the first and second time derivatives with respect to the barycentric rotating frame. Hence, the dynamics of a spacecraft in the CRF3BP are given by the kinematics in Eq. (45) and the combination of Eqs. (54) and (55), i.e.

$$\begin{aligned} {}^{\mathcal{C}}\Omega' &= -{}^{\mathcal{C}}\omega_{\mathcal{C}}^{\times}\, {}^{\mathcal{C}}\Omega \;-\; {}^{\mathcal{C}}I_G^{-1}\, {}^{\mathcal{C}}\Omega^{\times}\, {}^{\mathcal{C}}I_G\, {}^{\mathcal{C}}\Omega +\; {}^{\mathcal{C}}I^{-1}\, {}^{\mathcal{C}}\tau \\ {}^{\mathcal{C}}r'' &= -2\, {}^{\mathcal{C}}\omega_{\mathcal{C}}^{\times} r' - {}^{\mathcal{C}}\omega_{\mathcal{C}}^{\times\,2}\, {}^{\mathcal{C}}r + \frac{{}^{\mathcal{C}}f}{m} \end{aligned} \tag{56}$$

Note that ${}^{\mathcal{C}}\tau = R_{\mathcal{C}}^T\, {}^{\mathcal{N}}\tau$ and ${}^{\mathcal{C}}f = R_{\mathcal{C}}^T\, {}^{\mathcal{N}}f$. In the case of the CRF3BP, where the external effects are merely due to mutual gravitational forces and moments, ${}^{\mathcal{N}}\tau$ and ${}^{\mathcal{N}}f$ are replaced with Eqs. (39) and (41), respectively. Note again that Eq. (56), also denoted as the kinetic equations, are equivalent to the compact form in Eq. (52) which is similar in structure to Eqs. (16) and (28).

Classical CR3BP equations of motion: The most frequently investigated problem of this class is the special case CR3BP. In this problem, the spacecraft is considered as a point mass and, hence, all the inertia terms in Eqs. (39) and (41) are omitted. Let ${}^{\mathcal{C}}r = [x, y, z]^T$, where x is along r_{12} (Fig. 3), y is normal to x and in the motion plane of the primary bodies, and z is such that x, y, and z form a right-handed triad. Then, Eq. (56) simplifies to three scalar coupled second order differential equations of CR3BP of the form

$$\begin{aligned} & x'' - \omega_{\mathcal{C}}^2 x - 2\omega_{\mathcal{C}} y' + G\left[\frac{m_1}{\|r_{13}\|^3}\left(x+\frac{m_2}{m_1+m_2}r_{12}\right)+\frac{m_2}{\|r_{23}\|^3}\left(x-\frac{m_1}{m_1+m_2}r_{12}\right)\right]=0 \\ & y'' - \omega_{\mathcal{C}}^2 y + 2\omega_{\mathcal{C}} x' + G\left(\frac{m_1}{\|r_{13}\|^3}+\frac{m_2}{\|r_{23}\|^3}\right) y = 0 \\ & z'' + G\left(\frac{m_1}{\|r_{13}\|^3}+\frac{m_2}{\|r_{23}\|^3}\right) z = 0 \end{aligned} \tag{57}$$

which are well known.

NORMALIZATION OF THE EQUATIONS

The order of magnitude of orbital and attitude motions and their rates can be significantly different. Hence, in order to avoid numerical errors, the normalized versions of the equations developed in the previous sections are used to develop the simulations in the current investigation, where parameters that denote time (t), length (l), and mass (m) are replaced with their normalized counterparts

$\tilde{t} = t/t^*$, $\tilde{r} = l/l^*$, and $\tilde{m} = m/m^*$, respectively, where t^*, l^*, and m^* are normalizing constants. Additionally, note that the matrices of moment of inertia, the translational and rotational velocities, as well as forces and torques, are also normalized considering the units of each of those quantities in terms of time units (TU), length units (LU), and mass units (MU).

NUMERICAL SIMULATION RESULTS AND DISCUSSION

Consider that the inertial and barycentric rotating frames are initially aligned. The normalized initial conditions of the spacecraft in the Earth-Moon CRF3BP studied here are selected to be the same as those of a NRHO given in the Earth-Moon CR3BP, i.e.

$$\begin{aligned} \tilde{r}_0 &= [1.02300331117127, 7.92508194142745 \times 10^{-22}, -0.182765473770332]^T, \\ \tilde{V}_0 &= [0.3757 \times 10^3, 0.7514 \times 10^3, 1.1271 \times 10^3, -0.02286, -0.10009, 0.02377]^T \end{aligned}$$

with respect to the Earth-Moon barycentric rotating frame. Furthermore, the initial attitude of the spacecraft relative to the inertial frame, i.e., R at the initial time, is selected to be corresponding to the yaw-pitch-roll $(\pi/7, \pi/3, \pi/5)$ rad angles in the 3-2-1 set of Euler angles. The normalizing constants are $t^* = 3.7570 \times 10^5$ s (related to the orbital period of the Moon around the Earth), $l^* = 3.8475 \times 10^8$ m (the distance between the two primaries), and $m^* = 6.046 \times 10^{24}$ kg (the sum of the masses of the two primaries). Four different sets of mass and inertia properties are assumed for the spacecraft in the NRHO: Those of a cubesat (CS), the international space station (ISS), the James Webb Space Telescope (JWST), and a spacecraft with arbitrarily selected mass and inertia, hereinafter referred to as "Gateway" (GW). These are represented in Table 1, where the inertia matrices are in the spacecraft body frame.

Spacecraft	Mass (kg)	Inertia (kg.m^2)
CS	10	$\begin{bmatrix} 20 & 3 & 1 \\ 3 & 50 & 2 \\ 1 & 2 & 40 \end{bmatrix}$
JWST	6200	$\begin{bmatrix} 67946 & -83 & 11129 \\ -83 & 90061 & 103 \\ 11129 & 103 & 45821 \end{bmatrix}$
GW	40000	$\begin{bmatrix} 1 \times 10^6 & -1 \times 10^5 & 6 \times 10^4 \\ -1 \times 10^5 & 3 \times 10^6 & -6 \times 10^4 \\ 6 \times 10^4 & -6 \times 10^4 & 4 \times 10^6 \end{bmatrix}$
ISS	419725	$\begin{bmatrix} 10276978 & -1084837 & 597098 \\ -1084837 & 31940398 & -614081 \\ 597098 & -614081 & 40019058 \end{bmatrix}$

Table 1. Mass and inertia properties of the spacecraft in the NRHO

Numerical results are obtained using the integration of Eqs. (45) and (56). The torque and force models are obtained for three bodies using Eqs. (39) and (41), respectively. The same results may be obtained using the integration in the $\mathcal{N}$ frame via Eq. (20) and (28), and using the transformation from the $\mathcal{N}$ frame into the $\mathcal{C}$ frame via $g_{\mathcal{C}}^{-1}$. The system is propagated over 5 orbital periods using the ODE45 integrator. During the propagation, it is important for the orthogonality of the rotation matrices to be maintained. In order to ensure a negligible loss of orthogonality of the rotation matrix, the integration tolerance of 10^{-9} is used in the simulation results. The loss of orthogonality of the rotation matrix may be further reduced by reducing the integration tolerances. Other integration methods such as the variational integrator[26,27] avoid any orthogonality violation.

The NRHO orbit in question is shown in Fig. 4 for 'GW'. The difference between the orbits computed using the rigid-body spacecraft model discussed in this manuscript and a point-mass spacecraft with identical mass are indiscernible in the scale shown in this figure. The orbital differences between the point-mass and rigid-body models become noticeable when position and translational velocity errors are plotted. The norms of position and velocity errors between these two models (CR3BP and CRF3BP) are shown in Figs. 5 and 6, respectively, for the four spacecraft given in Table 1. Recall that CRF3BP considers the spacecraft as a rigid body, while in CR3BP the spacecraft is modeled as a point mass. The maximum errors for both position and translational velocities occur at the periapsis of the orbit, where the speed of the center of mass is maximum. It is also possible to observe that, without station-keeping maneuvers or control, the error has a tendency to increase with time. Additionally, the more deviated the moment of the inertia of spacecraft is from that of a cubic or spherical shape (which could otherwise be modeled as a point mass), the more significant the errors become as compared to a point-mass model. This is due to the fact that the initial conditions in the CRF3BP are selected to be the same as those of the CR3BP and that the rigid-body spacecraft follows its natural dynamics. This brings up the idea of finding customized families of halo orbits considering inertia properties of the spacecraft in future missions which would have the potential to reduce the control effort significantly. Finally, in order to study the attitude, rotational motions are plotted near periapsis and apoapsis for the four spacecraft in Figs. 7-10, where the orbit is shown in black and the body frame of the spacecraft at each time step is indicated by a set of right-handed coordinates represented in red, green, and blue. The plot in each panel of these figures is provided for 1250 s (selected arbitrarily) before and after periapsis or apoapsis passage. It can be seen that, for the same amount of time (i.e. 1250 s), the spacecraft makes about the same number of rotations near apoapsis as it does near periapsis.

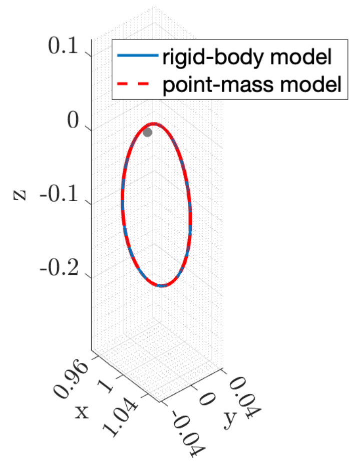

Figure 4. Normalized NRHO

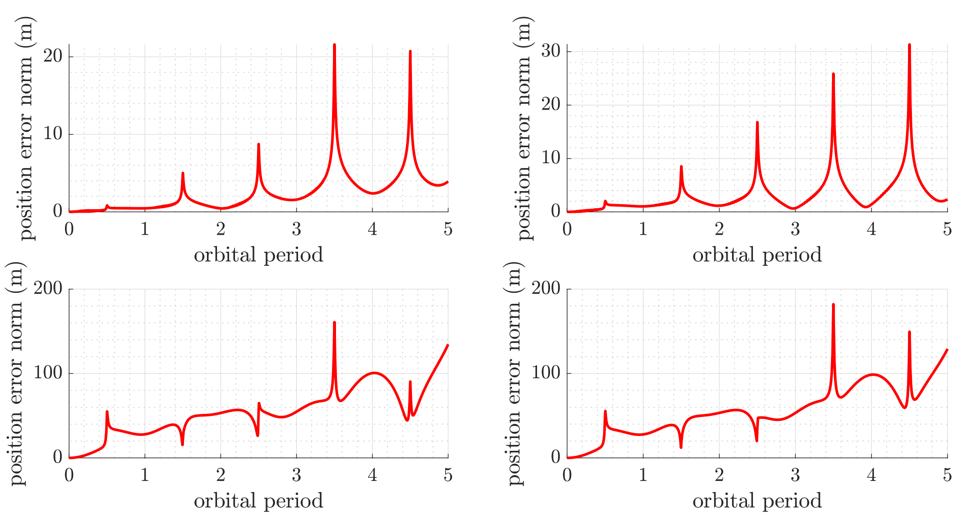

Figure 5. Position error norm: CS (upper-left), 'JWST' (upper-right), 'GW' (lower-left), and 'ISS' (lower-right)

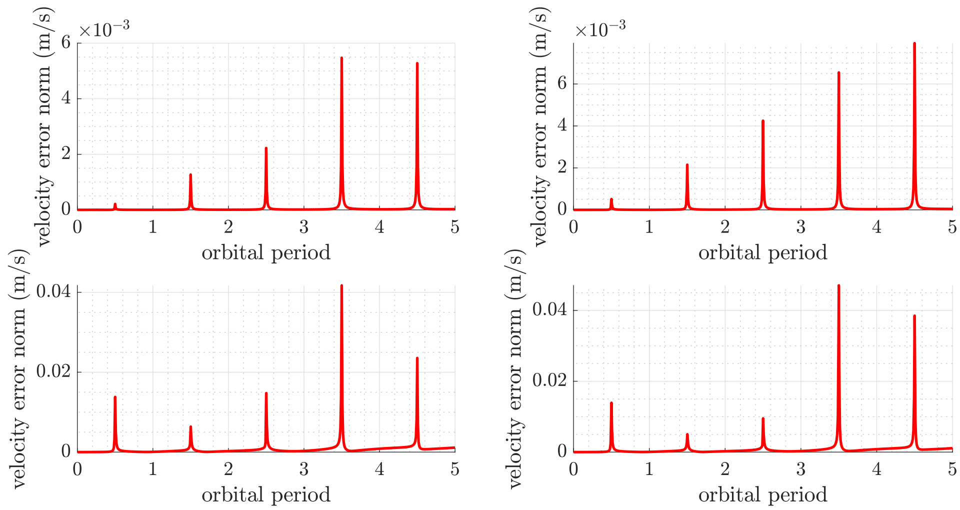

Figure 6. Velocity error norm: CS (upper-left), 'JWST' (upper-right), 'GW' (lower-left), and 'ISS' (lower-right)

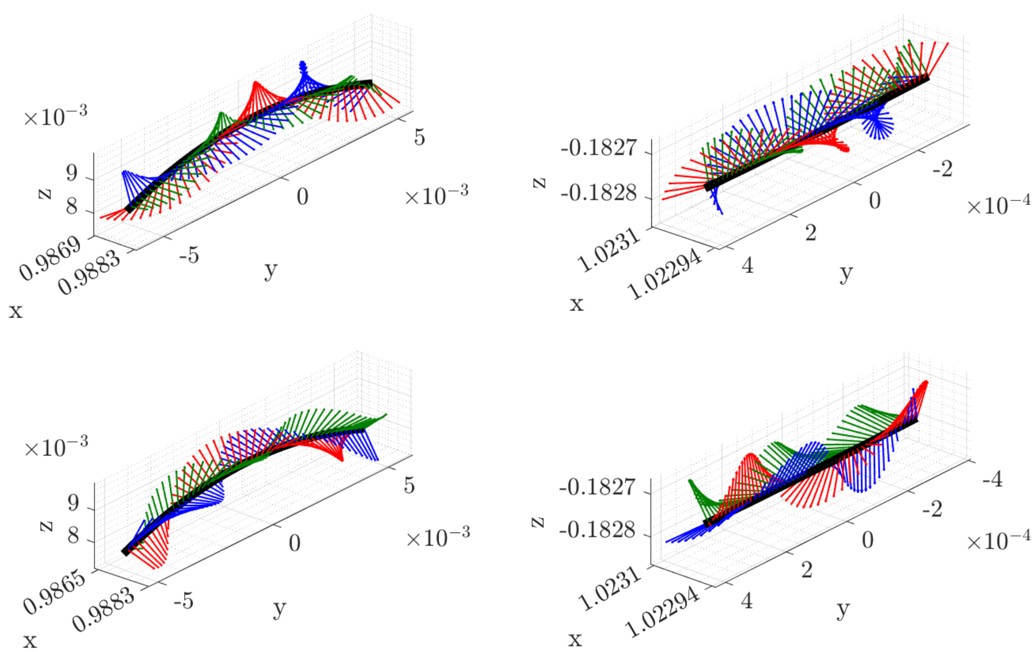

Figure 7. CS attitude near periapsis and apoapsis: First periapsis crossing (upper left), first apoapsis crossing (upper right), third periapsis crossing (lower left), and third apoapsis crossing (lower right)

CONCLUSIONS

A novel formalism was presented for the full N-body problem (FNBP) in the geometric mechanics framework of special Euclidean ($\mathsf{SE}(3)$) group and their tangent bundle $\mathsf{TSE}(3)$. The proposed formulation naturally accounts for the dynamical coupling of the orbital motion and attitude of the bodies. A general structure for rigid body dynamics formalism is presented and which is demonstrated to be independent of the choice of reference frame. In particular, the dynamics are formulated in different

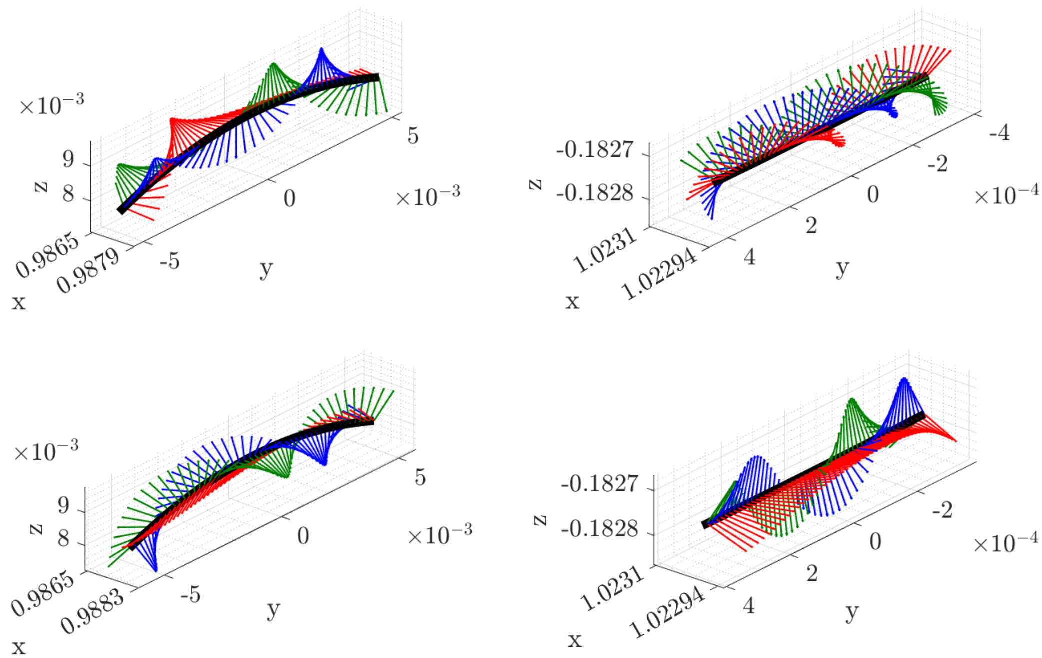

Figure 8. 'JWST' attitude near periapsis and apoapsis: First periapsis crossing (upper left), first apoapsis crossing (upper right), third periapsis crossing (lower left), and third apoapsis crossing (lower right)

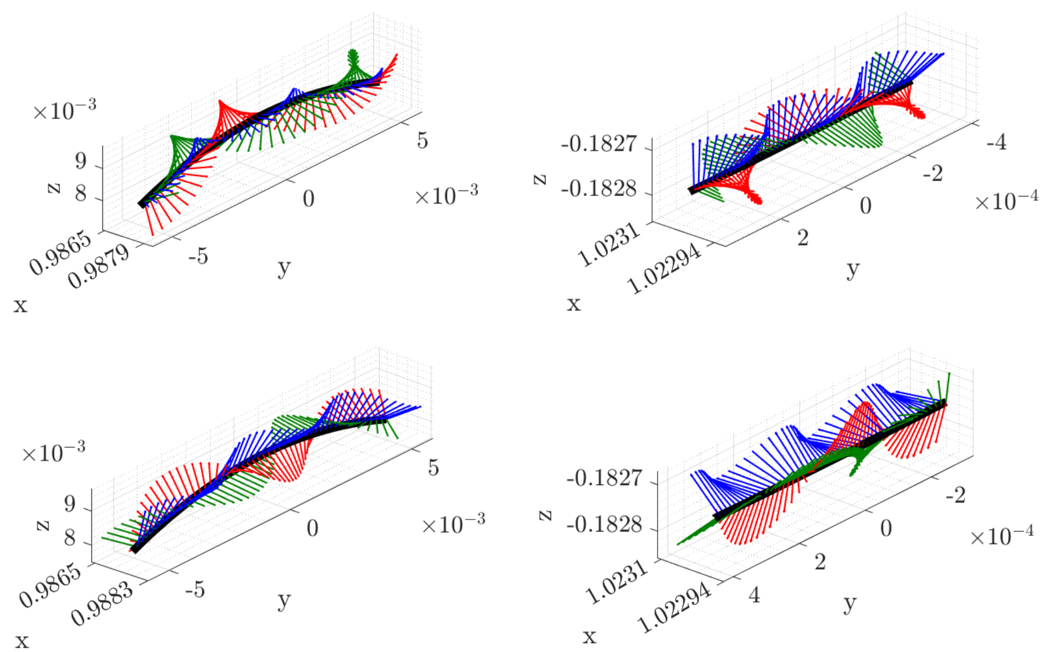

Figure 9. 'GW' attitude near periapsis and apoapsis: First periapsis crossing (upper left), first apoapsis crossing (upper right), third periapsis crossing (lower left), and third apoapsis crossing (lower right)

coordinate frames including a) the body frame and b) the inertial frame. The first set of formalism above

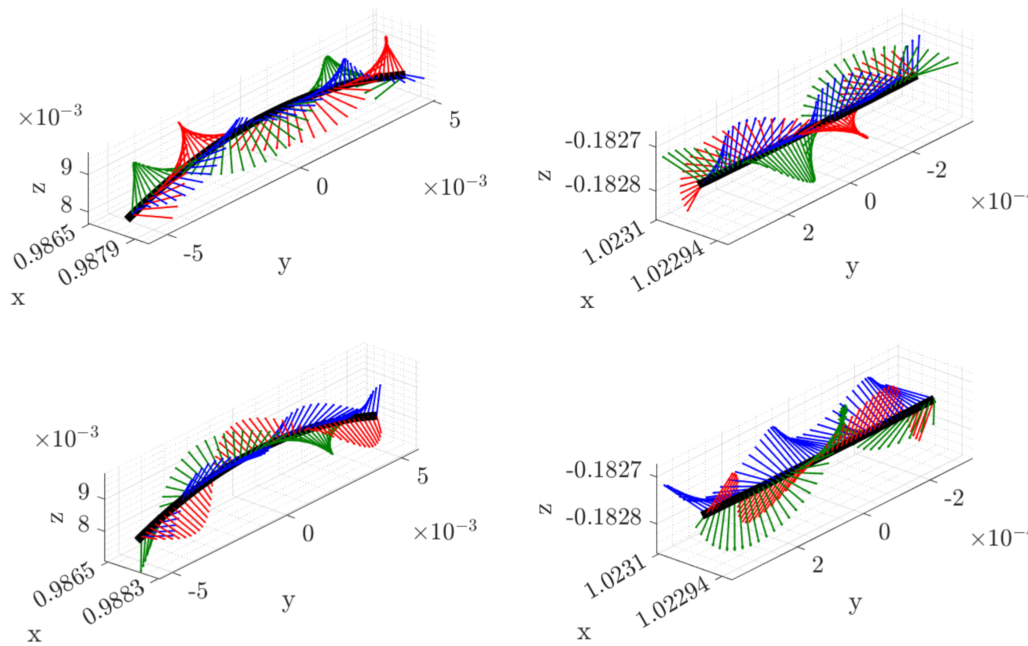

Figure 10. 'ISS' attitude near periapsis and apoapsis: First periapsis crossing (upper left), first apoapsis crossing (upper right), third periapsis crossing (lower left), and third apoapsis crossing (lower right)

is suitable for rigid-body space vehicle control design in which dynamics should be expressed in the body frame of the vehicle(s). The second set of formalism above is suitable when the motions of the celestial bodies are studied or when the dynamics are easier to be expressed in a known inertial frame.

Using the transformation between the inertial and the barycentric rotating frames, as well as leveraging the underlying assumptions behind the circular restricted full three-body problem (CRF3BP), the general dynamical representation proposed here for F*N*BP is revised to formulate the dynamics of CRF3BP, where the third body is considered as a rigid body, while it does not affect the motions of the other two primaries. Then, it is demonstrated that the CRF3BP formalism simplifies to recapture the well-known equations of motion in the conventional case of restricted three-body problem. The general formulation proposed here, when used for only three bodies, is applied to understand orbital-attitude coupling in the near-rectilinear halo orbit (NRHO). The comparison between CRF3BP and CR3BP using the CR3BP (i.e. point-mass approximation) initial conditions suggests either a) the need for computation of customized families of halo orbits considering inertia properties of each spacecraft assigned or b) the use of attitude-only control for station-keeping, which would require no extra fuel when using momentum exchange actuators, in future NRHO missions in order to reduce station-keeping costs which would otherwise be relatively large if the point-mass approximation were used.

REFERENCES

[1] P. Z. Schulte, P. T. Spehar, and D. C. Woffinden, "GN&C Sequencing for Orion Rendezvous, Proximity Operations, and Docking," *AAS Guidance, Navigation and Control Conference*, Vol. 172, 2020.

[2] L. Breger and J. P. How, "Safe Trajectories for Autonomous Rendezvous of Spacecraft," *J. of Guidance, Control, and Dynamics*, Vol. 31, No. 5, 2008, doi: 10.2514/1.29590.

[3] R. Serban, W. Koon, M. Lo, *et al.*, "Halo Orbit Mission Correction Maneuvers Using Optimal Control," *Automatica*, Vol. 28, 2000, doi: 10.1016/S0005-1098(01)00279-5, pp. 257–583.

[4] R. Farquhar, *The Control and Use of Libration-Point Satellites*. PhD thesis, Stanford University, Ph.D.thesis, 1969.

[5] M. Nazari, E. A. Butcher, and W. Anthony, "Earth–Moon L1 libration point orbit continuous station-keeping control using time-varying LQR and backstepping," *International J. of Dynamics and Control*, Vol. 5, No. 4, 2016, doi: 10.1007/s40435-016-0256-8, pp. 1089–1102.

[6] S. Lizy-Destrez, L. Beauregard, E. Blazquez, *et al.*, "Rendezvous Strategies in the Vicinity of Earth-Moon Lagrangian Points," *Frontiers in Astronomy and Space Sciences*, Vol. 5, No. 45, 2019, doi: 10.3389/fspas.2018.00045.

[7] D. Guzzetti and K. C. Howell, "Attitude dynamics in the circular restricted three-body problem," *Astrodynamics*, Vol. 2, No. 2, 2018, doi: 10.1007/s42064-017-0012-7, pp. 87–119.

[8] A. Colagrossi and M. Lavagna, "Dynamical analysis of rendezvous and docking with very large space infrastructures in non-Keplerian orbits," *CEAS Space Journal*, Vol. 10, No. 1, 2017, doi: 10.1007/s12567-017-0174-4, pp. 87–99.

[9] L. Scardovi, N. Leonard, and R. Sepulchre, "Stabilization of Three-Dimensional Collective Motion," *Communications in Information and Systems*, Vol. 8, 2008, pp. 473–500.

[10] A. Sarlette, S. Bonnabel, and R. Sepulchre, "Coordinated Motion Design on Lie Groups," *IEEE Transactions on Automatic Control*, Vol. 55, 2010, doi: 10.1109/TAC.2010.2042003, pp. 1047–1058.

[11] D. Lee, A. K. Sanyal, and E. A. Butcher, "Asymptotic Tracking Control for Spacecraft Formation Flying with Decentralized Collision Avoidance," *J. of Guidance, Control, and Dynamics, in press*, 2014, doi: 10.2514/1.G000101.

[12] M. Nazari, E. A. Butcher, T. Yucelen, and A. K. Sanyal, "Decentralized Consensus Control of a Rigid Body Spacecraft Formation with Communication Delay," *J. of Guidance, Control, and Dynamics*, Vol. 39, No. 4, 2016, doi: 10.2514/1.G001396, pp. 838–851.

[13] A. J. Knutson, *Application of Kane's method to incorporate attitude dynamics into the circular restricted three-body problem*. PhD thesis, Purdue University, Indiana, Jan. 2012.

[14] A. J. Knutson, D. Guzzetti, K. C. Howell, and M. Lavagna, "Attitude Responses in Coupled Orbit-Attitude Dynamical Model in Earth-Moon Lyapunov Orbits," *J. of Guidance, Control, and Dynamics*, Vol. 38, No. 7, 2015, doi = 10.2514/1.G000469, pp. 1264–1273.

[15] D. Guzzetti and K. C. Howell, "Natural periodic orbit-attitude behaviors for rigid bodies in three-body periodic orbits," *Acta Astronautica*, Vol. 130, 2017, pp. 97–113, https://doi.org/10.1016/j.actaastro.2016.06.025.

[16] L. Bucci, M. Lavagna, D. Guzzetti, and K. C. Howell, "Periodic orbit-attitude solutions along planar orbits in a perturbed circular restricted three-body problem for the Earth-Moon system," *Acta Astronautica*, Vol. 147, 2018, pp. 152–162, https://doi.org/10.1016/j.actaastro.2018.03.042.

[17] D. Guzzetti and K. C. Howell, "Attitude dynamics in the circular restricted three-body problem," *Astrodynamics*, Vol. 2, No. 2, 2018, pp. 87–119, 10.1007/s42064-017-0012-7.

[18] D. J. Scheeres, "Stability in the full two-body problem," *Celestial Mechanics and Dynamical Astronomy*, Vol. 83, 2002, pp. 155–169.

[19] D. J. Scheeres, "Relative equilibria in the full N-body problem with applications to the equal mass problem," *Recent Advances in Celestial and Space Mechanics*, pp. 31–81, Springer, 2016.

[20] D. J. Scheeres, "Hill Stability of Configurations in the Full N-Body Problem," *International Astronomical Union (IAU) Symposium*, Vol. 318, 2016, doi:10.1017/S174392131500719X, pp. 128–134.

[21] R. M. Murray, Z. Li, and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*. Boca Raton, FL: CRC Press, 1994.

[22] A. J. Maciejewski, "Reduction, relative equilibria and potential in the two rigid bodies problem," *Celestial Mechanics and Dynamical Astronomy*, Vol. 63, 1995, pp. 1–28.

[23] E. G. Fahnestock, *The Full Two-Body-Problem: Simulation, Analysis, and Application to the Dynamics, Characteristics, and Evolution of Binary Asteroid Systems*. PhD thesis, University of Michigan, 2009.

[24] T. Lee, M. Leok, and N. H. McClamroch, "Lie group variational integrators for the full body problem in orbital mechanics," *Celestial Mechanics and Dynamical Astronomy*, Vol. 98, 2007, doi: 10.1007/s10569-007-9073-x, pp. 121–144.

[25] B. S. McCann and M. Nazari, "Analysis of the Impact of Orbit-Attitude Coupling at Higher-Degree Potential Models on Spacecraft Dynamics," *The J. of the Astronautical Sciences, in press*.

[26] T. Lee, M. Leok, and N. H. McClamroch, "Lie group variational integrators for the full body problem," *Computer Methods in Applied Mechanics and Engineering*, Vol. 196, 2007, pp. 2907–2924.

[27] M. Nazari, E. A. Butcher, and A. K. Sanyal, "Spacecraft Attitude Fractional Feedback Control Using Rotation Matrices and Exponential Coordinates," *J. of Guidance, Control, and Dynamics*, Vol. 41, No. 10, 2018, pp. 2185–2198.